Customized!

Step-by-step instructions to reach your
Customized Goal selected by
DIAGNOSTIC MATH TESTS

Private Tutor for
SAT
Math Success

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Every reasonable effort has been made to make sure this book is free from errors. However, any inadvertent errors discovered are listed in the Corrections page of our web site www.privatetutor.us

Please feel free to report any errors by sending an e-mail to the author Dr. Gulden Akinci at

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Thank you.

Dedication

I would like to dedicate this book to the memory of my father

Hasan Karatepe

Teacher

Renown Special Education Expert

Author

Translator

Poet

Wonderful Father

Dedicated Husband

Acknowledgement

First and foremost I'd like to thank my husband, **Dr. Ugur Akinci**, for his support in several ways. As a writer himself, he edited every page of this book for language, style, format, and content several times. He designed the master pages, front and back covers, Table of Contents, Index, the Private Tutor$^{\text{TM}}$ logo, and some of the illustrations. He provided the proper software to write this book and helped me learn how to use it. He worked tirelessly to find the best solution to publish this book. He created and maintained the www.privatetutor.us web site to support this project. Most importantly, I'm grateful for his morale support. Without his loving care and patience this book could never have been created.

My special thanks go to **Ayla Ictemel**, **Prof. Mubeccel Demirekler**, and **Berna Unal** for their technical editing of various chapters and their friendship. Their careful and insightful editing corrected several errors and improved the quality of many questions and their solutions throughout this volume. Their sincere encouragement was and is crucial to the completion of the task.

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Finally, I'd like to thank **my students**, including my own son **Ersin Akinci**, for their inspiration and feedback during the design and preparation of this work. Their success gave me the energy I needed to continue and bring this projection to completion.

Needless to say, despite all the help and feedback received from all the above-mentioned individuals, the responsibility of any errors in this book ultimately rests with the author.

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1

INTRODUCTION

PRIVATE TUTOR for SAT Math Success is a SAT math preparation guide and workbook in one, written mainly for college-bound high school students. It helps you to become your own private tutor for SAT math preparation.

High SAT score is one of the most effective and quickest ways to impress colleges. By the time you are a senior, it is usually too late to increase your GPA, learn how to write excellent essays for college applications or join the school's varsity sports team. But if you know how to study, increasing your SAT math score shouldn't take more than one hour each week for about one to two months of study. For some students, this period can be as short as a few days.

Time is a scarce commodity for the high school juniors and seniors. To maximize their gain within minimum time, we have developed a method by examining several SAT tests. This method is tested with great success on students at different levels of preparation.

The Private Tutor Method teaches real math, as well as great test-taking techniques. Studying this book will increase your math grade and your score in other tests like SAT II math, ACT math, AP math and standardized math tests as well.

We are aware that many parents feel the pressure as much as (if not more than) their children. **This book is also for the parents who would like to help their children get ready for SAT.** Parents don't need to know any math to do so. Anybody who can follow simple directions can use this book to help students prepare for SAT. **Just follow the instructions in Chapter 3** and you will witness your child's improvement yourself.

The Private Tutor Method

This book helps each user to become his/her private tutor. It has all the tools necessary, including four SAT-like diagnostic tests, to enable the student to evaluate himself/herself, pick a realistic goal and follow the step-by-step, individualized, specific instructions towards that goal.

The instructions are presented at **six different levels** to cover all the users, from the lowest to the highest achievers.

The Private Tutor Method will help you to achieve the maximum score on the topics that you already know. It will also direct you to concentrate on your weak points at a level appropriate for you, **without wasting any time** in studying the subjects that you already know. **It is the only SAT preparation book with these features in the market today.**

To facilitate an accurate diagnosis of your strong and weak areas, **the SAT math subjects are divided into 50 unique math topics.** These topics do not always correspond to the "official" math subjects. We created them after carefully analyzing many students and SAT exams. "Numbers between –1 and 1" and "Formulation Only" are two such categories.

Each topic is covered with several examples. Practical exercises are provided to help you learn the subjects at a basic level. Finally, at the end of each chapter, you will find exercises similar to those in the real SAT. The solutions to these exercises are also provided.

There are about 1000 examples and exercises in the book covering these 50 topics. Each example and exercise is labeled to reflect its difficulty level, **Easy, Medium** and **Hard**, to allow you tackle the ones appropriate for you.

In addition, we also provide **unique, individualized test taking, guessing and time management techniques** for each student. Most of these techniques don't require deep math knowledge. They guide you to find the correct answer by using common sense and smart guessing.

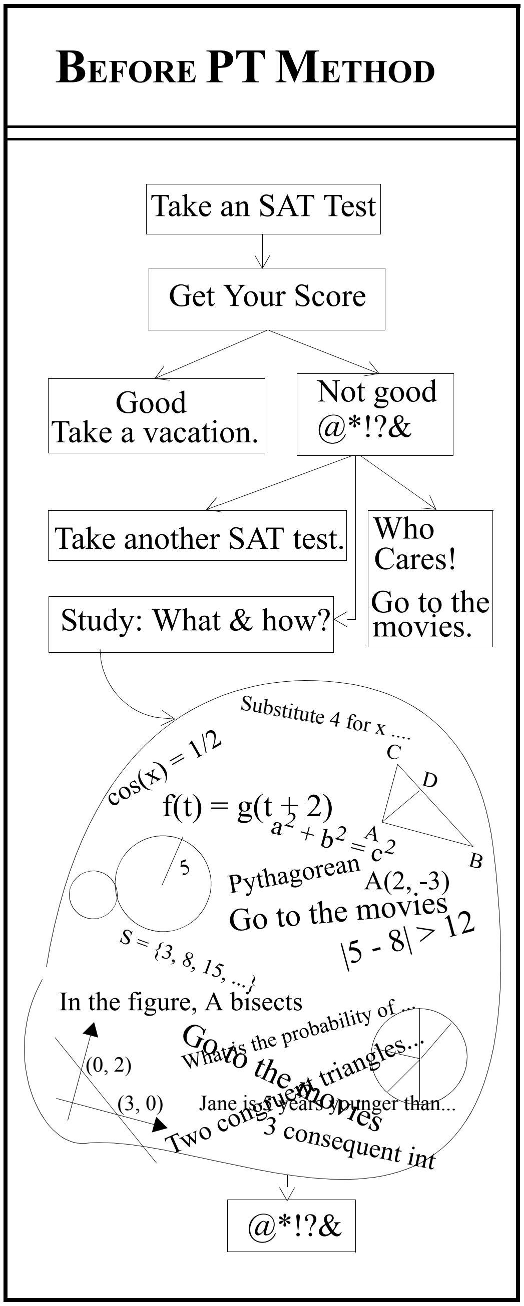
The book is designed so that each student can read it many times and get something different each time at a different level.

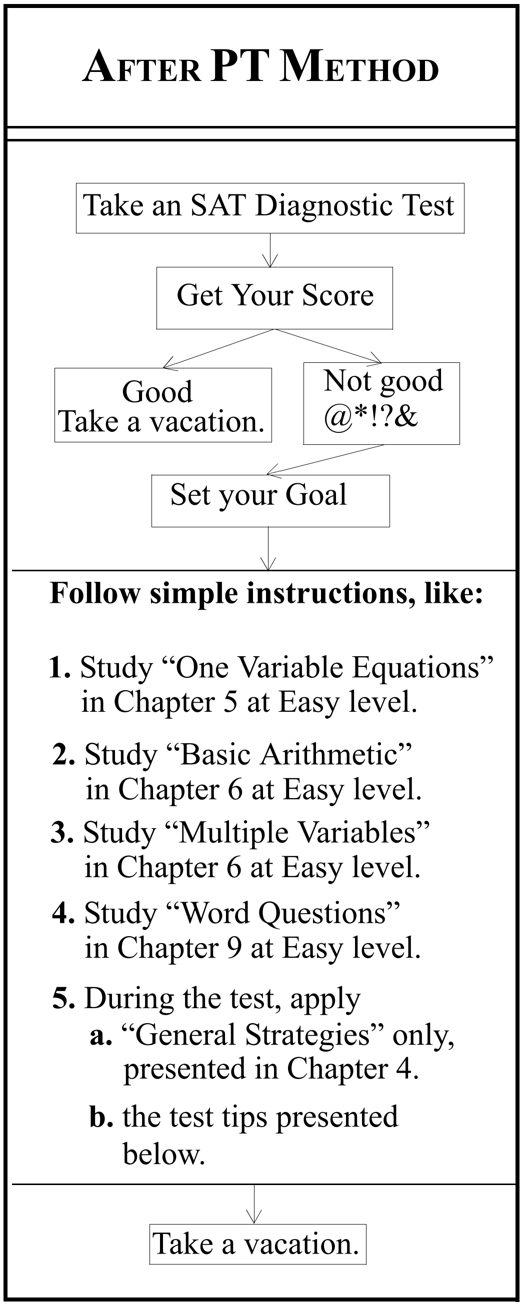
This future of the book helps the readers accomplish their goals at their own pace. If necessary, you can read the book more than once, each time at a different level until you achieve your ultimate goal.

The Private Tutor Method pays great attention to the real SAT test's format, number of questions on each math subject at each difficulty level.

We have spent utmost care to cover the changes to the SAT math exam, introduced in 2005.

Private Tutor Method Advantage





Follow simple instructions, like:

1. Study "One Variable Equations" in Chapter 5 at Easy level.
2. Study "Basic Arithmetic" in Chapter 6 at Easy level.
3. Study "Multiple Variables" in Chapter 6 at Easy level.
4. Study "Word Questions" in Chapter 9 at Easy level.
5. During the test, apply
   - **a.** "General Strategies" only, presented in Chapter 4.
   - **b.** the test tips presented below.

Summary of the Chapters

You are holding in your hands one of the most updated books available today on SAT math success.

Chapter 2, "About the SAT" provides information about SAT. It also helps the reader to understand the method of this book and how it relates to the actual SAT test.

Chapter 3, "Diagnostic Test" is the core chapter of the book. Here the student takes a diagnostic test, determines a realistic goal, and is instructed what subjects to study, at what level and in what order.

The instructions suggest not only how to prepare for the test, but also what questions to answer and what techniques to use during the test, all personalized for each student.

To get the most benefit from this book, you need to follow the instructions provided in this chapter closely. If you are a parent or a tutor, make sure that your student follows these instructions.

Chapter 4, "Non Academic Strategies" is a collection of test-taking and time-management strategies and techniques. Whenever appropriate, each technique is explained by examples. These techniques are not simple short cuts to increase SAT scores. Instead, they teach you how to think creatively.

Chapters 5 through 8, "Arithmetic", "Geometry", "Algebra" and "Others" present the basic information on 47 math subjects, several examples and exercises in each of the subjects. Your instructions will suggest which of these subjects you need to study. Each subject is complete so that you will not need any supplemental materials.

Chapter 9, "Word Questions": Roughly 1/4 of the SAT questions are word questions. Some of them are regular word questions and some others are descriptions of algebraic formulas or geometrical figures. Some of the students are good in math but not as good in verbal skills. We have written this chapter to provide an excellent opportunity to exercise your word skills.

Chapter 10, "Sample Tests" presents 4 sample SAT math tests. The answers, the solutions and all the tools necessary to analyze your performance are provided for each test. Again, you will be instructed when to take these tests.

Appendix A, "The Analysis Chart" provides extra help on analyzing your test results.

Appendix B, "Measuring Distances and Angles" presents methods to measure the distances and angles accurately by using only the materials available during the test.

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ABOUT SAT

General Information

The SAT is a well known college entrance exam developed by the College Entrance Examination Board. The SAT Test Development Committee, TDC, oversees the test and reviews the questions for the College Board.

Most colleges publicly make available the range of acceptable SAT scores for their institutions. You can check these requirements for your favorite colleges to have an idea about the effort you need to make to increase your SAT score to the level that these colleges require.

SAT score is only one of the many criteria that colleges look for in a candidate. Here is a list of the most commonly used criteria by most colleges in their admission process:

- High school academic record known as GPA (Grand Point Average).
- Letters of recommendation.
- One or more essays written by you.
- SAT I or ACT score.
- SAT II score (mathematics, writing, and a subject of student's choice are required by most colleges.).
- AP scores.
- Extracurricular activities (sports, social activities, academic activities, music and art).
- Leadership capacity.
- Awards.
- Motivation.
- Most private colleges also interview the student.

SAT Math Sections and Types of Questions

There are 3 math sections in each SAT. However, TDC tests future SAT math questions once in every three SAT by adding an extra math section. Therefore you will find **3 to 4 math sections** in the actual test. If there are 4 sections, your grade from one of these sections will not count toward your final SAT score. The extra section will be used in preparation of the future SATs.

<u>Do not try to guess which section is not for real.</u> You will waste valuable time and lose your concentration. The best is to try to answer all the questions as best as you can.

1) **The first math section** is 25 minutes long and has 20 multiple choice questions. For each question there are 5 possible answers.
2) **The second math section** is also 25 minutes long and has 18 questions. It has two parts.
    - **The first part** has 8 multiple choice questions. For each question there are 5 possible answers.
    - **The second part** has 10 grid-in questions. For these questions you are expected to solve the problem and punch your answer in the answer sheet.
3) **The third section** has16 questions and you are allowed 20 minutes to complete it. They are all multiple choice questions with 5 answer choices.

There are **a total of 54 questions** that will count toward your score and you will have **1 hour and 10 minutes** to answer them.

All the questions effect your SAT score equally. Four wrong answers cancels out one correct answer if the question is a multiple choice question. There is no penalty for wrong grid-in question. There is also no penalty for the questions that you skip.

Before the test, you should familiarize yourself with both types of questions and the instructions for each. You will have plenty of chances to do just that by following this book.

Difficulty Levels

TDC classifies the questions at **3 different levels of difficulty, “Easy” (E), “Medium” (M) and “Hard” (H)**. In general, each section starts with easy questions and the level of difficulty increases gradually from “Easy” to “Hard.”

The 30 - 50 - 20 Rule

For each section, approximately the first 30% of the questions are “Easy”, 50% are “Medium” and the last 20% are “Hard”. We call this the “30 - 50 - 20” rule.

The 30 - 50 - 20 rule is only an approximate representation of the real SAT. There are exceptions to this rule in each test. It is not uncommon to find a “Hard” question in the middle of a section.

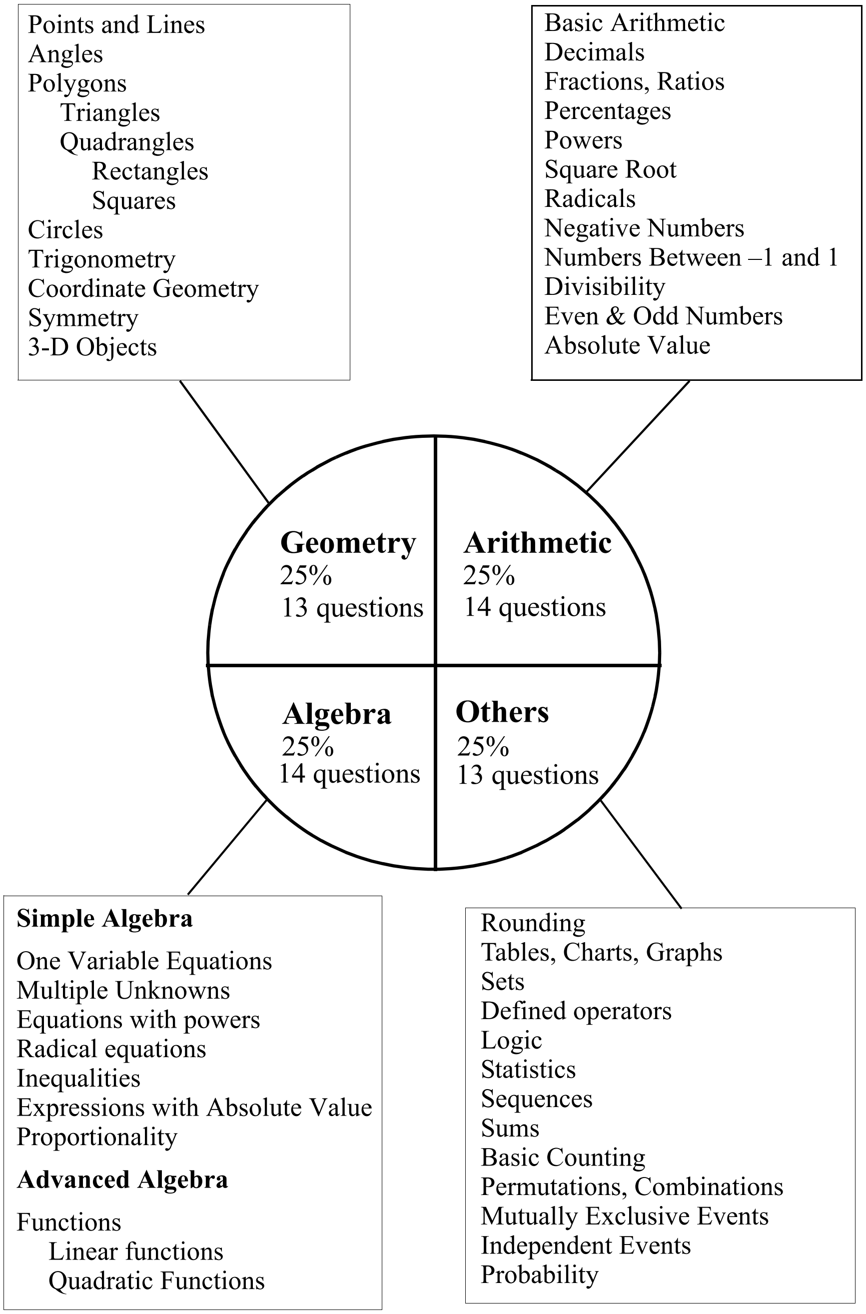
In the following chapters, we will utilize the 30 - 50 - 20 Rule to suggest individualized strategies to fit your individual needs.

Subjects of SAT

All the questions in SAT math test are in **4 general subjects**, which we further divide into 47 subcategories. The pie chart below shows the math categories and the number of questions asked in each category. Keep in mind that several questions fall into more than one category. Therefore the number of questions indicated in this chart is only an approximation. They are included to give you a rough idea.

In this book, **each of these 47 subjects are explained by examples and exercises** at three different difficulty levels.

It may look a lot, but fortunately **you don’t have to learn all the subjects shown in the Subject Chart to increase your SAT score.** For example, most of the Easy questions are in “Basic Arithmetic” and “One Variable Simple Equations.” If your score is low, all you need to do is learn these two subjects, and only at Easy level to increase your score substantially.



Subject Chart

3

DIAGNOSTIC TEST

Before using the methods and techniques in this book, you must take the Private Tutor diagnostic test which will give you the information you need to analyze yourself and help you choose the right strategy for your preparation.

Even if you have taken several SAT tests and know your score, you must still take this test because your final SAT score itself does not automatically help you understand which areas you need to improve for a higher score. Very often, two students with the same SAT score need completely different strategies to raise their SAT scores.

There are four diagnostic tests in Chapter 10. They are very similar to the real SAT in format and essence. They have the same sections with similar questions. Like in the real SAT, we have included the answer sheet, the difficulty level for each question, the Scoring Worksheet and the Score Conversion Table. In addition, we have provided the solutions to the questions and an "Analysis Chart" for each test.

Before the Test

Follow these instructions before you take the diagnostic test:

(1) Find a quiet room without any interference from your friends or family.
Turn off the TV, radio, music sets and telephones.

(2) Familiarize yourself with the overall format of the test. Although the diagnostic tests are very similar to the real SAT, there are few differences as well:

- **(a)** In each section, you'll see position markers reminding you where you are in the test: at the beginning of the section (**B**), after finishing one fourth of the questions (**1/4**), after finishing one third of the questions (**1/3**), after finishing the first half of the questions, (**1/2**), after finishing the two thirds of the questions, (**2/3**) and the last question (**L**). Later in this chapter you'll see how these markers are utilized.

  Note that **you will not see these marks on a real SAT**. You need to guess their aproximate positions on your test.

- **(b)** Test instructions and answer sheets are somewhat different in format between the diagnostic test and the real SAT.

(3) Make sure you have a clock that you can easily see during the test. A digital clock with big numbers is the best.

(4) The time allocated for each section is noted in the beginning of each section - just like it is in the real SAT. Set the clock to the time allowed for each specific section. This will prevent you from worrying about going overtime while making sure you spend the correct amount of time in each section.

During the Test

Follow these instructions during the diagnostic test:

(1) Don't take any breaks during the test. If you finish a section early, you can check your answers or just wait until the time allotted for that section is up. Imagine that you are taking the real SAT test. You are not allowed to eat, talk, walk out of the room or listen to music during the real test. So don't do these things during the diagnostic test either.

(2) During the test, don't try to check your answers against the answer key provided.

(3) Chapter 10 provides 4 different tests. Take the TEST 1 first. Once you finish the test, continue with "After the Test" section below and follow the instructions. These instructions will lead you step by step in analyzing and improving your test results.

After the Test

Follow these instructions after you take the diagnostic test:

(1) You can take a break, but not longer than 24 hours.

(2) After the break, check your results against the answer sheet provided at the end of each test and mark the wrong answers. Don't look up the solutions provided.

(3) Calculate your SAT score by using the tools provided at the end of each test.

(4) If you are satisfied with your results, or if you have taken all four diagnostic tests in this book, you can stop now and take the real SAT. Otherwise, continue with the following section.

Fill Out Your "Analysis Chart"

If your score is 490 or less go to the next section, "Set Your Goal."

If your score is more than 490, you need to fill the “Analysis Chart,” provided at the end of each test in Chapter 10. This chart is designed to help you understand your strong and weak points. The results will be used in “What To Do Next” section that follows.

Analysis chart has all the questions of the test grouped by subject, category and difficulty level. Questions that belong to more than one subject are repeated for both subjects.

To help you find all the occurrences of a wrong or missing answers, we have also provided a **Subject Table** for each test. This table gives the subjects and the difficulty levels of each question in the test. Find the subjects and the difficulty level of each incorrect or missing answer on the subject table and mark these subjects on the analysis chart.

See Appendix A for more information about the Analysis Chart.

Set Your Goal

You can decide on what action to take depending on where you are now, what your score is, and where do you want to be. You must set a realistic goal for yourself. Setting unrealistic goals usually leads to failure and frustration.

This book enables you to make gradual yet assured progress.

Each time you take a diagnostic test, adjust your goal to achieve a new and higher score. We suggest the below “Realistic Goal Table” for your convenience. It is designed after analyzing many SATs. Breakpoints and goals are set to maximize your gain in minimum time. Find your score in this table and select a realistic goal.

Realistic SAT Goal Table

Private Tutor Diagnostic Test Score	Realistic SAT Goal
Less than 250	300
250-300	400
300-400	490
400-490	550
490-560	620
560-620	670
620-670	720
670-720	760
720-760	Over 760

Customized Instructions

This section provides individualized, step by step directions. All you have to do is:

(1) Find your level below. Don’t be discouraged if you end up in the same level more than once. As long as you make progress, it is okay to go over the same material.

(2) Read the instructions provided from the beginning to the end.

(3) Go back to the beginning and follow the directions step by step.

Diagnostic Test Score 250 or Less

Your realistic SAT goal is 300.

(1) You need to learn some of the math subjects only at Easy level. You also need to learn to apply some of the test taking strategies. Below table shows which sections of which chapters you have to review. Study these subjects in the order given in this table. Examine all the examples and solve all the exercises of these subjects at Easy level.

Chapter	Section	Level
5	Addition, Subtraction, Multiplication, Division	Easy
7	Simple Algebra - One Variable Simple Equations	Easy
7	Simple Algebra - Multiple Unknowns	Easy
9	Word Questions	Easy
4	General Strategies	Easy

(2) If this is the first or second time you have reached this point in the book, go to step 4.

(3) If this is the third time you have reached this point in the book, study the solutions provided to all the Easy questions in the diagnostic test. Make sure that you understand the solutions and can relate them to the subject that you have studied. If you have an incorrect answer to a question, look at your solution and pinpoint your error.

(4) Read the "Test Tips" section below. Then go back to the beginning of this chapter and repeat the same test. You will see that your score will be higher. Depending on your new score, pick a new goal. Just follow the new instructions listed in the section written for your new goal. Good luck.

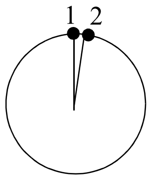
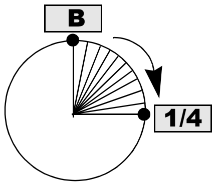
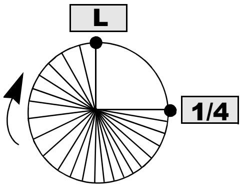
Test Tips for Diagnostic Score 250 of Less

(1) Remember that all you have to do to achieve your goal is to answer 3 questions correctly and not answer the questions that you don't know.

(2) It is very important that you take your time for each question. As explained in Chapter 4, questions are arranged from Easy to Hard. It is okay not to answer all the questions as long as you answer a few easy ones.

(3) Do not guess any of the answers. If you are not sure of the answer, just leave the question unanswered.

(4) Below instructions (second column) depends on what point you are at (first column) during each test section. Read and follow them carefully during the next test:

Questions You are Answering	What To Do?
 	• Answer only the first two questions of the section, if the section doesn't have two parts. If the section has two parts then answer only the first questions in each part. When you finish the first two questions in the section, stop and relax for a moment. Then go back and make sure that your answers are correct.
 	• Attempt to answer the rest of the questions only in the first one-fourth of each section, between B and 1/4 marks. • When you reach 1/4 sign, stop for a few moments and go back and try the questions that you couldn't do. • Don't guess your answers. If you are not sure, just skip the question. • When the alarm clock goes off, just stop. Leave the rest of the questions unanswered.
 	• Do not solve any of the remaining questions between the signs 1/4 and L.

Diagnostic Score greater than 250, but less than or equal to 300

Your realistic SAT goal is 400

To achieve your goal, your score must increase by more than 100 points. This is not easy, but you can do it. Here is how:

(1) You need to learn some of the math subjects at only Easy and/or Medium levels. You also need to learn to apply some of the test taking strategies. Below table shows which sections and chapters you need to study. Study these subjects in the order presented in this table. Start at Easy level first. Once you are finished studying all the subjects at Easy level, study them at Medium level from the beginning. Examine all the examples and solve all the exercises of these sections at the level provided in the last column.

Chapter	Section	Level
5	Addition, Subtraction, Multiplication, Division	Easy-Medium
7	Simple Algebra - One Variable Simple Equations	Easy-Medium
7	Simple Algebra - Multiple Unknowns	Easy-Medium
5	Decimals, Fractions, Ratios and Percentages	Easy-Medium
5	Powers	Easy-Medium
5	Square Root	Easy-Medium
5	Negative Numbers	Easy-Medium
6	Points and Lines	Easy-Medium
6	Angles	Easy-Medium
6	Polygons	Easy
6	Circles	Easy
7	Equations with Multiple Unknowns	Easy
7	Equations with Powers	Easy
7	Radical Equations	Easy
7	Advanced Algebra - Functions	Easy
8	Rounding	Easy-Medium
8	Data Representation	Easy-Medium
8	Sets	Easy-Medium
8	Logic	Easy
9	Word Questions	Easy-Medium
4	General Strategies	Easy
4	100% Accurate Guessing Methods (first 4 methods only)	Easy
4	Random Guessing Method	N/A
4	Time Management Methods	Basic

(2) If this is the first or second time you have reached this point, go to Step 4.

(3) If this is the third time you have reached this point in the book, study the solutions provided to all the Easy questions in the diagnostic test. Make sure that you understand the solutions and can relate them to the subject that you have studied. If you have an incorrect answer to a question, look at your solution and pinpoint your error.

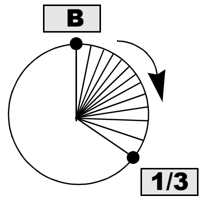
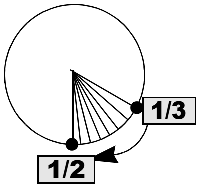
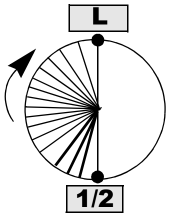
(4) Read the "Test Tips" below. Then go back to the beginning of this chapter and repeat the same test again. You will see that your new score is higher. Depending on your new score, pick a new goal. Just follow the new instructions for your new goal. Good luck.

Test Tips for Diagnostic Score between 250 and 300:

(1) It is very important that you take your time for each question. As explained in Chapter 4, questions are arranged from Easy to Hard. It is okay not to answer all the questions. All you have to do is to answer most of the easy ones.

(2) For multiple choice questions, eliminate the wrong answers only if you are 100% sure that the answers are wrong. If you have the slightest doubt, do not eliminate any answers. It is better not to answer such questions.

(3) Below instructions (second column) depend on what point you are at (first column) during each test section. Read and follow them carefully during the next test:

Questions You are Answering	**What To Do?**
	• When you finish one-third of the questions and reach the 1/3 mark for each section, stop and relax for a moment. Then go back and try the questions that you couldn't do or were not sure of. • If you couldn't reach the 1/3 mark before the time is up for the section, you need to speed up a little. This will happen automatically as you practice more. Don't rush your answers just to go faster. • Continue below only after you try your best for the first one-third of each section. • While guessing, use the methods you have learned in "100% Accurate Guessing Methods" section of Chapter 4.
	• When you finish the first half of the questions and reach the 1/2 mark, stop and relax for a moment. Then go back to those questions that you are not sure of. Check all of your answers again. • When the alarm clock goes off, just stop. • While guessing, use "Random Guessing" method.
	• Do not solve any of the remaining questions.

Diagnostic Score greater than 300, but less than or equal to 490

Your realistic SAT goal is 490-550.

(1) At this level, you can achieve your goal by correctly answering all of the Easy and most of the Medium questions.

To achieve your goal, study the sections in the order given in the below table. Study first all the Easy examples and solve all the Easy questions in these categories. Then study all the Medium examples and solve all the Medium questions.

Chapter	Section	Level
5	Arithmetic (all subsections)	Easy + Medium
6	Geometry (all subsections)	Easy + Medium
7	Algebra (all subsections)	Easy + Medium
8	Rounding	Easy + Medium
8	Data Representation	Easy + Medium
8	Sets	Easy + Medium
8	Defined Operators	Easy + Medium
8	Logic	Easy + Medium
8	Sequences	Easy + Medium
8	Statistics	Easy + Medium
8	Basic Counting	Easy + Medium
9	Word Questions	Easy + Medium
4	All Guessing Methods	Easy + Medium
4	Time Management Methods	Basic

(2) If this is the first or second time you have reached this point in the book, go to step 4.

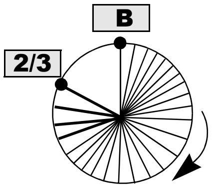
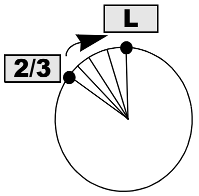
(3) If this is the third time you have reached this point in this book, study the solutions provided to all the Easy and Medium questions in the diagnostic test. Make sure that you understand the solutions and can relate them to the subject that you studied. If you have an incorrect answer to a question, look at your solution and pinpoint your error.

(4) Read the "Test Tips" below. Then you are ready to go back to the beginning of this chapter and repeat <u>the same</u> test again. You will see that your new score is higher. Depending on your new score, pick a new goal. Just follow the new instructions for your new goal. Good luck.

Test Tips for Diagnostic Score between 300 and 490:

(1) Eliminate the wrong answers only if you are 100% sure that the answers are wrong. If you have the slightest doubt, <u>do not</u> eliminate any answers. It is better not to answer such questions.

(2) Below instructions (second column) depend on what point you are at (first column) during each test section. Read and follow them carefully during the next test:

<table>
<tr><th>Questions You are Answering</th><th>What To Do?</th></tr>
<tr><td> </td><td>• <u>When</u> you finish Easy and Medium level questions and reach the 2/3 mark, stop and relax for a moment. Then go back and try the questions that you couldn't do or were not sure of.
• If you couldn't reach the 2/3 mark before the time is up for the section, you need to speed up a little. This will happen automatically as you practice more. Do not rush your answers just to go faster.
• If you couldn't eliminate all the wrong answers with 100% certainty and if the question is within the first one-third of the section, between B and 1/3 marks, apply the methods you have learned in "Finding the Best Possibility" section and make your best guess. Otherwise use the "Random Guessing" method.</td></tr>
<tr><td> </td><td>• You are now in the bonus zone. Answer as many questions as you can -- but only if you are 100% certain. It is better to leave these questions unanswered if you are not 100% sure of the answer.
• When the alarm clock goes off, just stop.</td></tr>
</table>

Diagnostic Score greater than 490, but less than or equal to 560

Your realistic SAT goal is 620.

At this level, you need to be able to answer all the Easy and Medium questions correctly. In addition, you should be able to answer or guess some of the Hard questions. These Hard questions will compensate for some of the errors you might have made at the easier levels.

(1) Fill your Analysis Chart. See Appendix A if you need help. Example 2 in this appendix is designed to help the students utilize the Analysis Chart at this level.

(2) Look at your Analysis Chart and identify all the subjects in which you have made a mistake or missed the questions at the Easy and Medium levels. Study these topics thoroughly. Solve all the Easy and Medium questions for these subjects. If necessary, refer to the text. Use the hints and finally, look up the answers and understand the solutions.

(3) Look at your Analysis Chart and find the subjects in which you did well. These are the subjects where you have answered at least one of the Hard questions correctly. If there is none, identify the subjects in which you have answered all of the Easy and Medium questions correctly. These are your strong areas. See Appendix A if you need help. Since you will need to answer some of the Hard questions as well, it is best to try the subjects that you are good at.

Go to the appropriate chapter and solve all the questions labeled Medium and Hard in these subjects. If necessary, refer to the text. Use the hints and finally, look up the answers and understand the solutions.

(4) If you have not yet read Chapter 4, *Guessing Methods and Time Management*, read it now. Study all the guessing methods and solve all the questions at the Easy and Medium levels and learn basic time management techniques.

(5) If you could not finish the test on time, pay special attention to the Basic Time Management methods. Make sure to apply them in the next test.

(6) Go to the same test and follow the instructions below:

- **(a)** Without looking at the solutions, answer all the questions that you have missed or answered incorrectly.
- **(b)** If you guessed the answers wrong, think about how you can improve your guessing techniques. Also try to guess the answers for the questions that you have missed. Try to find a suitable method from Chapter 4 to guess the answers better.
- **(c)** Study the solutions given for the questions that you have missed or answered incorrectly. If you have an incorrect answer to a question, compare your solution to the one given in this book, and pinpoint your error.

(7) Read the "Test Tips" below. Then you are ready to go back to the beginning of this chapter and repeat the procedure with the next test in Chapter 10. You will see that your new score is higher. Depending on your new score, pick a new goal. Just follow the new instructions for your new goal. Good luck.

Test Tips for Diagnostic Score between 490 and 560:

(1) Don't panic if the time is running out. Do not guess the answer just for the sake of guessing. If you can't eliminate at least one of the answers with 100% confidence, leave the question unanswered.

(2) Use "Random Guessing" technique for the Hard questions after the $\boxed{2/3}$ mark.

Diagnostic Score greater than 560, but less than or equal to 670

Your realistic SAT goal is a score in between 670-720.

(1) Look at your Analysis Chart and identify all the subjects in which you have answered one or more questions incorrectly or skipped. Some subjects are not included in every diagnostic test. To make sure that you know these subjects, identify them on your Analysis Chart as well.

The only topics you may skip are combinations, permutations, independent events and Hard probability examples and exercises in Chapter 9.

See Appendix A if you need help. Example 1 in this appendix is designed to help the students utilize the Analysis Chart at this level.

(2) Go to the appropriate chapter and study all the subjects at all levels that you have identified in the previous step. Solve all the exercises. If necessary, refer to the text. Use the hints and finally, look up the answers and understand the solutions.

(3) If you have not yet read Chapter 4, *Guessing Methods and Time Management*, read it now. Study all the examples and solve all the questions at all levels. Concentrate on the techniques that you have not used in your test. Try to identify the questions that you could have answered correctly if you had applied any of the techniques explained in this chapter.

(4) If you could not finish the test on time, pay special attention to both the Basic and Advanced Time Management methods. Make sure to apply them in the next test.

(5) Go back to the same test and follow the instructions below:

   (a) Answer all the questions that you have missed or answered incorrectly without looking at the solutions.

   (b) If you guessed the answer wrong, think about how you can improve your guessing techniques. Also try to guess the answer for the questions that you have missed. Try to find a suitable method from Chapter 4 to guess the answer better.

   (c) Study the solutions given for the questions that you have missed or answered incorrectly. If you have an incorrect answer to a question, compare your solution to the one given in this book, and pinpoint your error.

(6) Read the "Test Tips" below. Then you will be ready to go back to the beginning of this chapter and repeat the procedure with the next test in Chapter 10. You will see that your new score is higher. Depending on your new score, pick a new goal. Just follow the new instructions for your new goal. Good luck.

Test Tips for Scores between 560 and 670:

At this level, managing time becomes an important factor for success. If you are not monitoring your timing, start monitoring now.

Record your time 3 times in each section when you see the $\boxed{\textbf{B}}$, $\boxed{\textbf{1/2}}$ and $\boxed{\textbf{L}}$ marks. You will need these measurements to apply some of the advanced time management techniques.

(1) If you can't eliminate at least one of the answers with 100% confidence, leave the question unanswered. While guessing, use the "Random Guessing" technique for the last part of each section after you reach $\boxed{\textbf{2/3}}$ mark.

(2) **Always answer a grid-in question.** You are not penalized for the wrong answers in this section. Even if you are clueless, make a reasonable guess. For example, if the question is about an adult's age, enter your own mother's age. If it is about the speed of a car, answer "42 miles/hr." If it is about the radius of a circle, look at the figure and make a guess.

(3) Check your Analysis Chart again. If your wrong answers are randomly spread at all levels, that means you probably lose your concentration from time to time. If this is the case, make sure that you are constantly aware of the question and the answer you provide. If you feel that your mind is wandering, take a few seconds off and answer the question again.

Randomly-spread wrong answers may also mean that you are overconfident. Overconfidence will make you careless. That's why you may be overlooking a smart twist in the question or in the answer choices. Sometimes due to overconfidence, you may even miss some of the information provided in the question. Pay proper attention to each question, at all levels.

Diagnostic Score greater than 670

Your realistic SAT goal is a score greater than 720.

At this level, you are able to answer about 90% of all the math questions correctly. In each test there are two or three questions that are tricky. The reason you are missing the remaining three or four questions is probably due to the lack of enough practice in a particular subject. More often than not, "accidental" errors are due to unclear concepts or not-so-accurate methods you may be using.

Even at this level you may need more practice. Suppose you are not very clear about the questions that involve powers. You may successfully find the correct answers for the Easy questions by substituting numbers but you are more likely to make mistakes in harder questions. To be able to get a perfect score, you need to use proper methods.

Follow the instructions below:

(1) Follow all the instructions including the "Test Tips" given in the previous section ("Diagnostic Score more than 560, less than or equal to 670") with two exceptions:

  (a) Don't use the "Random Guessing" method. Instead, when you have to guess, apply the methods you have learned in the "Finding the Best Possibility" section in Chapter 4.

  (b) Don't exclude any of the subjects when you are identifying your weak points. At this level, you need to know all the subjects at all levels.

(2) Study all the Very Hard examples and exercises provided in Chapters 5, 6, 7, 8 and 9. They will not be in your real SAT, but if you pursue perfection, you need to go beyond what is expected of you.

(3) For the problem areas in your Analysis Chart, read the solutions to the problems labeled Medium and Hard even if you have correctly solved them. This book's approach may help you learn new and more efficient ways of solving problems, which will help you during the test.

(4) A very good method of learning a subject is to teach it. Write 3 questions at each of the three levels on your weak subjects and answer them.

(5) Read the "Test Tips" below. Then you will be ready to go back to the beginning of this chapter and repeat the procedure with the next test in Chapter 10. You will see that your new score is higher.

Test Tips for Diagnostic Score greater than 670:

(1) At the end of each section and after finishing your corrections stop and relax for a moment. Then, if you still have time, review the answers you have provided to the last third of the questions and verify them.

(2) If you are a student with a goal of perfect score 800, and if you are in the last section of the test, and you think that you have answered all the questions in the previous sections correctly, you can guess the answer to one question even if you can not eliminate any of the choices with 100% certainty.

4

PRACTICAL STRATEGIES

In this chapter, you'll learn test taking techniques that will greatly enhance your score. You can use these strategies and the methods if you can't answer a question in a normal way. Some of the techniques do not require any knowledge of the subject; some others do.

The techniques described here are no substitute for solving a question in the "good old fashion" way. Use these methods only if you can't solve the problems by using proper methods which utilize your math knowledge and reasoning ability.

You may be familiar with some of these techniques. You may even think that the method used is the proper way to solve the problem. To demonstrate the difference, we include the proper solution to each example.

You can find several exercises to practice in Chapters 5 to 10. Use the techniques you learn here in the chapters ahead and during the next diagnostic test whenever you can't solve a problem by using your math knowledge alone.

We have also included both basic and advance time management techniques in this chapter. It is very important that you learn them before the test and apply them during the test.

General Strategies

A good grasp of all the math subjects is a great advantage. However, almost nobody has the desire, ability or time to acquire perfect knowledge. Even when you know all the material, the questions in SAT do not very openly and directly test your math knowledge. You may know the subject, but still you may not be able to answer the questions correctly, mostly because of the way they are asked. Methods below will help you increase your score substantially. If you think math is boring, you can study these methods and have fun while raising your score.

Before the Test

Strategy 1 - Get Regular Exercise

Exercise regularly for at least 3 weeks before the test. Half an hour a day of running, soccer playing, swimming or any other sport that you like will do. It will increase you energy level and raise your metabolic rate and sharpen your mind. If you are not a physically active person, make sure that you don't overdo it. Never start vigorous exercise just two or three days before the test. Don’t exercise the day before the test.

Strategy 2 - “Tools of the Trade”

Make sure you have two number 2 pencils, a pencil sharpener, an eraser that erases well and your calculator. Make sure your calculator has fresh batteries. Don't use a brand new calculator. Use a calculator that you are comfortable with. Set these necessary tools aside the day before the test.

Strategy 3 - Learn Your Way to the Test Location

Make sure you know the location and the time of the test. Also make sure that you know how to get there. Prepare the directions a day before the test. If you want to start the exam with a peaceful mind, make sure how to get there on time.

Strategy 4 - Sleep Well

Set the alarm clock to one hour before the time you plan to leave home. Sleep well the night before the test. There is no point studying just the night before the test. A good night’s sleep will help you more.

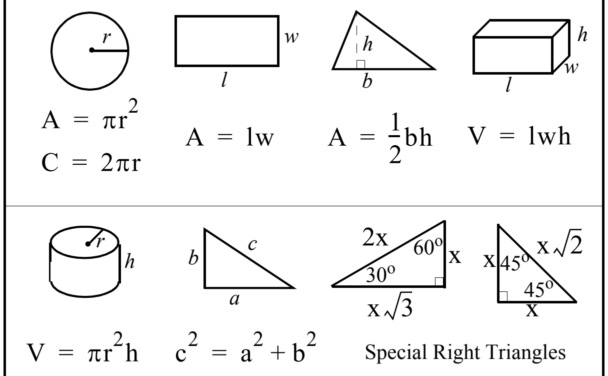
Strategy 5 - Get Up On Time & Dress Comfortably

On the test day, wake up 1 hour before you need to leave home. Have a good breakfast. Dress comfortably for the test. The test is 3 hours and 20 minutes long. It is long enough to make you feel cramped. If your outfit is too warm, too skimpy on a cold day, too tight or your shoes are hurting it will effect your performance.

Strategy 6- Learn the Information Provided in SAT

SAT does not require you to memorize long formulas or unreasonable information. To help you answer some of the questions that may require memorization, SAT offers some of the information you will need. At the beginning of each math section, you will see the below “Reference Information.” However, you are expected to know how to use the given information. Therefore it is important that you familiarize yourself with it before the test. There is not much time to learn such formulas during the test. Below is the reference information provided at the beginning of each math section.

Reference Information



$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$

$V = \pi r^2 h$

$c^2 = a^2 + b^2$

The number of degrees of arc in a circle is 360.

The sum of the measures of the angles of a triangle is 180 degrees.

In the above figure, A, V and C stand for the area, volume and circumference.

Strategy 7- Familiarize Yourself with the Forms in the Test

It is important that you know how to fill in your name, social security number, etc. correctly. Also make sure that you know how to fill in the answers for both multiple choice and grid-in questions.

Together with each of the four tests in Chapter 10, we provide set of test forms that you will find in the real test. This will give you a chance to practice the forms ahead of time.

During the Test

Strategy 1 - Answer the Questions in the Order They Are Asked

In general, not all the time, the questions are organized from Easy to Hard. Starting from the beginning will give you a chance to finish the easier questions correctly and score some easy points. Hard questions do not count more than the easy ones. Answering the easy questions first will also help you warm up and prepare your mind for the difficult ones.

Strategy 2 - Read the Questions Carefully

This is one of the reasons why many of the students lose points. You may think that it is easy to read and understand the questions, but it requires concentration to really understand them. To draw your attention, most of the time, the "unusual" words are underlined or capitilized in the question.

Examples:

1. (Easy)
Jim, Joe and John are 3 brothers. Jim is older than John. Joe is younger than Jim. Joe is 13 years old. Which of the below statements is WRONG?

(A) Jim is older than Joe.
(B) John is younger than Jim.
(C) Jim is older than 13.
(D) John is younger than 13.
(E) Jim is the oldest brother.

Solution:

From the data, you don't know who is older, Joe or John. So you can't conclude that John is younger than 13. The answer is (D).

In this question, you are asked the WRONG choice, not the correct one. This is an unusual situation. In most cases you are asked to identify the correct choice. Hence the word "WRONG" is capitalized to get your attention.

2. (Medium)
If $a^2 > a$, which of the following must always be true?

I. $a > 1$
II. $a < -1$
III. $a^3 > a$

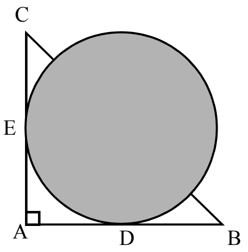
(A) I only
(B) II only
(C) III only
(D) I or II
(E) I and III

Solution:

The important words to notice in the question are "must" and "always". If you did not notice these words, you could be confused since all of the answers seem to be correct, but only case (D) is always true.

For example if $a = -2$, $a^2 = 4$. Since $4 > -2$, $a^2 > a$. On the other hand, $a^3 = -8$. Since $-8 < -2$, III is not correct.
For $a > 1$ and $a < -1$, a^2 is always more than a.

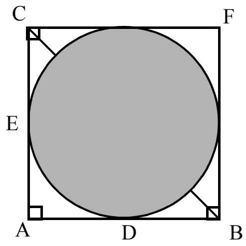
3. (Medium)
The diameter of the below circle = 4. ABC is a right triangle with AC = AB. $\overline{AC}$ and $\overline{AB}$ are tangent to the circle at points E and D, respectively. AE = EC.



What is the area of unshaded region in ΔABC ?

Solution:

The important word here is unshaded, and it is underlined.



To solve the problem, let's draw perpendicular lines to $\overline{CF}$ and $\overline{BF}$ so that the circle is inscribed in the square formed.

The unshaded area =
$\frac{1}{2}$(Area of the square - Area of the circle).

AC = AB = Diameter of the circle = 4

The unshaded area =
$$\frac{1}{2}(4^2 - \pi \cdot 2^2) = \frac{1}{2}(16 - 4\pi) = 2(4 - \pi) \cong 1.72$$

4. (Medium)
If x is a non-negative integer, which of the following can be the value of $x^3 + 1$?

I. 0
II. 1
III. 9

(A) I only
(B) II only
(C) III only
(D) II and III
(E) I and III

Solution:

Case I: If $x^3 + 1 = 0$ then $x = -1$

Case II: If $x^3 + 1 = 1$ then $x = 0$

Case III: If $x^3 + 1 = 9$ then $x = 2$

Since x is non-negative, both Case II and III are correct. So the answer is (D).

If you are not careful, you may think that x is positive, and decide that only III is correct. However, the question states that x is non-negative, meaning that it can be positive or zero. Therefore both case II and III are correct. So the answer is (D), not (C).

Strategy 3 - A Question Is Not Easy Just Because College Board Says So

A question is not easy just because College Board has placed it at the beginning of a section. What's more, "increasing level of difficulty" rule does not hold all the time. In each test, there are some hard questions toward the beginning of the section and there are easier questions toward the end of the section. During the test, your own opinion is the one that counts. So do not panic if you can't answer an "Easy" question.

Similarly, don't underestimate the questions at the beginning of the section. Answer all the questions carefully. Each question has a clever twist appropriate for its difficulty level. Wrong answers that look correct are included in the answer choices to trick you. Remember that College Board prepares these tests every year several times. They are very familiar with the student mentality and their weak points. They make sure that careless thinking will lead you to one of the wrong answer choices.

Strategy 4 - Utilize All the Information

Use all the information given in the question and in the answer. College Board rarely provides any information that is not necessary for the solution. Sometimes it is easy to miss unimportant looking sentences.

Examples:

1. (Easy)
John walks from A to B in 10 minutes. If his speed is 30 meters/minute, how many feet are there between A and B? (1 feet = 0.3048 meter).

Solution:

$$\text{distance(d)} = \text{speed(s)} \times \text{time(t)} \Rightarrow$$
$$d = \frac{30}{0.3048} \times 10 \cong 984 \text{ feet}$$

If you are not careful and miss the conversion factor provided in the question, your answer will be $30 \cdot 10 = 300$ feet. This is a wrong answer and in a multiple choice question, you can be sure that it will be one of the answer choices.

2. (Hard)
Joe correctly solved the 79% of the questions and had a score of 79 on the first math test. In the second test he could solve 5/4 of what he had solved on the first test. In the third and final test he did well but he could not learn his grade. However he had an "A" (average of 3 exams is 90 or above up to 100) for the quarter. Which of the following is the range of his grade for the third test? Grades are approximated to the nearest integer for each exam. All tests have a maximum score of 100 points and all questions in the test effect the final test grade equally. For example 64% correct answer will result in a score of 64.

(A) More than or equal to 90
(B) More than 91
(C) More than 92
(D) More than or equal to 92.25
(E) More than 93

Solution:

1st test score = 79

2nd test score = (5/4)79 = 98.75
Since it is rounded to the nearest integer, the 2nd test grade = 99

3rd test score = x
$((79 + 99 + x)/3) \geq 90 \Rightarrow 178 + x \geq 270 \Rightarrow$
$x \geq 92$

You don't need to find the upper limit of x, because you know that it is 100.
The answer is $100 \geq x \geq 92$, so the answer is (B).

Notice that the way that the correct answer is provided is not usual. If you did not notice that more than 91 is equivalent to 92 and above, you will be confused and probably make a mistake.

If you did not notice that the grades are rounded to the nearest integer for each test, the answer would be
$(79 + 98.75 + x)/3 \geq 90 \Rightarrow 177.75 + x \geq 270 \Rightarrow$
$x \geq 92.25$, which is (D).

3. (Hard)
Joe had 79 on the first math test. In the second test he scores 5/4 of what he scored in the first test. In the third and final test he did well but he could not learn his grade. However, he had an "A" (average of 3 exams is 90 or above, up to 100) for the quarter. If the final average grade is rounded to the nearest integer, which of the following is the range of his grade for the third test?

(A) More than or equal to 90
(B) More than or equal to 90.75
(C) More than 91
(D) More than or equal to 91
(E) More than or equal to 92.25

Solution:

1st test score = 79

2nd test score = (5/4)79 = 98.75

3rd test score = x

Since the final grade is rounded to the nearest integer, if the final grade is between 89.5 - 90, it will be rounded to 90. Hence the average of the 3 grades must be 89.5 or higher for Joe to get an A for the quarter.
$((79 + 98.75 + x)/3) \ge 89.5$ ➔
$177.75 + x \ge 268.5$ ➔ $x \ge 90.75$

So the correct answer is (B).

If you did not notice that the final average grade is rounded to the nearest integer, the result would be $(79 + 98.75 + x)/3 \ge 90$ ➔ $177.75 + x \ge 270$ ➔ $x \ge 92.25$, which is (E).

Strategy 5 - Guess Only If You Can Eliminate at Least One of the Answers with 100% Certainty

For multiple choice questions, if you can eliminate one of the wrong answers with 100% accuracy, guess the answer as best as you can from the remaining possible answers. Guessing the answer after you eliminate one of the answers gives you a statistical advantage.

However, sometimes the answers look correct when they are not. This is by design. If you are not careful, you are more likely to guess the wrong answer and lose your statistical advantage.

In this chapter we have described three different types of guessing methods.

Follow the directions presented in the Chapter 3, "Individualized Step by Step Directions" sections of each level, to choose the preferred method of handling the questions that you are not sure of.

Strategy 6 - Write the Formulas

Sometimes the formulas are described in words. Write them down clearly as you read the question. Make sure that they correspond to the description provided in the question.

Example:

1. (Medium)
What is the number which is 5 less than the 3/4 of one half of 12?

Solution:

Let x be the solution. Then
$x = (3/4) \cdot (12/2) - 5 = (9/2) - 5 = -0.5$

You will find several exercises to practice your formulation skills in Chapter 9, "Word Questions."

Strategy 7 - Draw a Figure and Write the Data on the Figure

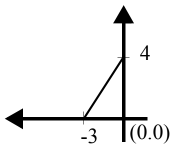
Sometimes the values of distances, angles etc. on a figure are provided in words, without the figure. Draw the intended figure and write down these values on it clearly. Try your best to make the figure reflects the values provided. The visual aid helps you understand the question better.

Example:

1. (Medium)
ABC is a triangle with vertices (0, 4), (-3, 0) and (0, 0). What is the area of ΔABC ?

Solution:

Let's first draw the triangle ΔABC as shown in the figure. From the figure, it is clear that ABC is a right triangle with area = $\frac{1}{2}(3 \cdot 4) = 6$



You will find several exercises to practice your drawing skills in Chapter 9, "Word Questions."

Guessing Methods

In this section we will present several guessing methods that you can use during the test. You can try them to gain time if you can not answer the question directly. Occasionally, it is easier to guess than to solve.

These techniques are divided into three groups. They are:

a. **100% Accurate Guessing Methods!**
This group consists of the techniques that will give you the correct answer with 100% accuracy even if you don't know how to sove the problem properly. Try these methods first.

b. **Finding the Best Possibility Methods**
These methods provide techniques to guess the best possible answer. If you can not "guess" the answer with 100% accuracy, try these methods.

c. Random Guessing Method
This method explains how to guess without using any math knowledge. Refer to the instructions in Chapter 3 to decide if and when to use this method.

100% Accurate Guessing Methods

Method 1 - Substitution of Answers

Sometimes, one or more equations are provided with one or more unknowns. You are asked to find the value of the unknowns. If the question is a multiple choice question, substitute each of the answers given to the equation(s) and find the one that satisfies the equation.

This is a very powerful method for most of the algebra questions, but it may take longer to find the correct answer. You need to go through all the answers until you find the correct one.

Examples:

1. (Easy)
$2a - 5b - 7 = 2$ and $b = -2a$, then $a = ?$

(A) 8/9
(B) -8/9
(C) 9/12
(D) -5/12
(E) 5/12

Solution by Substituting the Answers:

(A) $a = 8/9$ ➔
$b = -2a = -2 \times \frac{8}{9} = -\frac{16}{9}$ ➔
$2a - 5b - 7 = \frac{16}{9} + \frac{80}{9} - 7 = \frac{11}{3} \neq 2$ ➔
(A) is not the answer.

(B) $a = -8/9$ ➔ $b = -2a = 2 \times \frac{8}{9} = \frac{16}{9}$ ➔
$2a - 5b - 7 = -\frac{16}{9} - \frac{80}{9} - 7 = \frac{53}{3} \neq 2$
➔ (B) is not the answer.

(C) $a = 9/12 = 3/4$ ➔
$b = -2a = -2 \times \frac{3}{4} = -\frac{3}{2}$ ➔
$2a - 5b - 7 = \frac{3}{2} + \frac{15}{2} - 7 = 2$ ➔
Since the result of the substitution is 2 as it is supposed to be, the answer is (C).

Proper Solution:

$2a - 5b - 7 = 2$ and $b = -2a$ ➔ $2a + 10a - 7 = 2$ ➔ $12a = 9$ ➔ $a = 9/12$

2. (Medium)
$2^x = 8$. Which of the following is the value of x?

(A) -3
(B) -2
(C) 0
(D) 2
(E) 3

Solution by Substituting the Answers:
Let's substitute all the answers to the equation one by one until we find the correct answer. You can use your calculator if necessary.

(A) $2^{-3} = 0.125$, not 8
(B) $2^{-2} = 0.25$, not 8
(C) $2^{0} = 1$, not 8
(D) $2^{2} = 4$, not 8
(E) $2^{3} = 8$

The answer is (E).

Proper Solution:

Without trying all the other answer choices, you should be able to recognize that 8 is the 3rd power of 2, ie., $8 = 2^3$. ➔ $x = 3$. So the answer is (E).

Exercises:

1. (Easy)
$2n = 6k$ and $2k + n = 3$, then $k = ?$

(A) 3
(B) 2/5
(C) 3/5
(D) -3/5
(E) -3

2. (Medium)
$9^3 = 3^{-x}$, then $x = ?$

(A) 3
(B) -3
(C) 6
(D) -6
(E) 27

3. (Medium)
If $3^{2x+1} = 9^2$, then $x = ?$

(A) 0.5
(B) 1
(C) 1.5
(D) 2
(E) 2.5

4. (Medium)
A car dealer sold 3500 cars in 3 years. Starting from the first year, they double their sales record each year. Which of the following can be the number of cars sold in the 3^{rd} year?

- **(A)** 500
- **(B)** 1000
- **(C)** 1500
- **(D)** 1800
- **(E)** 2000

Hint for substituting the answers: Divide the answer for each case with 2, and do so twice to find the 2nd and 1st year production. Then add all three years' total production. Whichever case ads up to total production of 3500 cars is the correct answer.

Answers:

1. (C); **2.** (D); **3.** (C); **4.** (E)

Method 2 - Solve by Example

Sometimes the answers to a question are expressions formed with letters, rather than numbers. If you don't feel comfortable dealing with letters, you can substitute a number for the letter(s) in the question and the answer choices to find the correct answer.

Example:

1. (Medium)
Mary spends x hours to finish her homework. Joe finishes his homework in 10% less time than Mary. Jill's homework time is the average of Mary's and Joe's time. How long it takes for Jill to finish her homework?

- **(A)** (7/5)x
- **(B)** (19/20)x
- **(C)** x/2
- **(D)** (9/20)x
- **(E)** 9/10

Solution by Example:

Let's assume that x = 10 (Mary spends 10 hours to finish her homework!). Note that 10 is an easy number to work with and it makes no difference whether you pick 1 hour, or 10 hours or 50000 hours. Just pick a number that is easy to work with.

Joe spends 10% less time. 10% of 10 is 1. So Joe spends 9 hours to finish his homework.

Jill spends the average of 10 and 9 hours. So she spends (10 + 9)/2 = 9.5 hours.

Now you need to substitute 10 for x in the answers, and see which one yields 9.5.

(A) (7/5)10 = 70/5 = 14, not 9.5

(B) (19/20)10 = 19/2 = 9.5 ➔
(B) is the correct answer.

It is not very likely but sometimes the numbers that you picked may be a special number that makes a wrong answer look right. To make sure this is not the case, you need to check the other answers and prove that they are wrong.

(C) 10/2 = 5, not 9.5

(D) (9/20)10 = 4.5, not 9.5

(E) 0.9, not 9.5

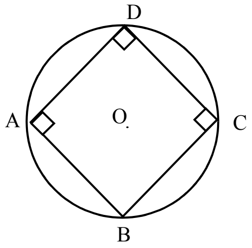
Proper Solution:

Joe spends 10% less than Mary spends to finish his homework ➔ Joe spends (9/10)x hours.
Jill spends the average of the two ➔

Jill spends $\frac{x + (9/10)x}{2} = \frac{19x}{20}$ hours. ➔
The answer is (B).

2. (Hard)
ABCD is a square inscribed in the circle O.

If the area of ABCD is a, what is the circumference of the circle O?



- **(A)** $\pi\sqrt{a}$
- **(B)** $\pi\sqrt{\frac{a}{2}}$
- **(C)** $\pi\sqrt{2a}$
- **(D)** $2\pi\sqrt{a}$
- **(E)** $\frac{\pi}{2}\sqrt{a}$

Solution by Example:

Let's assume that a = 9, a number for which the square root is easy to find. ➔

Side lengths, AB and BC, of the square is $\sqrt{9} = 3$

By using the Pythagorean theorem, the diagonal, $AC = \sqrt{AB^2 + BC^2} = 3\sqrt{2}$

Then the circumference of the circle is $\pi AC = 3\pi\sqrt{2}$

Now let's see which of the answers match this result. At this point we can start with case (A) and continue downward. But because the actual answer has the term $\sqrt{2}$ in it, we can possibly save time if we start with answer (C), since it also has the term $\sqrt{2}$ in it.

Substitute 9 for a:
(C) $\pi\sqrt{2a} = \pi\sqrt{2 \times 9} = 3\pi\sqrt{2}$
The answer is (C). As you can see in this example,

if you make some reasonable assumptions, you can find the correct answer quicker.

Proper Solution:

If the area of the square is a, we don't need to calculate the side length to find the diameter, d, of the circle O (or equivalently, the diagonal of the square ABCD). The area of the square is twice the area of the triangle ABC.

Let the diameter of the circle be d.

The area of the square = $2\left(\frac{AC \times OB}{2}\right)$ =

$AC \times OB = d \times \frac{d}{2} = \frac{d^2}{2} = a$ ➔ $d = \sqrt{2a}$ ➔

circumference = $\pi d = \pi\sqrt{2a}$

The answer is (C).

Practice Exercises:

1. (Easy)

$\frac{3a^3 - 3y}{3 + a^3} = 3$, then y = ?

(A) -3
(B) 3
(C) 9
(D) -9
(E) -1/3

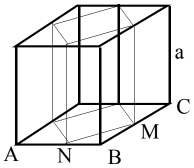
2. (Medium)

If x and y are two negative numbers and $x^2 - 12y + 4 = (3y - 2)^2$, then y = ?

(A) x/3
(B) -x/3
(C) 3x
(D) -3x
(E) -3

3. (Medium)

As shown in the figure, a prism is inscribed in a cube of edge a. AN = NB = BM = MC = a/2 What is the volume of the prism?



(A) $a^3/3$

(B) $\frac{3a^3}{\sqrt{2}}$

(C) $\frac{a^3}{\sqrt{2}}$

(D) $\frac{a^3}{2}$

(E) $\frac{3a^3}{4}$

Hint for solving by example: Assume a = 2. Calculate the volume of the prism. Then compare it with the answer choices until you find a match.

Answers:

1. (A); **2.** (A); **3.** (D)

Method 3 - Eliminate by Example

Sometimes the question is about the range of an expression or an unknown. In these cases, if you don't know the proper solution, try to find one example that makes the answer choice WRONG by substituting values for the variable in the expression. If you can find just one example that makes an answer choice wrong, you can eliminate that choice. This is a very powerful method for most of the algebra questions. However, sometimes it may take a long time to eliminate the wrong answers.

Example:

1. (Medium)

n is an even integer. 7n - 375 must always be which of the following.

(A) A negative even integer.
(B) A negative odd integer.
(C) A prime number.
(D) An even integer.
(E) An odd integer.

Elimination by Example:

It is quite easy to see that (A) and (B) are both wrong because for a large enough n, 7n - 375 will be positive. For example for n = 100, 7n - 375 = 325, which is positive.

This example alone proves that both (C) and (D) are wrong as well, because 325 is neither a prime number nor an even number.

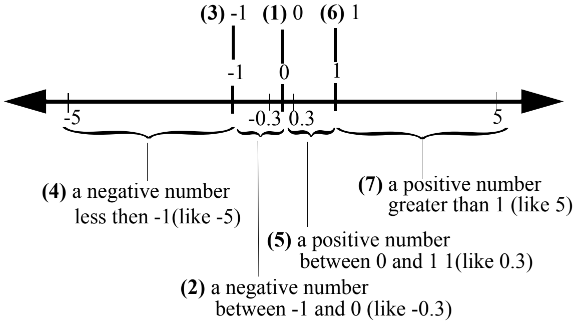
So the answer is (E).

Hard Questions:

Hard questions (the last 20% of the questions in each section) may be tricky. To eliminate the wrong answers, you need to consider values from 7 different range of numbers before you decide that a given answer is correct. For Hard questions, when you substitute, do NOT start from a positive number. Most of the time, you will arrive at the conclusion faster if you follow the order provided below.

(1) 0
(2) A negative number between -1 and 0 (like -0.3)
(3) -1
(4) A negative number less than -1 (like -5)
(5) A positive number between 0 and 1 (like 0.3)
(6) 1
(7) A positive number greater than 1 (like 5)

These ranges are also shown on the number line below from 1 to 7:



Example:

1. (Hard)
x is a real number. Which of the following is always true?

(A) $x^2 > x$
(B) $\sqrt{x} < x$
(C) x^2 is positive
(D) x^2 is non-negative
(E) All of the above.

Elimination by Example:

Step 1: Let x = 0, then

(A) is not correct because $x^2 = 0$ equals to x but not greater than x.

(B) is not correct because $\sqrt{x} = 0$ equals to x but not less than x.

(C) is not correct because $x^2 = 0$, not positive.

(D) is correct, because $x^2 = 0$, a non-negative number.

You don't need to go any further, you found the correct answer, (D), in just a single step.

If you start from a more common choice, like positive x, you would spend more time and are more likely to answer incorrectly. Here is why:

Step 1: Let x = 2, then

(A) $2^2 > 2$, which is correct.

(B) $\sqrt{2} < 2$, which is correct.

(C) 2^2 is positive.

(D) 2^2 is non-negative.

All of the 4 choices seem to be correct.

At this point if you decide that the answer is (E), "All of the above", you would have been wrong. You need to try other possibilities to make sure that for all real values of x, all 4 choices are indeed correct.

Step 2: Let x = -2, then

(A) $(-2)^2 > -2$, which is correct.

(B) $\sqrt{-2}$ is not real.

(C) $(-2)^2$ is positive.

(D) $(-2)^2$ is non-negative.

So we eliminated (B) and (E). Now the possible answers are (A), (C) or (D).

Step 3: Let x = 0.5, then

(A) $(0.5)^2$ is less than 0.5

(C) $(0.5)^2$ is positive.

(D) $(0.5)^2$ is non- negative.

Now we eliminated (A). Possible answers are (C) or (D).

Step 4: Let x = 0, then

(C) $0^2 = 0$ is not positive.

(D) 0^2 is non- negative.

So the answer is not (C). The answer is (D).

As you can see in this example, you can easily make a mistake if you don't try several different values but jump to the conclusion quickly.

2. (Medium)
The median of 5 numbers will remain the same if

(A) All the numbers are multiplied by 2.
(B) All the numbers are multiplied by -2.
(C) All numbers are decreased by 2.
(D) Largest number is squared.
(E) Smallest number is decreased by 2.

Elimination by Example:

Let's assume that the numbers are (-3, 0, 0.5, 10, 21) The median is 0.5.

(A) If you multiply all the numbers by 2, you will get (-6, 0, 1, 20, 42). The new median is 1 and is different from 0.5. So (A) is not the answer.

(B) If you multiply all the numbers by -2, you will get (6, 0, -1, -20, -42). The new median is -1 and is different from 0.5. So (B) is not the answer.

(C) If you decrease all the numbers by 2, you will get (-5, -2, -1.5, 8, 19). The new median is -1.5 and is different from 0.5. So (C) is not the answer.

(D) If you square the largest number, you will get (-3, 0, 0.5, 10, 441). The new median is 0.5 and is the same as the original one.

It looks like you found the answer, but to be 100% sure you need to examine case (E) as well.

(E) If you decrease the smallest number by 2, you will get (-5, 0, 0.5, 10 and 21). The new median is 0.5 and is the same as the original one.

If you can't decide which one is the correct answer, you can make a guess between (D) and (E). However the correct answer is (E), because if we have another set of numbers, (D) will be wrong.

Let's consider (-3, 0, 0.5, 0.6 and 0.7). The median is 0.5.

Examine case (D) again:

If you square the largest number, you will get (-3, 0, 0.5, 0.6 and 0.49). Let's arrange the numbers from smallest to largest: (-3, 0, 0.49, 0.5 and 0.6). The new median is 0.49 and is different from 0.5. Note that the numbers between -1 and 1 are discussed in Chapter 5, section "Numbers Between -1 and 1". Read this section if you could not eliminate (D).

3. (Hard)
If integer $x > 100$ is divisible by 3 and 8, which of the following numbers x may <u>not</u> be divisible by?

- **(A)** 2
- **(B)** 6
- **(C)** 12
- **(D)** 18
- **(E)** 24

<u>Elimination by Example:</u>

Assume x = 216, which is divisible by both 3 and 8 and is larger than 100.

If you divide 216 by all the answer choices from (A) to (E), you will see that it is divisible by all. So to find the correct answer, we need to find a new example.

Assume x = 168, which is divisible by both 3 and 8 and it is larger than 100.

Again if you divide 168 by all the answer choices from (A) to (E), you will see that it is divisible by all except 18. So the answer is (D).

<u>Proper Solution:</u>

If a number is divisible by a second number, it is also divisible by the second number's integer factors. So if x is divisible by 8, it is also divisible by 2 and 4.

If a number is divisible by more than one number, it is also divisible by the multiple of these numbers as long as one is not an integer power of the other. So if x is divisible by 2, 3, 4, and 8, it is also divisible by $3 \times 2 = 6$, $3 \times 4 = 12$ and $3 \times 8 = 24$.

Therefore, x is always divisible by 2, 6, 12 and 24, but not 18. So the answer is (D).

Method 4 - Calculate Expressions

Sometimes the answers are provided as an expression of numbers. In these cases, calculate the expression in the question and in each answer choice to get the correct answer. This is a very powerful method for arithmetic questions but like the previous techniques, it may take more time to calculate all the answers. Once you find the correct answer, you don't need to go any further.

Example:

1. (Medium)

$$\frac{2^3 \times 8^2}{4 \times 2^2} = ?$$

- **(A)** 2^6
- **(B)** 2^5
- **(C)** 4^2
- **(D)** 8
- **(E)** 4

<u>Solution by Calculation:</u>

If you are not sure how to simplify the terms of the above expression and not so familiar with the powers, calculate the expressions in the question and in the answer choices until you find a match. Use your calculator.

The expression $\frac{2^3 \times 8^2}{4 \times 2^2}$ evaluates to 32.

(A) 2^6 evaluates to 64, not the correct answer.

(B) 2^5 evaluates to 32, so it is the correct answer. You do not need to go any further.

<u>Proper Solution:</u>

$$\frac{2^3 \times 8^2}{4 \times 2^2} = \frac{2^3 \times (2^3)^2}{2^2 \times 2^2} = \frac{2^9}{2^4} = 2^5$$

Practice Exercises:

1. (Easy)

$\frac{54}{150} \neq$?

- **(A)** 0.36
- **(B)** 27/75
- **(C)** $3^2/5^2$
- **(D)** $(3/5)^2$
- **(E)** $(3/5)^3$

2. (Medium)
If x = 3, y = 2, then $x^y y^x =$

- **(A)** 36
- **(B)** 216
- **(C)** $3^3 2^2$
- **(D)** 2×6^2
- **(E)** $\frac{6^3}{2}$

Answers: 1. (E); **2.** (D)

Method 5 - Trial and Error

In this method you guess the answer by substituting numbers for the unknowns of the equation until you satisfy the equation.

This technique can be used when you can not substitute the answers back to the equation because of the nature of the question or because it is a “grid-in” question and there are no answers. It may take time until you find the correct answer but in most cases a few trials are sufficient.

In this method, you try to find the value of an unknown by getting progressively closer to the desired result. We will explain this with two simple examples.

Examples:

1. (Easy)
If -2y - 3 = -1, y = ?

Solution by Trial and Error:

Try y = 0 ➔ -2y - 3 = -3, which is not -1.

Let’s increase y and try y = 1 ➔
-2y - 3 = -2 - 3 = -5, which is further away from the desired result of -1.

Since increasing y made the situation worse, let’s decrease y and try y = -1. If y = -1 then
-2y - 3 = 2 - 3 = -1 which is the correct answer.

Proper Solution:

-2y - 3 = -1 ➔ -2y = 3 - 1 = 2 ➔ y = -1

2. (Easy)
If 3x - 5/2 = 2, what is x?

Solution by Trial and Error:

Try x = 0 ➔ 3x - 5/2 = -5/2 which is much less than 2.

Let’s increase x and try x = 1 ➔
3x - 5/2 = 3 - 5/2 = 1/2, which is still less than 2 but we are closer to 2 now.

Let’s increase x even more and try x = 2 ➔
3x - 5/2 = 6 - 5/2 = 7/2, which is more than 2. It means x has to be between 1 and 2.

Let’s try x = 3/2 ➔ 3x - 5/2 = 9/2 - 5/2 = 4/2 = 2
So x = 3/2 is the answer. Notice that 3/2 = 1.5 and it is in between 1 and 2.

Proper Solution:

$$3x - 5/2 = 2 \Rightarrow 3x = 2 + \frac{5}{2} = \frac{9}{2} \Rightarrow x = \frac{\left(\frac{9}{2}\right)}{3} = \frac{3}{2}$$

Comment:
In this method, you need to make a judgement on which value to start with. In most cases it is best to follow the steps given below:

1. Try 0
2. Try a small positive value, like 1. If you are closer to the result, try a greater positive value.
3. Continue to increase the value until you start getting away from the result.
4. If you are not closer to the result, try a negative value, like -1.
5. Continue to decrease the value until you start getting away from the result.

3. (Hard)
$x^2 + 4x + 4 = 0$, what is 2x - 2

Solution by Trial and Error:

Let $y = x^2 + 4x + 4$

First we have to find the value of x that makes y = 0

Let x = 0 ➔ y = 4 not 0

Let x = 1 ➔ y = 1 + 4 + 4 = 9 not 0 and we are even further away from 0. So let's try a lower value.

Let x = -1 ➔ y = 1 - 4 + 4 = 1 not 0 but we are closer. So let's try even a lower value.

Let x = -2 ➔ y = 4 - 8 + 4 = 0
Correct value of x = -2 ➔
$2x - 2 = 2 \cdot (-2) - 2 = -6$

Proper Solution:

$y = x^2 + 4x + 4 = (x + 2)^2 = 0$ ➔ x + 2 = 0 ➔
x = -2 ➔ 2x - 2 = -6

Practice Exercise:

1. (Medium)

In the figure if the area of the triangle is 18 and c = 2a, then a = ?

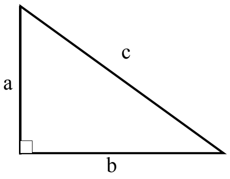


Figure is not drawn to scale.

- **(A)** 2.44
- **(B)** 3.46
- **(C)** 4.56
- **(D)** 5.66
- **(E)** 5.98

Guessing Hint: The area of this triangle is ab/2 = 18. This is your first equation to satisfy. Start with a value of a and b that makes ab/2 = 18. One value is a = b = 6. Then check if these values satisfy the second equation, c = 2a. Note that $c^2 = a^2 + b^2$. Try other values, like a = 5 or a = 4 until you satisfy both equations or come close enough to one of the answer choices.

Answer: 1. (C) (You can easily see that the answer is in between 4 and 5, so it must be 4.56.)

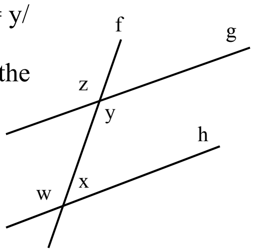
Method 6 - Elimination of Wrong Answers

Sometimes it is possible to eliminate all the wrong answers by using common sense, logic and some knowledge of the subject.

Example:

1. (Medium)

In the figure, g || h and x = y/3. Which of the below answer choices <u>can not</u> be the value of z?



- **(A)** y
- **(B)** y - x - 180 + w
- **(C)** 2y - w
- **(D)** 2y - 3x
- **(E)** w

<u>Solution by Elimination:</u>

You can eliminate (A), because z and y are reverse angles and they are equal.

You can eliminate (E), because g and h are parallel lines, hence y = w

You can eliminate (C), because w = y and 2y - w = 2y - y = y

You can eliminate (D), because y = 3x and 2y - 3x = 2y - y = y

So the answer is (B).

Let's calculate (B) to prove the result. You can write all the terms in the expression in terms of x by substituting 3x for y and w, and 4x for 180. Note that w + x = 180 = 3x + x = 4x.

y - x - 180 + w = 3x - x - 4x + 3x = x
x is not equal to z.

Method 7 - Redraw the Figure to Scale

If you can not answer a geometry question that has a figure marked as "Figure is not drawn to scale.," redraw the figure to scale. Once you redraw it, the answer may seem obvious to you. College Board would rather have you calculate the answer, as opposed to figuring it out by just looking at the figure.

Redrawing the figures will take some valuable time unless you practice it before the test. In many cases, you can redraw the figure freehand, without using any special tools. To be able to do that, develop a feeling for the angles, 90°, 60°, 45°, 30°. Practice drawing these angles and see how large they are.

If you need to redraw the figure accurately, you need to mark distances and angles accurately. Read Appendix A to learn how to mark the distances and the angles.

Examples:

1. (Easy)

Line m and line n are NOT parallel.
Which of the following CAN NOT be x?

- **I.** 39°
- **II.** 40°
- **III.** 41°

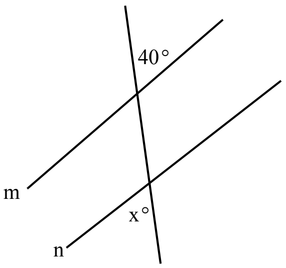


Figure is not drawn to scale.

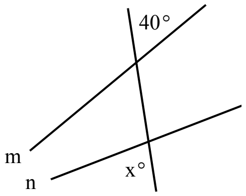
- **(A)** I only
- **(B)** II only
- **(C)** I and II
- **(D)** I and III
- **(E)** I and II and III

<u>Solution:</u>

Because lines m and n look parallel, you may think that x is 40°. In this case your answer would be (D), which is the wrong answer.

If you draw the figure again, and make m and n obviously not parallel (the way they look in the figure) you will notice that x can not be 40 degrees.

The answer is (B), not (D).



2. (Medium)

Line AC intersects line BE at point O.

$\overrightarrow{OD}$ bisects $\angle EOC$

$\overrightarrow{OF}$ bisects $\angle EOA$

What is the value of y?

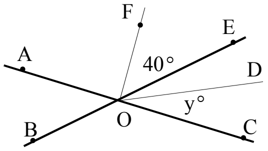
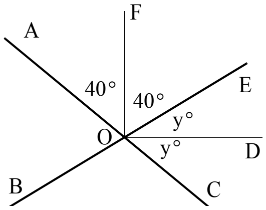


Figure is not drawn to scale.

Solution:

Here, we redraw the figure so that OD truly bisects $\angle EOC$ and FO truly bisects $\angle EOA$ as shown in the figure.



In this figure, you will easily see that OF is perpendicular to OD and $y = 90° - 40° = 50°$

Notice that our figure is not to scale either. $\angle FOE$ is not exactly 40^o. But even this simple redrawing is helpful to find the correct answer.

Of course it is also easier to see that $40° + 40° + y + y = 180°$ ➔ $y = 50°$

3. (Medium)

In the figure,

AB = BC = BD

and

$\overline{BD}$ bisects $\overline{AC}$

and

$\angle DBC = 50°$

What is the value of $\angle BAC$?

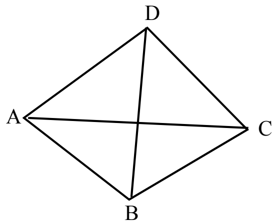
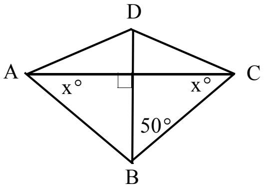


Figure is not drawn to scale.

Solution:

Let's redraw the figure.

When you redraw the figure, you realize that $\overline{AC}$ and $\overline{BD}$ has to be perpendicular.



You also realize that $\angle DBC = \angle DBA = 50°$ and

Let $\angle BAC = \angle ACB = x°$ ➔

x + x + 100 = 180 ➔ x = 40

4. (Hard)

In the figure, $\overleftrightarrow{EG}$ bisects $\overline{BC}$ at H (not shown in the figure) and

$\overleftrightarrow{AG}$ intersects $\overline{DC}$ at J (not shown in the figure).

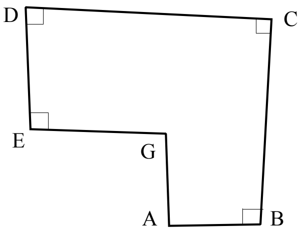


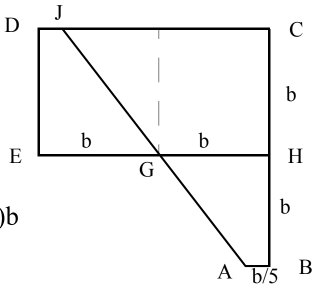
Figure is not drawn to scale.

The area of ABCJ is 50, AB is 1/5 of BH, and EG = CH

What is the area of EGJD?

Solution:

Let's redraw the figure to scale and write down the data on this figure:



It is now quite clear that DJ is b/5 and JC = 2b - b/5 = (9/5)b

The area of ABCJ =

$\left(\left(\frac{b}{5}+\frac{9b}{5}\right)\cdot 2b\right)/2 = 50$ ➔ $2b^2 = 50$ ➔ $b = 5$

The area of EGJD =

$\left(\left(\frac{b}{5}+b\right)\cdot b\right)/2 = \frac{3b^2}{5} = 15$

As you can see in these examples, absolute accuracy in redrawing is not necessary. Try your best to be accurate, but do not spend too much time on it.

Method 8 - Rotate and Slide

Rotate and/or slide the figure to help find the correct answer.

Sometimes it helps just to rotate the page to see the figure from a different angle. Other times you may need to redraw to rotate or slide part of the figure. Our eyes are used to seeing certain shapes in a certain way, because they are drawn that way in the textbooks. It helps to frame the question in a familiar form to help you solve it.

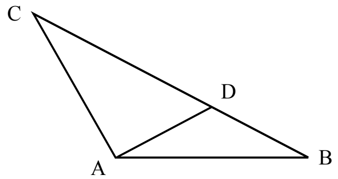
Examples:

1. (Medium)

The area of $\Delta ADC = 18$

BC/DC = 3/2

What is the area of ΔABD?

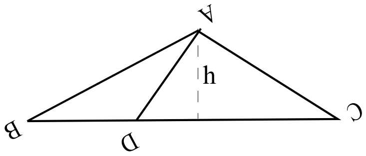


Solution:

It is natural to think that, to be able to calculate the area of ΔABD, you need to know the base length and the height of the ΔABD.

However, it looks impossible to calculate these distances from the data.

But, if you rotate the book so that A is on top and $\overline{BC}$ is horizontal,



as shown in the figure, you will see the answer quickly because you are accustomed to look at the triangles that way.

Now you can see that these two triangles have the same height, h, and base length, CD, of ΔADC is twice as long as the base length, BD, of ΔABD. So (The area of ΔABD) =
(The area of ΔADC)/2 = 18/2 = 9

2. (Hard)

$\overline{AD} \parallel \overline{BC}$,
BC = EG = (1/2)AD, CD = 5

What is the value of FG?

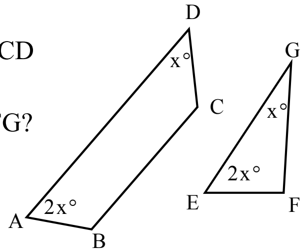
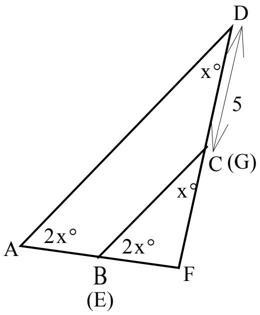


Figure is not drawn to scale.

Solution:

It may seem to be a difficult question. But let's slide the triangle toward the trapezoid such that B coincides with E and C coincides with G as shown in the figure.



We can do this because BC = EG, $\overline{AD} \parallel \overline{BC}$ and

$\angle CBF = \angle DAB = 2x°$ and
$\angle BCF = \angle ADC = x°$ ➔ $\overline{BC} \parallel \overline{EG}$ ➔

AFD forms a triangle as shown in the figure.

Now it is easy to realize that $\Delta AFD \sim \Delta EFG$ ➔

$$\frac{AD}{EG} = \frac{FD}{FG} = 2 \text{ ➔}$$

$$FG = \frac{FD}{2} = \frac{DC + FG}{2} = \frac{5 + FG}{2} \text{ ➔ } FG = 5$$

Try the technique demonstrated in this example when you see two or more detached shapes in the figure.

Method 9 - Measure Distances, Angles and Areas

If you are careful enough, all the measurement techniques can be very accurate. Sometimes they are faster than the actual calculations. For many students, this may be the only option to answer even the toughest geometry questions. In addition, this is one of the few techniques that you can use for the grid-in questions with sometimes 100% accuracy.

To be able to use the method, the figure has to be drawn to scale. College Board warns you if the figure is not drawn to scale. If you see this warning you may have to redraw the figure and make it to scale, or you have to make some adjustments to your answer. We will provide examples for each case.

Measuring Distances

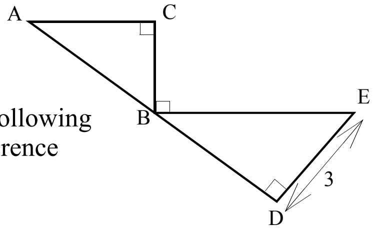
Measuring distances are explained in Appendix B with examples. Here is another example.

Example:

1. (Hard)

B bisects AD,
The area of
$\Delta BDE = 6$

Which of the following is the circumference of ΔABC ?



(A) 8

(B) 9

(C) 9.6

(D) 10

(E) 10.5

Solution by measuring:

Step 1: Make a ruler of length 6 by using your answer sheet and the known distance DE. This procedure is explained in Appendix B.

Step 2: Align your ruler on the paper with AB and measure it. You will see that AB equals or very close to 4. Then align your ruler with AC and BC and measure their length. You will see that they are around 2.3 and 3.1 respectively. So the circumference of ABC = 4 + 2.3 + 3.1 = 9.4

The closest answer is 9.6. So we guessed that the correct answer is (C).

Proper Solution:

Area of $\Delta BDE = 6$ ➔ $DE \times (BD/2) = 6$ ➔
$BD = 6 \times 2/3 = 4$

$BE^2 = 4^2 + 3^2 = 25$ ➔ BE = 5

B bisects $\overline{AD}$ ➔ AB = 4

$\angle C$ and $\angle CBE$ are right angles ➔ $\overline{AC} \parallel \overline{BE}$ ➔
$\angle CAB = \angle EBD$ ➔ $\Delta ACB \sim \Delta BDE$ ➔

AC/BD = AB/BE ➔ $AC = 4 \cdot \frac{4}{5} = \frac{16}{5} = 3.2$
and

CB/AB = DE/BE ➔ $CB = 4 \cdot \frac{3}{5} = \frac{12}{5} = 2.4$

The circumference of $\Delta ABC = 4 + 3.2 + 2.4 = 9.6$

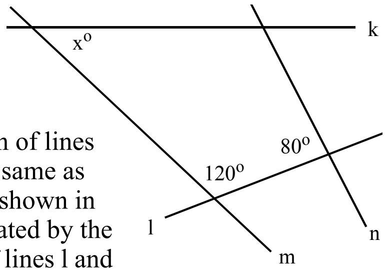
Measuring Angles

In Appendix B, we have explained how to measure an unknown angle with an example. Here is another example.

Example:

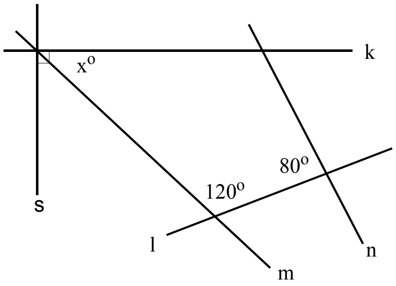
1. (Hard)

If the angle (not shown in the figure) created by the intersection of lines m and n is the same as the angle (not shown in the figure) created by the intersection of lines l and k, what is the value of x?



Solution by measuring:

Step 1: We know that x is less than 90°. In fact if we create a 90° by drawing line s, as shown below, we can guess that x is close but less than 45°.



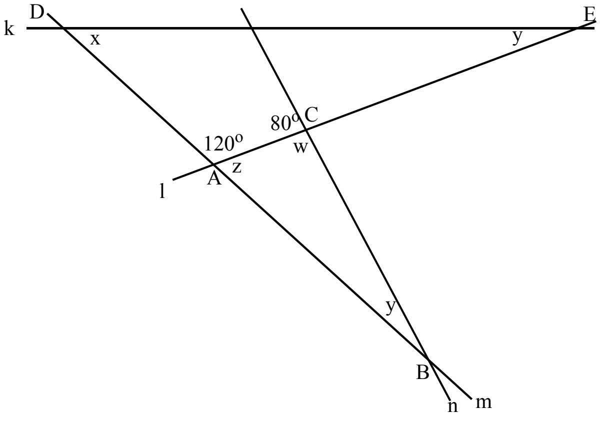
Step 2: Let's try to see if x = 40. You have a 80° angle in the figure. Half of it is 40°. Use the method explained in Appendix B to get the half of 80°.

Step 3: Mark the 40° on your answer sheet.

Step 4: Compare x with 40° to see that x is indeed 40°.

Proper Solution:

Step 1: As shown in the below figure, let's extend lines m, n, l and k to get the 2 angles mentioned in the question.



Step 2: From the figure, z = 180° - 120° = 60° and w = 180° - 80° = 100°

Step 3: y = 180° - 60° - 100° = 20° because z, y and w are the inner angles of the triangle ABC.

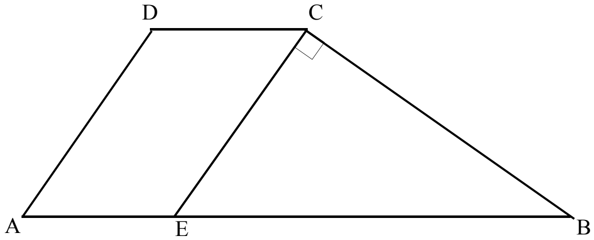
Step 4: x = 180° - 120° - 20° = 40°, because x, y and 120° are the inner angles of the triangle DEA.

Measuring Areas

You can calculate an area if you can measure the proper distances in a figure. However your error margin increases especially when the distances you measure get longer. You must be quite precise in your measurements.

Example:

1. (Hard)



In the above figure, $\overline{AE} \parallel \overline{DC}$, AE = DC = 2. EC = 3 and the area of $\Delta EBC = 6$

Which of the following best represents the area of ABCD?

- **(A)** 12
- **(B)** 11
- **(C)** 10
- **(D)** 9
- **(E)** 8

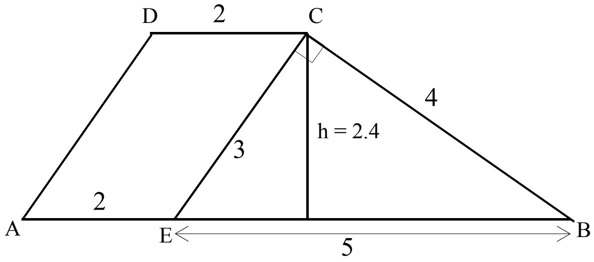
Solution by Measuring:

Make a ruler of length 6, by using the known distances of EC = 3 and/or AE = 2.

Measure EB and h to see that they are very close to 5 and 2.4 respectively. So the area of the trapezoid ABCD = (5 + 2 + 2)2.4/2 = 10.8.

The closest answer is (B).

Proper Solution:



The Area of EBC = $\frac{EC \times CB}{2} = \frac{3}{2} \times CB = 6$ ➔ CB = 4

Calculate EB by using the Pythagorean Theorem:

$EB^2 = EC^2 + CB^2 = 9 + 16 = 25$ ➔
$EB = \sqrt{25} = 5$

The area of $\Delta EBC = \frac{EB \times h}{2} = \frac{5}{2} \times h = 6$ ➔
$h = 12/5 = 2.4$

The area of $ABCD = \frac{(AB + DC)h}{2} =$
$(7 + 2) \times 2.4 \times \frac{1}{2} = 10.8$

The closest answer is (B).

Note that the actual, precise answer is not one of the answer choices. But 11 represents the area of ABCD the best.

Finding the Best Possibile Answer

Sometimes you can not find the answer with 100% accuracy. In these cases, you should try to eliminate the wrong answers as much as you can and pick your best guess from the remaining possible answers. In this section we will present methods that help you find the best choice.

Method 1 - Don't select the same answer for four consecutive questions

For example, if you have already answered (B) for the previous three questions, eliminate (B).

Method 2 - Answer that is very different from the others, usually is NOT the correct answer.

Note that "None of the above," "All of the above," "Not enough information" type of answers are excluded from this method. They are always different from the other answers, but they can be the correct answer.

Examples:

1. (Medium)
The addition of four prime numbers is 23. What is the median of those prime numbers and 5?

(A) 1
(B) 2
(C) 3
(D) 5
(E) 11

Solution:

In this example, 11 is most likely not the answer, because it is very different from the other answers.

The correct answer is 5. In fact the four prime numbers are 2, 3, 7and 11.

2. (Medium)
There are 5 two-color marbles in a bag. They are white-red, blue-white, yellow-blue, yellow-red and blue-green. If you pick one marble randomly, which of the following will most likely be the marble that you pick?

(A) A marble with red.
(B) A marble with blue.
(C) A marble with white.
(D) A marble with green.
(E) A marble with both white and blue.

Solution:

The answer is probably not (E) because in all the other answer choices only one color is specified, but in (E) two colors are specified, hence it is very different than the others.

The answer is (B) because there are 3 marbles with blue color. All the other colors are on one or two marbles. So blue is the most probable color to pick.

Method 3 - Obvious answers to Hard questions

Difficult questions usually require smart reasoning. Ordinary thinking will lead you to the wrong answer. If the answer to a "Hard" question looks "obvious" to you and if you have arrived at the answer easily, without understanding why the question is labeled Hard, your answer is probably wrong. Therefore eliminate that answer.

Examples:

1. (Medium)
In a classroom with 20 students, the average math grade is 60, and in another class with 10 students, the average math grade is 72. What is the average math grade if you combine the two classes into one class?

(A) 63
(B) 64
(C) 65
(D) 66
(E) 67

The obvious answer seems to be (D), because it is the average of 60 and 72. But this is a wrong answer.

Solution:

Addition of all the grades in class of 20 students is
$20 \times 60 = 1200$
Addition of all the grades in class of 10 students is
$10 \times 72 = 720$

Sum total of all the grades = 1200 + 720 = 1920

Total number of students = 10 + 20 = 30

The average math grade in the combined class = 1920/30 = 64

So the answer is (B).

2. (Hard)

In the figure, what is the length of the longest line segment you can draw in or on the rectangular prism?

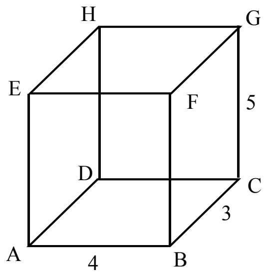


Figure is not drawn to scale.

(A) 5
(B) $\sqrt{34}$
(C) $\sqrt{41}$
(D) $\sqrt{50}$
(E) $\sqrt{52}$

(A) can be the answer you may have picked because it is the longest side of the rectangular prism. It looks obvious, but it is the wrong answer.

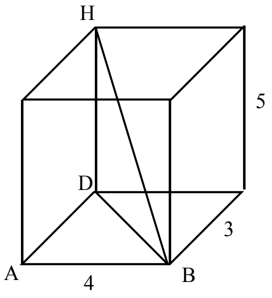
(B) can be the answer you may have picked because it is the length of the diagonal line segment, BG, of one side and it looks the longest line one can draw on the prism. However, it is not the longest line segment. Once again, the answer is obvious, but it is the wrong answer.

Perhaps (C) is the answer you have picked because it is the length of the diagonal line segment of one side, AF, and it is the longest line one can draw on the prism. However, it is not the longest line segment on or in the prism. This answer does require some insight but not enough. It is the wrong answer.

Did you pick (E) simply because it is the largest number in the answer choices? But it is not the correct answer either.

Solution:

The longest line segment is HB. It is shown in the figure.



You can calculate its length by using Pythagorean Theorem.

$HB^2 = BD^2 + HD^2 = (AB^2 + AD^2) + HD^2 =$

$16 + 9 + 25 = 50$ ➔ $HB = \sqrt{50}$

So the answer is (D).

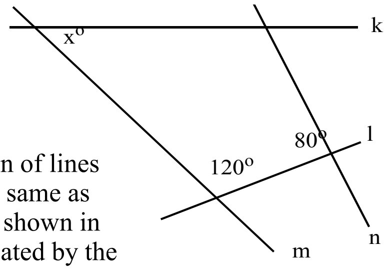
Method 4 - Visual Estimation of Angles, Distances and Areas

Whenever a figure is drawn to scale, you can visually estimate angles, distances and areas.

Examples:

1. (Hard)

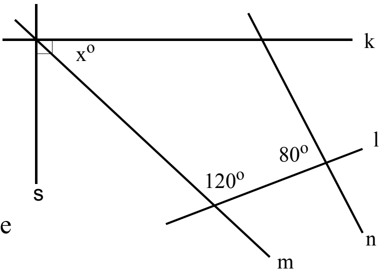
If the angle (not shown in the figure) created by the intersection of lines m and n is the same as the angle (not shown in the figure) created by the intersection of lines l and k, what is the value of x?



(A) 30
(B) 35
(C) 40
(D) 45
(E) 50

Solution by visual estimation:

You know that x is less than 90°. In fact if you can create a 90° angle by drawing line s, as shown in the figure.



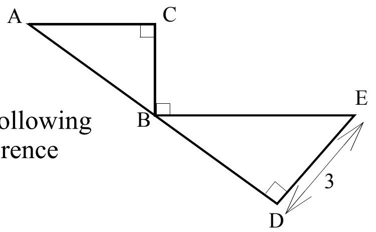
You can see that x is close to but less than 45°.

You can guess that it is either 35 of 40. So the answer is (B) or (C). From the previous section, “Measuring Angles”, Exercise 1, we know that the answer is (C).

2. (Hard)

B bisects AD, The area of $\Delta BDE = 6$

Which of the following is the circumference of ΔABC?



(A) 8
(B) 9
(C) 9.6
(D) 10
(E) 10.5

Solution by visual estimation:

$\Delta BDE = 6$ ➔ $\frac{BD \cdot DE}{2} = \frac{3}{2}BD = 6$ ➔ $BD = 4$

B bisects AD ➔ $AB = BD = 4$

It is obvious that AC + CB > AB ➔
AC + CB > 4 ➔

The circumference of ΔABC =
AB + AC + CB > 4 + 4 = 8 ➔
You can eliminate (A), and guess your answer from the remaining 4 choices.

If you wish, you can go further:
AC looks very close to DE = 3. Let's assume that AC = 3 ➔ $BC^2 = AB^2 - AC^2 = 16 - 9 = 5$ ➔

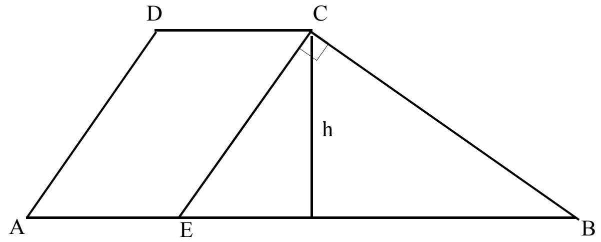
$BC = \sqrt{5} \cong 2.24$ ➔

The circumference of ΔABC =
AB + AC + CB = 4 + 3 + 2.24 = 9.24

This answer is in between 9 and 9.6. So you can guess your answer from two choices, (B) or (C), which is better than having 4 choices.

From the earlier section, "Measuring Distances", Exercise 1, we know that the answer is (C).

3. (Hard)



In the above figure, $\overline{AE} \parallel \overline{DC}$, AE = DC = 2.
EC = 3 and the area of ΔEBC = 6

Which of the following best represents the area of ABCD?

- **(A)** 12
- **(B)** 11
- **(C)** 10
- **(D)** 9
- **(E)** 8

Solution by Visual Estimation I:

h is in between EC(= 3) and AE(= 2). Let's assume that it is 2.5. ➔
The area of AECD = 2h = $2 \cdot 2.5$ = 5 ➔
The area of ABCD =
The area of AECD + The area of ΔEBC =
5 + 6 = 11

So the answer is (B).

Solution by Visual Estimation II:

The area of AECD looks a bit less than the area of ΔEBC. So the answer is less than but close to 6 + 6 = 12. At this point you can guess that it is either 11 (B) or 10 (C).

From the earlier section, "Measuring Areas", Exercise 1, we know the answer is (B), not (C).

Method 5- "All of the above"

When "All of the above" is one of the answer choices, be extra careful in eliminating any answer. This choice is included not because College Board ran out of possible answers. They include this answer when one or more of the answers seem to be obviously wrong, but in reality they are correct.

If "All of the above" is one of the answer choices, you can eliminate this answer under the following conditions:

1. When you can eliminate one wrong answer with 100% accuracy, "All of the above" can not be the correct answer.

2. When two of the answer choices contradict each other. In this case, since both of them can not be correct, there is at least one wrong answer among the answer choices. So "All of the above" can not be the correct answer.

Examples:

1. (Medium)
a and b are two consecutive positive integers. Which of the following is wrong?

- **(A)** ab is even
- **(B)** ab^2 is even
- **(C)** a + b is even
- **(D)** a - b - 1 is even
- **(E)** All of the above

Solution:

Since one of the consecutive integers is odd and the other is even, and since the multiplication of an even and odd number is always even, (A) is not wrong. Therefore "All of the above" can not be correct. You should eliminate (E) as soon as you eliminate (A) when you realized that (A) can not be wrong.

(B) is not wrong, because ab^2 is multiplication of three numbers, a, b and b, and one of them is always even, hence the multiplication is also even.

(C) is wrong, because addition of an even and an odd number is always odd, not even.

So the answer is (C).

2. (Medium)

$$\frac{2^8 \times 8^2}{4} = ?$$

- **(A)** 2^6
- **(B)** $\frac{2^{10}}{4}$
- **(C)** $2^5 + 2$
- **(D)** 2^{12}
- **(E)** All of the above

If you look at all the choices, they may be confusing at first. However, it is easy to see the difference between (A) and (D). 2^6 is never equal to 2^{12}. Hence one or both of these answers must be wrong. Therefore, not all of the answer choices can be correct. So you can eliminate (E) for sure.

You can also eliminate (C), because it is different from the other answers. It is the only one with an addition. So you can pick your answer among (A), (B) and (D).

Solution:

$$\frac{2^8 \times 8^2}{4} = \frac{2^8 \times (2^3)^2}{2^2} = \frac{2^8 \times 2^6}{2^2} = 2^{12}$$

So the answer is (D).

3. (Hard)
If x is a positive number, which of the following is always true?

I. $x^2 > x$
II. $x > \sqrt{x}$
III. $(x + 1)^2 > (x + 1)$

(A) I only
(B) II only
(C) III only
(D) I and II
(E) I, II and III

Solution:

Let's assume that x = 4 (or any positive number greater than 1). In this case all the answer choices are correct. Based on this answer if you decide that (E) (all 3 choices) is correct, you would be wrong.

Let's try some other values of x. For x < 1, (A), (B), (D) and of course (E) is wrong. The correct answer is (C) because x + 1 is always greater than 1 and square of numbers greater than 1 is always greater than themselves.

Notice that in this example, "All of the above" comes in a different format.

Method 6 - "None of the above"

When "None of the above" is one of the answer choices, be extra careful in determining the correct answer. Just like "All of the Above", this choice is again included not because College Board ran out of possible answers. They include this answer choice in the following cases:

1. When one of the answers seems obviously correct, but in reality it is wrong.

2. None of the answers looks correct, but actually, one of them is correct.

Examples:

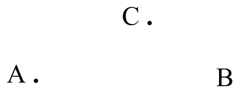
1. (Medium)
A, B and C are three distinct points. Which of the following must be correct?

I. AB < AC + CB
II. $AB^2 = AC^2 + CB^2$
III. AB = AC + BC

(A) I only
(B) II only
(C) III only
(D) I and II
(E) None

Solution:

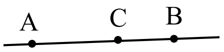
Let's draw the 3 points as shown in the figure.



It looks obvious that I is correct, and II and III are wrong. If you agree with this assessment, you would choose (A) as your correct answer, and you would be wrong.

A student who is partially familiar with Pythagorean theorem may think that II is correct as well. If you agree with this statement, your answer would be (D), and you would be wrong again.

However, A, B, and C can be anywhere. For example they can be on the same line as shown in the figure.



In this case, I and II are wrong and III is correct. Once you could see all the possibilities, you can easily find out that the correct answer is (E), "None."

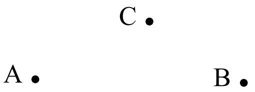
2. (Medium)
A, B and C are three distinct points. Which of the following cases is possible?

I. BC = AC + AB
II. $BC^2 = AC^2 + AB^2$
III. AC = AB

(A) I only
(B) II only
(C) I and II
(D) I and II and III
(E) None

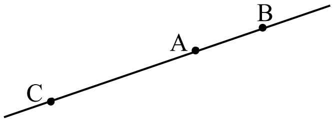
Solution:

Let's draw the 3 points as shown in the figure.

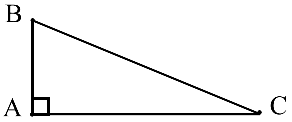


In this case they all seem to be wrong. If you answer (E) without thinking further, you would be wrong. In fact all three cases are possible. Let's examine each one.

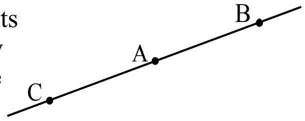
I. This case is possible when all three points are aligned the way shown in the figure.



II. This is possible when the three points are arranged the way shown in the figure.



III. This case is possible when all three points are aligned the way shown in the figure where A, B and C are on the same line and A bisects $\overline{CB}$.



The answer is (D).

Method 7 - "Not enough information"

Be very careful if one of the answers include the phrase "There is not enough information to answer the question." Do not choose this answer only because you are not sure of the other answers. To solve a question correctly is hard enough, but in this case, you need to prove that the question is not **solvable**. This is even harder.

Example:

1. (Hard)
A circle is inscribed in an equilateral triangle with side length a. What is the area of the circle?

(A) $\pi\left(\frac{a}{2}\right)^2$

(B) $\pi\left(\frac{a}{3}\right)^2$

(C) $\pi\frac{a^2}{3}$

(D) $\pi\frac{a^2}{12}$

(E) Not enough information is provided to answer the question.

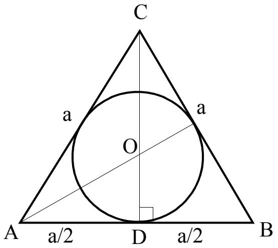
Solution:

It looks like it is impossible to get the answer with such little information. If this is the reason you answer (E), you would be wrong.

Here is the solution:

Let's draw the figure first.

To calculate the area of the circle you need to find the radius, r = OD, of the circle first.



ΔABC is equilateral ➔ $\angle A = \angle B = \angle C = 60^o$

ΔABC is equilateral ➔ Angle bisector $\overline{CD}$ passes through the center of the circle, O, and it is perpendicular to $\overline{AB}$ as shown in the figure.

$\overline{AO}$ bisects $\angle CAB$ ➔ $\angle OAB = 60/2 = 30^o$

ΔABC is equilateral ➔ D bisects $\overline{AB}$ ➔
AD = a/2

ΔAOD is a right triangle ➔

$$\tan(\angle OAD) = \tan(30) = \frac{OD}{AD} = \frac{r}{a/2}$$

On the other hand,

$$\tan(30) = \frac{\sin(30)}{\cos(30)} = \frac{(1/2)}{((\sqrt{3})/2)} = \frac{1}{\sqrt{3}} \text{ ➔}$$

$$r = \frac{a}{2\sqrt{3}} \text{ ➔}$$

The area of the circle = $\pi r^2 = \pi\frac{a^2}{12}$

The answer is (D), not (E).

Method 8 - Never Leave the Grid-In Questions Unanswered

There is no penalty for incorrect answers for grid-ins. Earlier in this chapter, we suggested guessing techniques, like measuring distances, angles, areas, that are also appropriate for this group of questions. Use them whenever appropriate.

If nothing else, use your common sense. Here are some examples.

Examples:

1. (Medium)
Mary is 2 years older than her brother Jack. If Jack's age is 2/3 of Mary's age, how old is Jack?

Guess 1:

If you don't even have a clue, guess an integer number appropriate for a kid's age from 2 to 12, like 1, 2, 3, 4, 6 or 8 etc. There is 1 in 12 chance that it may be the correct answer. Since for grid-in questions there is no penalty, you lose nothing even if your answer is wrong.

Note that there is no clue that Mary and Jack are kids. They can be teenagers or adults as well. Also note that their age may not be integers. But these are the assumptions you have to make to come up with a reasonable guess.

Guess 2:

You can increase your chances of solving the question correctly if you think a bit further. Since Jack's age is 2/3 of Mary's age, assuming integer ages, Mary's age must be divisible by 3. So your guess for Mary's age can be 3, 6, 9, 12 etc. and for Jack's age 1, 4, 7, 10 etc. This time you have a better chance than 1 in 12. It is 1 in 4.

Proper Solution:

Let m and j be the Mary's and Jack's age respectively.
Mary is 2 years older than her brother Jack ➔ m = j + 2
Jack's age is 2/3 of Mary's age ➔ j = 2m/3, then
Substituting j from last equation in to the first equation: m = 2m/3 +2 ➔
m/3 = 2 ➔ m = 6 ➔ j = 6 - 2 = 4

So the answer is 4

2. (Medium)
If 2/3 of 3/4 is subtracted from 3 what is the result?

If you don't have a clue, you should still guess the answer if the question is a grid in question.

Guess 1:

Since something positive is supposed to be subtracted from 3, the answer must be a number less than 3. Since the number which is subtracted looks like a fraction, the answer is probably not an integer. It can be something like 1.5. If you guessed 1.5, you would have been wrong. But again, you lose nothing by being wrong.

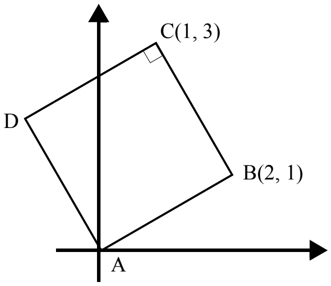
Guess 2:

Since the number which is subtracted from 3 is a fraction of 3/4, it can't be any greater than 3/4 = 0.75. Hence the answer must be greater than $3 - 0.75 = 2.25$, but less than 3. It could be 2.5. If you guessed 2.5 you would have been correct.

Proper Solution:

2/3 of 3/4 is $\frac{2}{3} \cdot \frac{3}{4} = 0.5$
So the answer is 3 - 0.5 = 2.5

3. (Medium)
In the figure
$\overline{AB} \parallel \overline{DC}$ and
$\overline{AD} \parallel \overline{CB}$.
What is the area of ABCD?



Guess 1:

The above figure looks like a square. The area of a square is the square of one side. Since all the coordinates in the figure are between 1 and 3, the area must be a number between $1^2 = 1$and $3^2 = 9$ You could guess 2, 3, 4, 5 etc. Of course the area does not have to be an integer, but you are only guessing.

Guess 2:

The above figure looks like a square. Let's assume that ABCD is a square and AB = BC. Since point A is at the origin, it is easier to calculate AB.
$AB = \sqrt{2^2 + 1^2} = \sqrt{5}$

So the area of ABCD = $\sqrt{5} \cdot \sqrt{5} = 5$. This is the correct answer.

Proper Solution:

$C = 90^o$, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{CB}$ ➔
ABCD is a rectangle.
$AB = \sqrt{2^2 + 1^2} = \sqrt{5}$
$CB = \sqrt{(2-1)^2 + (1-3)^2} = \sqrt{5}$ ➔
The area of ABCD = $\sqrt{5} \times \sqrt{5} = 5$

Random Guessing Method

If you couldn't eliminate all the wrong answers with 100% accuracy, but eliminated at least one of the wrong answers with certainty, it is to your advantage to guess the correct answer from the remaining answer choices.

In this situation, most students will choose a most likely answer. Unfortunately, this most likely answer is frequently the wrong answer.

In SAT the wrong answers are not chosen randomly, but prepared so that each one corresponds to the answer the student will get if he/she makes a probable mistake. Since you don't know how to solve the problem, it may be more likely that you will arrive at these wrong answers by following semi-accurate logic or knowledge. Therefore sometimes it is better for you to guess the answer randomly without applying any "logical reasoning."

However, it is very difficult to be "random" once you read the question and the answer choices. You always tent to favor one of the answer choices over the others. **To eliminate your bias, decide on the answer before you take the test, and if you are instructed to use the random guessing method, always pick that choice each time you have to guess the answer.** For example, you can decide to pick (C) as your answer if you have to make a guess. If (C) is not one of the answers that you eliminated with 100% accuracy, make it your answer. If case (C) is eliminated, then, pick the next available choice, in this case it is (D).

You are advised to use this method back in Chapter 3, whenever appropriate. If your instructions asks you to use Random Guessing Method, use it as explained here. Otherwise, use the other methods provided in this chapter.

Time Management

Basic Strategies

Strategy 1 - Do not rush and make careless mistakes

Depending on your goal, you may not even have to try all the questions. Even if your goal is high, careless mistakes will not help you. For all the questions that you answered, make sure that you are aware of one of the following:

a) You answered a question correctly.
or

b) You guessed and marked the question as "G", stands for "Guessed."
or

c) You could not answer the question and marked it as "U", stands for "Unanswered."

Strategy 2 - Just pass

If you can't answer a question, don't spend too much time on it. Use the guessing techniques provided in this chapter and guess an answer if you can.

If you still can not answer the question, mark the question with the letter "U" and move on to the next question.

Strategy 3 - Don't panic

Don't panic if you are running out of time. Do not guess the answer just for the sake of guessing. If you can not eliminate at least one of the answers with 100% confidence, leave the question unanswered.

Strategy 4 - "G"s and "U"s from the beginning

At the end of the section, if you have time, first revisit the questions marked "G" and "U" from the beginning of the section and try to solve them or eliminate more wrong answers.

Strategy 5 - One question at a time

At the end of each section, after reviewing all the questions marked "G" or "U", if you still can't answer some of them, don't panic. Take a few deep breaths. Stretch your legs and arms. Then concentrate only on one of them, the one that seems the easiest to you.

Don't jump from one question to the other. Remember that once the test is over, most students are able to answer a good number of questions that they could not answer during the test. It is the time pressure and the stress of a long test that make you tired and think not clearly.

Strategy 6 - Do not constantly check the time

The alarm or the test instructor will let you know when the time is up. Checking the time constantly wastes time and disturbs your concentration.

Strategy 7 - Use your pen as you read the question

Make a habit of writing down the formulas, sketching the figures and placing values on the figures as you read the question. In the beginning it will take more time, but with a little practice, it will increase your speed without compromising the accuracy. Writing down the formulas, sketching the figures etc., helps you visually and forces you to notice the details. It eliminates confusion and panic.

Strategy 8 - Use a calculator

Calculators are not necessary but sometimes they are faster. They also eliminate some simple arithmetic errors.

In this book, we indicated when using a calculator would be to your advantage, and when it would not be.

Use a calculator that you are familiar with. SAT is not the time or place to learn how to use a new calculator.

Advanced Strategies

These advance time management strategies are for students who aim very high scores in SAT. Apply them only if you are instructed to do so in Chapter 3.

Strategy 1 - Solve the easy and medium questions properly

Solve the Easy and Medium questions properly as opposed to guessing the answer by using the methods described in this chapter. These methods are very powerful but most of the time, applying them takes time. Proper solutions to all the examples are given in this chapter. If you have the habit of using substitution, trial and error, calculation of all the answer choices etc. methods, try to break your habit and learn how to solve the Easy and Medium questions properly.

Strategy 2 - Mark your time

Try to finish the first half of each section in 30% of the time. In your diagnostic test 1/2 mark is provided for each section. Mark your time on the test at the beginning and at the 1/2 mark. After you finish the test, check if you could reach the 1/2 mark in 30% of the time allocated for that section. If not, you must practice on Strategy 1 very rigorously.

Strategy 3 - Short Arithmetic with Long Numbers

It is rare but not altogether impossible to have long numbers or too many numbers to add, subtract, multiply or divide in SAT. These situations increase the chance of making errors even when you use a calculator.

Sometimes you can choose the correct answer by looking at only one digit of the answer choices. These methods are only for the advanced students. You need to be very sure of your procedure to use this method. If your procedure is wrong, you may end up arriving at the wrong answer without even noticing it. Here are the 2 methods for long numbers:

1. Whenever you arrive at an answer by multiplying, adding or subtracting two or more numbers, you can check only the last digit.

Example: (Easy)
In the figure, $AB = 7$,
$BC = 2AB$ and $AD = BC - 2$

What is the volume of the rectangular prism?

(A) 1176
(B) 1008
(C) 490
(D) 1298
(E) 900

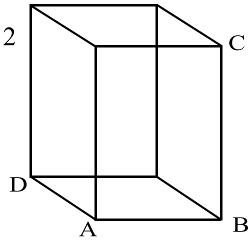


Figure is not drawn to scale.

Short Solution:
$BC = 14$, $AD = 12$,
Volume = $7 \times 14 \times 12$. The last digit of this multiplication is the last digit of $7 \times 4 \times 2$, which is 6. So the answer is (A) because it is the only one that ends with 6. You don't need to do the full calculation.

2. Whenever you arrive at an answer by dividing two numbers, you can only check the first digit.

Example: (Medium)
Mary sleeps 10% less than Susan. If Susan sleeps 2900 hours a year, what is the average number of hours Mary sleeps in a week? Approximate your answer to the nearest integer.

(A) 46
(B) 47
(C) 48
(D) 49
(E) 50

Short Solution:
Mary sleeps $(90/100)2900 = 2610$ hours a year. She gets $2610/52 = 5u$ hours a week, where u is an unknown non-negative integer. The answer has to start with 5. So the answer is 50.

Practice Exercise:

1. (Medium)
Which of the following sets has an integer average?

(A) {10, 7, 3, 8, 1}
(B) {125, 284, 2, 1, 3}
(C) {5, 12, 15, 2, 3}
(D) {1, 4, 5, 8, 3}
(E) {1, 2, 3, 4, 7}

Hint: You don't need to calculate the addition of the elements of each set. All the sets have 5 elements. The addition of the elements has to be divisible by 5, that is, the units digit of the addition has to be 0 or 5.

Answer:
1. (B)

Strategy 4 - Use Approximate Numbers

Sometimes, when the answers are not too close to one another, you can successfully find the correct answer by using approximate numbers to save time.

Example: (Medium)

In a town, there is one ice cream shop per 2050 population. Each person in the town eats on the average 2 scoops of ice cream per week. If the ice cream consumption is 1,400,000 scoops a year, which of the following represents the number of ice cream shops in town best?

(A) 7
(B) 14
(C) 35
(D) 70
(E) 350

Short Solution:

Since the answer options all seem to be approximate, we don't have to worry about the details of the numbers too much.

To make the calculations easy, assume that there are 50 weeks in a year. Then each person eats about 100 scoops of ice cream a year. This means there are about 1,400,000/100 = 14,000 people in the town. If we also assume that there are approximately one ice cream shop per 2000 (not 2050) people, we can calculate that there are about 14,000/2000 = 7 ice cream shops in the town. So the answer is (A).

Note: Do not make such calculations without writing down the numbers. It is very easy to make a mistake when there are too many zeros involved. Make sure that you count the number of zeros correctly.

Strategy 5 - Solve the Question Partially

You can save some time by solving the question partially. Find the maximum and/or minimum values to eliminate some of the wrong answers.

Example: (Hard)

An equilateral pentagon is inscribed inside a circle of radius 2. What is the area of the pentagon?

(A) 13
(B) 12.5
(C) 9.5
(D) 8
(E) 7.5

Short Solution:

Let's draw the figure first.

The area of the pentagon must be less than the area of the circle as shown in Figure A. The area of the circle is

$\pi r^2 \cong 12.57$

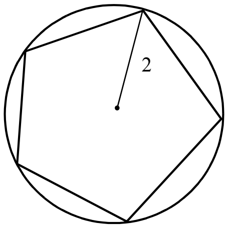


Figure A

It also must be more than the area of a square inscribed in the same circle. Such a square is shown in Figure B.

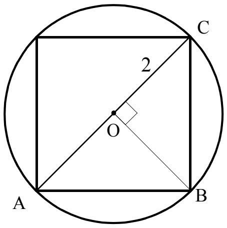


Figure B

The area of this square is 2 times the area of triangle ABC, which is

$AC \times OB = 4 \times 2 = 8$

So the answer is in between 8 and 12.57 which is (B) or (C). You can argue that (B) is too close to the upper limit. So you can guess the answer correctly as (C).

Note here that the area of the square is obtained by calculating the area of a triangle. This is an unconventional way, but it is accurate and it saves a lot of time.

5

ARITHMETIC

Arithmetic is one of the most important math subjects in the SAT. How important? About one-quarter of the questions are on arithmetic.

Most of the Easy SAT questions are in the first two sections of this chapter. The students with math scores less than 500 should pay special attention to them. The methods suggested in these sections will also help the advanced students and enable them to increase their speed. The remaining 6 sections are very important to the more advanced students.

To make it easy for you to learn the arithmetic subjects, we have used the "Learn by Example" method: the basic principle is simply stated and then explained by examples.

Some sections of this chapter may require knowledge of Simple Algebra, One Variable Simple Equations, presented in Chapter 7. If you have problem in understanding this chapter, first read that brief section in Chapter 7.

We have also provided Practice Exercises for most of the sections. These exercises are developed to help you practice your basic knowledge of the subject.

SAT questions are usually designed to measure your reasoning skills. We have provided these kind of exercises at the end of the chapter. Answers and the solutions to these questions are also provided at the end.

Basic Arithmetic - Addition, Subtraction, Multiplication, Division

There are some question in the SAT that involve only the basic operations: addition, subtraction, multiplication and division. These questions are usually at the Easy level. Here is what you need to know about them:

Addition

Addition is a communicative operator:

a + b = b + a

In an addition, you can interchange the positions of the terms without affecting the result.

Examples:

1. (Easy)
3 + 2 = 2 + 3 = 5

2. (Easy)
4 + 7 + 2 = 7 + 4 + 2 = 2 + 4 + 7 =
4 + 2 + 7 = 7 + 2 + 4 = 2 + 7 + 4 = 13

Subtraction

Subtraction is <u>not</u> a communicative operator:

a - b = -(b - a)

In a subtraction, you <u>cannot</u> interchange the positions of the terms without changing the result.

Example: (Easy) 3 - 2 = 1, while 2 - 3 = -1

Practice Exercises:

1. (Easy) 87 - 91 =

2. (Easy) 3 + 8 - 18 =

3. (Easy) -77 + 10 =

4. (Easy) -8 - 7 =

Answers: a. -4; **b.** -7; **c.** -67; **d.** -15

Multiplication

Multiplication is a communicative operator.

$a \times b = b \times a$

In a multiplication, you can interchange the positions of the terms without affecting the result.

Examples:

1. (Easy)
$3 \times 2 = 2 \times 3 = 6$

2. (Easy)
$4 \times 7 \times 2 = 7 \times 4 \times 2 = 2 \times 4 \times 7 =$
$4 \times 2 \times 7 = 7 \times 2 \times 4 = 2 \times 7 \times 4 = 56$

Multiplication of two positive numbers is a positive number.

Example: (Easy)
$2 \times 6 = 12$

Multiplication of two negative numbers is a <u>positive</u> number.

Example: (Easy)
$(-5) \times (-2) = 10$

Multiplication of a negative number with a positive number is a negative number.

Example: (Easy)
$(-7) \times 11 = 11 \times (-7) = -77$

Multiplication of any number with zero yields zero.

Examples:

1. (Easy)
$99 \times 0 = 0$

2. (Easy)
$0 \times 0.5 = 0$

3. (Easy)
$\frac{3}{5} \times 0 = 0$

4. (Easy)
$-8 \times 0 = 0$

Multiplication of any number with 1 equals to itself.

Examples:

1. (Easy)
$99 \times 1 = 99$

2. (Easy)
$1 \times 0.5 = 0.5$

3. (Easy)
$\frac{3}{5} \times 1 = \frac{3}{5}$

4. (Easy)
$(-888) \times 1 = -888$

Practice Exercises:

1. (Easy)
$12 \times 5 = ?$

2. (Easy)
$8 \times (-4) = ?$

3. (Easy)
$(-2) \times 4 \times (-1) \times (-3) = ?$

4. (Easy)
$128 \times 2 \times 89 \times 0 \times 65 = ?$

Answers: 1. 60; **2.** -32; **3.** -24; **4.** 0 (Since one of the terms in the expression is 0, the result is 0.)

Division

Division of Two Integers:

Division of two integers, x and y can be expressed as follows:

$\frac{x}{y} = w + \frac{r}{y}$, where x, y, r, and w are integers.

"w" is the whole part and "r" is called the remainder.

Note that $0 < r < y$

Examples:

1. (Easy)
19/5 = 3 + 4/5
The whole part is 3 and the remainder is 4.

2. (Easy)
137/3 = 45 + 2/3
45 is the whole part and 2 is the remainder.

Note that the remainder is always greater than 0 and less than the denominator.

Representations of a division:

The division of integers can be written in three different ways.

x/y or $\frac{x}{y}$ or $x \div y$ are all divisions of x by y.
The first number or the number at the top position, x, is called the **numerator** and the number at the bottom, y, is called the **denominator**.

Examples:

1. (Easy)
In 19/5, 19 is the numerator and 5 is the denominator.

2. (Easy)

In $\frac{13}{36}$, 13 is the numerator and 36 is the

denominator.

Division is <u>not</u> a communicative operator:

$\frac{a}{b} \neq \frac{b}{a}$

In a division, you <u>cannot</u> interchange the positions of the terms without changing the result.

Example: (Easy) 4 / 5 = 0.8, while 5/4 = 1.25

Division of two positive numbers yields a positive number.

Example: (Easy) $6 \div 2 = 3$

Division of two negative numbers yields a <u>positive</u> number.

Example: (Easy) $(-5) \div (-2) = 2.5$

Division of a negative number by a positive number or division of a positive number by a negative number yield a negative number.

Examples:

1. (Easy) $(-77)/11 = -7$

2. (Easy) $9/(-3) = -3$

Division of any number with 1 equals to itself.

Examples:

1. (Easy)
$99 \div 1 = 99$

2. (Easy)
$0.5 \div 1 = 0.5$

3. (Easy)
$\frac{3}{5} \div 1 = \frac{3}{5}$

4. (Easy)
$(-888) \div 1 = -888$

Division of any number with -1 equals to the negative of itself.

Examples:

1. (Easy)
$99 \div (-1) = -99$

2. (Easy)
$0.5 \div (-1) = -0.5$

3. (Easy)
$\frac{3}{5} \div (-1) = -\frac{3}{5}$

4. (Easy)
$(-888) \div (-1) = 888$

Examples:

1. (Easy)

$$\frac{5}{3} = \frac{5 \times 4}{3 \times 4} = \frac{20}{12} = \frac{5 \times (-7)}{3 \times (-7)} = \frac{-35}{-21}$$

Addition and Subtraction of Divisions:

To add or subtract two divisions:

1. Make both divisions' denominators the same without changing their value.

   The easiest way of doing this is to multiply the numerator and the denominator of each division by the denominator of the other. Here is how:

   $$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b}$$

   $$\frac{a}{b} - \frac{c}{d} = \frac{a \times d}{b \times d} - \frac{c \times b}{d \times b}$$

2. Add or subtract the numerators of the divisions and keep the denominator as is in the result. Here is how:

   $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad + bc}{bd}$$

   $$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{cb}{db} = \frac{ad - cb}{bd}$$

Examples:

1. (Easy)

$$\frac{2}{3} + \frac{3}{4} = \frac{8 + 9}{12} = \frac{17}{12}$$

2. (Easy)

$$\frac{3}{4} - \frac{2}{3} = \frac{9 - 8}{12} = \frac{1}{12}$$

Division of Additions/Subtractions:

When there are division of additions or subtractions, you can first calculate the additions or subtractions, and then perform the division.

Examples:

1. (Easy)

$$\frac{4 + 3}{2} = \frac{7}{2} = 3.5$$

2. (Easy)

$$\frac{4 - 3}{2} = \frac{1}{2} = 0.5$$

However, you can also calculate the individual divisions first and then add or subtract the results. Here is how:

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \text{ and } \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$$

Examples:

1. (Easy)

$$\frac{4 + 3}{2} = \frac{4}{2} + \frac{3}{2} = 2 + 1.5 = 3.5$$

2. (Easy)

$$\frac{4 - 3}{2} = \frac{4}{2} - \frac{3}{2} = 2 - 1.5 = 0.5$$

Multiplication of Divisions:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Example: (Easy)

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$$

Division of Divisions

$$\frac{c}{\frac{a}{b}} = \frac{c}{a/b} = \frac{c}{a \div b} = c \times \frac{b}{a}$$

Example: (Easy)

$$\frac{5}{2/9} = 5 \times \frac{9}{2} = \frac{45}{2}$$

$$\frac{a/b}{c} = \frac{a \div b}{c} = \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

Example: (Easy)

$$\frac{5/2}{9} = \frac{5}{2 \times 9} = \frac{5}{18}$$

$$\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{a/b}{c/d} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example: (Easy)

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

Practice Exercises:
(Don't use your calculator.)

1. (Easy)
$72 \div 9 = ?$

2. (Easy)
$(-55) \div 5 = ?$

3. (Easy)
$1 \div (-2) = ?$

4. (Easy)
$\frac{5}{2 \div 9} = ?$

5. (Easy)
$\frac{6}{6/7} = ?$

6. (Easy)
$12 \div \frac{2}{5} = ?$

7. (Easy)
$\frac{10}{\frac{8}{9}} = ?$

8. (Easy)
$\frac{\frac{10}{8}}{9} = ?$

9. (Easy)
$\frac{-4/7}{3/(-8)} = ?$

10. (Easy)
$\frac{7}{8} + \frac{2}{10} = ?$

11. (Easy)
$5 \div 6 + 7 \div 8 = ?$

12. (Easy)
$\frac{1}{3} - \frac{2}{5} = ?$

13. (Easy)
$\frac{7}{4} + \frac{1}{3} = ?$

14. (Easy)
$\frac{7}{4} - \frac{10}{3} = ?$

15. (Easy)
$\frac{7}{4} - \frac{10}{(-3)} = ?$

16. (Easy)
$-10 + \frac{2}{3} = ?$

17. (Easy)
$\frac{1}{7} \times \frac{3}{8} \times 5 = ?$

18. (Easy)
$\frac{1}{2} \div \frac{3}{4} = ?$

19. (Easy)
$\frac{3 \div (-4)}{(-5) \div (-2)} = ?$

Answers: 1. 8; **2.** -11; **3.** -0.5; **4.** 45/2; **5.** 7; **6.** 30; **7.** 45/4; **8.** 10/72; **9.** 32/21; **10.** 43/40; **11.** 41/24; **12.** -1/15; **13.** 25/12; **14.** -19/12; **15.** 61/12; **16.** -28/3; **17.** 15/56; **18.** 2/3; **19.** -3/10

Order of Operations

Expressions without Parentheses

If there are multiple arithmetic operations in an expression with no parentheses, they are performed in the following order:

1 Multiplications & Divisions	2 Additions	3 Subtractions

From Left to Right →

Example: (Easy)

$$\underbrace{3 \times 2}_{6} + \underbrace{6 \div 3}_{2} - \underbrace{5 \times 4}_{20} = 6 + 2 - 20 = 8 - 20 = -12$$

Expressions with parentheses

When there are parentheses in an expression, first calculate the terms inside the parentheses.

Examples:

1. (Easy)
$(1 + 3) \times 2 = 4 \times 2 = 8$

2. (Easy)
Consider the two expressions below:

$(3 + 3 \times 7) \times 2 = (3 + 21) \times 2 = 24 \times 2 = 48$

$(3 + 3) \times 7 \times 2 = 6 \times 7 \times 2 = 84$

The two expressions have the same terms and similar looking operators. The only difference is in the position of the parentheses. However, they evaluate to different numbers.

Practice Exercises:

1. (Easy)
$(8 - 5) \times (-3) \times 3 = ?$

2. (Easy)
$(10 - 12) \div 2 + (-4) \times 2 \div 2 = ?$

3. (Easy)

$$\frac{(98-90)\times(27+3)}{30}\times\frac{1}{(27-19)}=?$$

4. (Easy)

$$\frac{(98-90)\times 2+14}{30}\times\frac{1}{(28-19)}=?$$

Answers: 1. -27; **2.** -5; **3.** 1; **4.** 1/9

Simplifying the Expressions

The ability to simplify long expressions with large numbers will increase your speed and eliminate the careless errors you can make when entering the large numbers into your calculator. In the beginning simplifying long expressions may seem harder than using your calculator. But after a few exercises you will become very fast and reap the benefits.

Identical multipliers in the numerator and denominator of a division cancel each other.

Sometimes the terms are ready to simplify but many times you need to rearrange them to obtain the identical terms. Here are some examples.

Examples:

1. (Easy)

$$\frac{3843}{517}\times\frac{517}{3843}=1$$

To calculate this expression, you don't have to divide or multiply large numbers. 517 in the first term cancels the 517 in the second term and 3843 in the numerator cancels 3843 in the denominator.

2. (Easy)

$$\frac{0.98\times 1200\times 3}{98\times 600}=\frac{100\times 0.98\times 1200\times 3}{100\times 98\times 600}=$$

$$\frac{98\times 600\times 2\times 3}{100\times 98\times 600}=\frac{6}{100}=0.06$$

In this example, we follow the below steps to simply the expression.

a. Multiply the numerator and the denominator by 100 to make 0.98 same as 98 in the denominator.

b. Multiply 0.98 by 100 and replace 1200 with 2×600 in the numerator.

c. Cancel 98 and 600 from both the numerator and the denominator to get $6/100$.

d. Express the result as a decimal number by dividing 6 by 100.

3. (Easy)

$$\frac{(3+5)\times 38}{8\times 2}=19$$

In this example, we add 3 and 5, since they are in a parenthesis, then cancel 8s from both sides and divide 38 by 2 to get 19 in one step.

You cannot simplify the terms if either the numerator or the denominator has additions or subtractions outside the parentheses.

Example: (Easy)

If the question in Example 3 is written as:

$$\frac{3+5\times 38}{8\times 2}=?$$

with no parentheses, it is wrong to divide 38 by 2.

Instead, the result of this expression is:

$$\frac{3+5\times 38}{8\times 2}=\frac{3+190}{16}=\frac{193}{16}$$

Identical terms on each side of the equality or the inequality sign cancel each other.

Example with multiplication: (Easy)

$34\times 1567\times 2\times 88$ is larger than $2\times 34\times 88\times 1565$ because 1567 is greater than 1565 and all the other terms are the same and cancel each other.

No multiplication is necessary to compare these two expressions.

Example with addition: (Easy)

When there are additions or subtractions in the expression, you can cancel the identical terms that contain only multiplications or divisions on each side of the equation.

$34\times 1567+2\times 88$ is greater than $88\times 2+34\times 1565$ because 1567 is greater than 1565.

Note that we first cancel one term, 88×2, since it is common to both expressions and contains no addition or subtraction. Then we compare 34×1567 with 34×1565

It is wrong to cancel 34 from both of the expressions before canceling the term 88×2

No addition or multiplication is necessary to compare these expressions.

Practice Exercises:

Don't use your calculator to finish these exercises. You will solve them faster and make less errors if you don't use a calculator.

1. (Easy)
Which of the following expressions is bigger?
3×17 or 17×3

2. (Easy)
Which of the following expressions is smaller?
$3 \times 17 \times 1005 \times 88$ or $17 \times 88 \times 3 \times 1003$

3. (Easy)
$\frac{4355}{879} \times \frac{879}{4355} = ?$

4. (Easy)
$\frac{17}{655} \times \frac{6550}{170} = ?$

5. (Easy)
$\frac{17}{655} \times \frac{6550}{17} = ?$

6. (Easy)
$\frac{950 \times 88}{17} \times \frac{170}{10 \times 44} = ?$

Answers: 1. They are equal; **2.** $17 \times 88 \times 3 \times 1003$; **3.** 1; **4.** 1; **5.** 10; **6.** 1900

Decimals, Fractions, Ratios and Percentages

Decimals

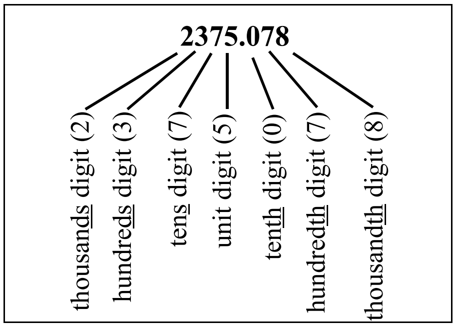
Definition

Decimals are numbers expressed with a decimal point. For example 1.9, 33.0, 0.03 are all decimal numbers.

Digits of the Decimals

Sometimes you will find some questions in the SAT asking the digit of a decimal number. Here is what you need to know:

Consider the decimal number 2375.078



Example: (Easy)
Which of the following numbers have number 3 in the thousandth digit?

- **(A)** 3000.000
- **(B)** 300.000
- **(C)** 0.300
- **(D)** 0.030
- **(E)** 0.003

Solution:
(E) is the correct answer.

Practice Exercise:

1. (Easy)
Which of the following numbers has 2 in both hundreds and hundredth digit?

- **(A)** 22.222
- **(B)** 222.2
- **(C)** 222.22
- **(D)** 22.2
- **(E)** 2.2

Answer: (C)

Comparing Decimals

Decimals are compared digit by digit from left to right. If one of the numbers does not have a digit in the beginning or at the end, substitute a zero for it. As you go from left to right, whichever number has the first larger digit, that is the bigger number.

First example is an obvious case, but it demonstrates the method.

Examples:

1. (Easy)
Which is greater, 366.99 or 1367.01?

Solution:
To make it easier to notice the differences, do the following:

1. Write the numbers in two lines
2. Align the decimal points.

3. Place a zero for the missing digit at the beginning of 366.99.

0 3 6 6 . 9 9
1 3 6 7 . 0 1

It is quite obvious that the second number is larger, because the thousands digit for the second number, 1, which is the left-most digit, is larger than the thousands digit, 0, of the first number.

2. (Easy)
Which one is greater, 0.00009 or 0.0001?

Solution:

To make it easier to notice the differences, do the following:

1. Write the numbers in two lines
2. Align the decimal points.
3. Place a zero for the missing digit at the end of 0.0001

0 . 0 0 0 0 9
0 . 0 0 0 1 0

Once again the second number is larger than the first, because the first 4 digits from the left are the same, but the 5th one is 1 in the second number and 0 in the first number.

Practice Exercises:

1. (Easy)
Which is greater, 0.09999 or 0.10001?

2. (Easy)
Which is greater, 0.33333 or 0.3333?

3. (Easy)
Which is greater, 0.010101 or 0.10101?

Answers: 1. 0.10001; **2.** 0.33333; **3.** 0.10101

Basic Operations with Decimal Numbers

All the operations with decimal numbers can be performed by using a calculator. Sometimes, using the calculator too much may increase a chance of error or it may slow you down, but not when you are dealing with the decimals. Therefore use your calculator to add, subtract, multiply or divide the decimals.

Practice Exercises:

1. (Easy)
0.1 + 0.11 + 0.001 = ?

2. (Easy)
4131.3131 + 313.1313 - 4444.4444 = ?

3. (Easy)
0.001 - 10.0 = ?

4. (Easy)
0.0003 x 100 = ?

5. (Easy)
$\frac{0.07}{1000} = ?$

6. (Easy)
$13.27 \times 1.1 = ?$

7. (Easy)
$-4.8 \times 2.2 = ?$

8. (Easy)
$\frac{95.4}{0.04} = ?$

9. (Easy)
$\frac{-0.65}{-2.6} = ?$

Answers:
1. 0.211; **2.** 0; **3.** -9.999; **4.** 0.03; **5.** 0.00007; **6.** 14.597; **7.** -10.56; **8.** 2385; **9.** 0.25

Fractions

Definition

A fraction is a division that represents a part of a whole. It can be represented in different ways:

x/y or $\frac{x}{y}$ or are all fractions.

As it is for divisions, the first number or the number at the top position, x, is called the **numerator** and the number at the bottom, y, is called the **denominator**.

Two examples of fractions are: 3/4 and $\frac{5}{8}$

Since the fractions are nothing but divisions, you can refer to the previous section for addition, subtraction, multiplication and division of fractions.

Fraction of a Number, Fraction of a Fraction

Fraction of a number:

a/b of c is $\frac{a}{b} \times c$

Example: (Easy)

4/5 of 3 is $\frac{4}{5} \times 3 = \frac{12}{5}$

Fraction of a fraction:

$\frac{a}{b}$ of $\frac{c}{d}$ is $\frac{a}{b} \times \frac{c}{d}$

Example: (Easy)

2/3 of 2/5 is $\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$

Practice Exercises:

Don't use your calculator.

1. (Easy)
What is $\frac{2}{3}$ rd of 138?

2. (Easy)
What is $\frac{1}{5}$ th of $\frac{255}{10}$?

Answers: 1. 92; **2.** 5.1

Mixed Numbers

Definition

When a number contains both an integer and a fraction, it is called a mixed number.

Example: (Easy)

$3\frac{4}{5}$ is a mixed number.
3 is the integer and 4/5 is the fractional part.
Note that $3\frac{4}{5} = 3 + \frac{4}{5}$

Mixed Number - Fraction Conversions

Fraction to Mixed Number:

You can convert a fraction to a mixed number. When the numerator (top) of a fraction is larger than the denominator (bottom), the value of the fraction becomes more than one. For example, 7/5, 19/2, 14/4 are all more than one.

Fractions which are greater than one can be expressed as mixed numbers, containing the whole integer part and the fraction part.

Examples:

1. (Easy)
7/5 = $1 + \frac{2}{5} = 1\frac{2}{5}$
Note that 2 is the remainder of the division of 7 by 5.

2. (Easy)
19/2 = $9 + \frac{1}{2} = 9\frac{1}{2}$

3. (Easy)
14/4 = $3 + \frac{2}{4} = 3 + \frac{1}{2} = 3\frac{1}{2}$

Mixed Number to Fraction:

You can also convert a mixed number to a fraction. The denominator of the fraction remains the same. To find the numerator of the fraction, just multiply the integer part with the denominator of the mixed number and add the numerator to it.

Examples:

1. (Easy)
$4\frac{2}{9} = 4 + \frac{2}{9} = \frac{4 \times 9 + 2}{9} = \frac{38}{9}$

2. (Easy)
$3\frac{1}{11} = \frac{3 \times 11 + 1}{11}$

3. (Easy)
$12\frac{2}{3} = \frac{3 \times 12 + 2}{3} = \frac{38}{3}$

If there is a mixed number in the questions, convert it to a fraction first. It is easier to work with a fraction.

Comparing Fractions

Comparing fractions may be confusing in some cases. Let's examine six cases:

Case 1: Equal Fractions

Two fractions, $\frac{a}{b}$ and $\frac{c}{d}$ are equal if ad = bc

Examples:

1. (Easy)
5/8 = 10/16 are equal, because
$5 \times 16 = 8 \times 10 = 80$

2. (Easy)
2/7 and 3/8 are not equal because $2 \times 8 = 16$ but $3 \times 7 = 21$

3. (Medium)
John has $7 and spends 3/4 of his money for food. Kim wants to spend 1/2 of her money for food. If John and Kim spend equal amount of money for food, how much money did Kim have?

Solution:

John spends $7 \times \frac{3}{4} = \frac{21}{4}$ dollars for food.

Let's assume Kim has m dollars.

This means she spends $\frac{m}{2}$ dollars for food.

Since John and Kim spend equal amounts,

$$\frac{21}{4} = \frac{m}{2}$$

Since these are equal fractions,

$21 \times 2 = 4 \times m$ ➔ $m = 42/4 = 10.5$

The answer is $10.50

Case 2: **Fractions with Identical Denominators**

If two fractions have the same denominator, the fraction with the bigger numerator is the bigger number. You don't need to convert each fraction to a decimal number to compare.

Example: (Easy)

$\frac{3}{5} < \frac{4}{5} < \frac{6}{5} < \frac{31}{5}$, because $3 < 4 < 6 < 31$

Case 3: **Fractions with Identical Numerator**

If two fractions have the same numerator, the fraction with the bigger denominator is the smaller number. You don't need to convert each fraction to a decimal number to compare.

Example: (Easy)

$\frac{5}{8} > \frac{5}{9} > \frac{5}{18} > \frac{5}{400}$ because $8 < 9 < 18 < 400$

Case 4: **Fractions with Different Numerators and Denominators**

When both the numerators and the denominators of two fractions are different, you can use two different methods to compare these fractions.

Method 1:

Treat the fractions as divisions and use your calculator to divide them and find their decimal equivalents. Then compare the two decimal numbers.

Example: (Easy)

Which is smaller, 5/7 or 3/4?

Solution:

5/7 = 0.7143 and 3/4 = 0.75

Now it is easy to see that 5/7 is smaller of the two.

Method 2:

1. Make the denominators of both fractions the same, without changing the fractions' values.

   The easiest way of doing this is by multiplying both the numerator and the denominator of each fraction by the denominator of the other fraction.
2. Compare only the numerators of the two fractions.

Examples:

1. (Easy)

Which is greater? 2/3 or 3/4?

Solution:

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \text{ and } \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

Since 9 is greater than 8, 3/4 is greater than 2/3.

2. (Easy)

Which is smaller, 5/7 or 3/4?

Solution:

$$\frac{5}{7} = \frac{5 \times 4}{7 \times 4} = \frac{20}{28} \text{ and } \frac{3}{4} = \frac{3 \times 7}{4 \times 7} = \frac{21}{28}$$

Since $20/28$ is smaller than $21/28$, 5/7 is smaller than 3/4.

Case 5: **Comparing Mixed Numbers**

a. When one of the fractions is a mixed number, convert the mixed number to a fraction and then compare the two.

Example: (Easy)

Are the two numbers, $5\frac{1}{3}$ and $\frac{32}{6}$, equal?

Solution:

First convert the mixed number to its fraction equivalent:

$$5\frac{1}{3} = \frac{5 \times 3 + 1}{3} = \frac{16}{3}$$

Now you need to compare 16/3 with 32/6. If these two fractions are equal, the following must be true: $16 \times 6 = 3 \times 32$. Indeed, both side of the equation is 96.

Hence $5\frac{1}{3}$ and $\frac{32}{6}$ are equal fractions.

b. If both numbers are in a mixed number format, first compare the integer parts. Whichever has the bigger integer part is the bigger of the two. You don't need to consider the fraction part.

Example: (Easy)

$7\frac{5}{8} < 8\frac{1}{8}$, because $7 < 8$

If the integer parts are the same, compare the fraction parts. Whichever number has the bigger fraction part is the bigger of the two.

Example:(Easy)
$7\frac{5}{8} > 7\frac{1}{8}$, because 5 > 1

c. Use Your Calculator
You can use your calculator to convert the mixed numbers to decimals. Here is how:

1. Convert the mixed number as an addition of an integer and a fraction.

2. Use your calculator to calculate the division and the addition.

Example: (Easy)
Are the two numbers, $5\frac{1}{3}$ and $\frac{32}{6}$, equal?
Solution:
Let's convert both numbers to their decimal equivalents.

$$5\frac{1}{3} = 5 + \frac{1}{3} = 5.3333 \text{ and } \frac{32}{6} = 5.3333$$

So the answer is yes, $5\frac{1}{3}$ and $\frac{32}{6}$ are equal.

Exercises:

1. (Easy)
Write 3 numbers equal to the following numbers. Each set of 4 equal numbers should have two numbers in fraction form, one in decimal form and one in mixed number form.

a. 3/2
b. $7\frac{3}{4}$
c. 10.8

2. (Easy)
Which is smaller? 9/8 or 10/9?

3. (Easy)
Which number is greater, 4/5th of 6 or 4/6th of 5?

4. (Easy)
Which number is greater, 4/5th of 6/7 or 6/7th of 4/5?

5. (Easy)
Organize the following numbers from the smallest to the biggest.

1/2, 2/3, 10/25, 1004/1800, 4/18, $2\frac{1}{3}$, 0.046, 5, 1.8, 1567/5, $20\frac{1}{3}$

Answers:

1. a. $1\frac{1}{2}$, 6/4, 1.5; **b.** 31/4, 62/8, 7.75;
c. $10\frac{4}{5}$, 54/5, 108/10; **2.** 10/9; **3.** 4/5th of 6;
4. They are equal.**;** **5.** 0.046, 4/18, 10/25, 1/2, 1004/1800, 2/3, 1.8, $2\frac{1}{3}$, 5, $20\frac{1}{3}$, 1567/5

Order of Operations With Fractions

Fractions are just numbers. Use the same rule that you use for numbers.

When there are parentheses in an expression, first calculate the terms inside the parentheses. Perform the operations starting with multiplications and divisions, then additions, and finally subtractions.

Examples:

1. (Easy)

$$\frac{2}{4} \times \frac{6}{8} + 5 - \frac{1}{4} = \frac{3}{8} + 5 - \frac{1}{4} =$$

$$\frac{43}{8} - \frac{1}{4} = \frac{43-2}{8} = \frac{41}{8}$$

2. (Easy)

$$\left(\frac{1}{2} + \frac{3}{2}\right) \times 2 = \frac{4}{2} \times 2 = 4$$

3. (Medium)
Consider the two expressions below:

$$\left(3 + \frac{3}{5} \times \frac{7}{8}\right) \times 2 = \left(3 + \frac{21}{40}\right) \times 2 =$$

$$\frac{141}{40} \times 2 = \frac{141}{20}$$

and

$$\left(3 + \frac{3}{5}\right) \times \frac{7}{8} \times 2 = \frac{18}{5} \times \frac{7}{8} \times 2 = \frac{126}{20}$$

The two expressions have the same terms with similar operators. The only difference is the position of the parentheses in the expression, which changes the results completely.

Practice Exercises:

1. (Easy)
$(3 \times 5 + 8)/4 - 8 =$

2. (Easy)
$-3 \times (5 + 8/4 - 8) =$

3. (Easy)
$3 \times 5 + 8/4 - 8 =$

4. (Hard)

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} =$$

5. (Hard)

$$\frac{m}{n+\frac{1}{m+\frac{1}{n}}} =$$

Answers: **1.** -9/4; **2.** 3; **3.** 9; **4.** 5/8; **5.** $\frac{nm^2+m}{mn^2+2n}$

Solution for Question 4:

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}} = \frac{1}{1+\frac{1}{\left(1+\left(\frac{1}{\left(1+\frac{1}{2}\right)}\right)\right)}} =$$

$$\frac{1}{1+\frac{1}{\left(1+\frac{1}{\left(\frac{3}{2}\right)}\right)}} = \frac{1}{1+\frac{1}{\left(1+\frac{2}{3}\right)}} = \frac{1}{1+\frac{1}{\left(\frac{5}{3}\right)}} = \frac{1}{\left(1+\frac{3}{5}\right)} =$$

$$\frac{1}{\left(\frac{8}{5}\right)} = \frac{5}{8}$$

Solution for Question 5:

$$\frac{m}{n+\frac{1}{m+\frac{1}{n}}} = \frac{m}{n+\frac{1}{\left(m+\frac{1}{n}\right)}} = \frac{m}{n+\frac{1}{\left(\frac{(nm+1)}{n}\right)}} =$$

$$\frac{m}{n+\frac{n}{nm+1}} = \frac{m}{\left(\frac{n(nm+1)}{nm+1}+\frac{n}{nm+1}\right)} = \frac{m}{\left(\frac{mn^2+2n}{nm+1}\right)} =$$

$$\frac{nm^2+m}{mn^2+2n}$$

Decimal - Fraction Conversions

Sometimes the multiple choices are given in fraction or decimal form. To recognize the correct answer, you must know how to convert fractions to decimals and vice versa.

Fraction to Decimal

This is very easy. A fraction is a division. Use your calculator and divide the numerator by denominator.

Example: (Easy)
3/4 = 0.75

Decimal to Fraction

This is also very easy. If the decimal number is less than one, divide the integer on the right side of the decimal point by

- 10 if it has only 1 digit on the right side of the decimal point.
- 100 if it has only 2 digits on the right side of the decimal point.
- 1000 if it has only 3 digits on the right side of the decimal point, etc..

If necessary, simplify the fraction.

Examples:

1. (Easy)
0.2 = 2/10 = 1/5

2. (Easy)
0.11 = 11/100

3. (Easy)
0.903 = 903/1000

If the decimal number is greater than one, remove the decimal point and write this integer as the numerator of the fraction. As the denominator, write

- 10 if the decimal number has only 1 digit on the right side of the decimal point.
- 100 if the decimal number has only 2 digits on the right side of the decimal point.
- 1000 if the decimal number has only 3 digits on the right side of the decimal point, etc.

This procedure is equivalent to first converting the decimal to a mixed number and then converting the mixed number to a fraction. Note that a decimal number greater than 1 is the sum of the whole part and the decimal part. The whole part is the integer part of a mixed number. You can find the fraction part of the mixed number by converting the decimal part to a fraction by using the method described above.

Examples:

1. (Easy)
$12.32 = 12\frac{32}{100} = \frac{1232}{100} = \frac{308}{25}$

2. (Easy)
$27.001 = 27\frac{1}{1000} = \frac{27001}{1000}$

Practice Exercises:

1. (Easy)
Convert the following fractions to a decimal.

11/34 =

87/3 =

2/5 =

2. (Easy)
Convert the following decimals to a fraction.

0.44 =

3.023 =

23.1 =

3. (Easy)
Put the following numbers in order from smallest to the biggest.

$3\frac{2}{5}, 4\frac{1}{5}, \frac{18}{5}, 3.5, \frac{18}{6}$

Answers:
1. 11/34 = 0.3235, 87/3 = 29, 2/5 = 0.4
2. 0.44 = 11/25, 3.023 = 3023/1000, 23.1 = 231/10
3. $\frac{18}{6}, 3\frac{2}{5}, 3.5, \frac{18}{5}, 4\frac{1}{5}$

Ratios

Ratios and fractions are very similar. From a purely mathematical point of view, there is no difference between the two. Both are divisions. Therefore everything that is presented in the previous section, Fractions, is also applicable to the ratios.

However, there is a linguistic difference between the two:

The word "**fraction**" is used to describe a portion of a whole. For example, consider the statement "1/3 of all the fruits are apples." Here 1/3 is a fraction of the fruits: One in every 3 fruits is an apple.

The word "**ratio**" is used to compare the quantities of two parts. For example, consider the statement "The ratio of apples to oranges is 1/3." Here 1/3 is a ratio: For each apple, there are three oranges.

You can express a ratio in different ways:

Like a division - "The ratio of apples and oranges is 1/3"

In words - "The ratio of apples and oranges is one to three."

With colon symbol ":" - "The ratio of apples and oranges is 1:3"

Example: (Medium)
If the ratio of boys to girls in a classroom is 4 to 5, and if the total student count is 27, how many girls and boys are there in this classroom?

Solution:
If the ratio of boys to girls in a classroom is 4 to 5, the fraction of boys is 4/9 and the fraction of girls is 5/9. In other words, out of 9 students 4 of them are boys and 5 of them are girls. Hence

the number of boys = $\frac{4}{9} \times 27 = 12$ and

the number of girls = $\frac{5}{9} \times 27 = 15$

Percentages

Some of the examples and exercises in this section require you to be familiar with simple algebra. If you have a difficulty in understanding them, study "One Variable Simple Equations" in Chapter 7.

x Percent of y

x% of y is $\frac{x \times y}{100}$

Examples:

1. (Easy) 8% of 90 is $\frac{8 \times 90}{100} = 7.2$

2. (Easy) 120% of 90 is $\frac{120 \times 90}{100} = 108$

x is What Percent of y?

Sometimes the question provides the value, and asks for the percentage. General form of these questions is "x is what percent of y?"
Answer is "x is $\frac{100x}{y}$ % of y."

Examples:

1. (Easy)
16 is what percent of 80?

Solution:

$$\frac{16 \times 100}{80} = 20\%$$

2. (Easy)
Carol had \$50.00. She spends \$12.00 for school play tickets. What percent of her money does she still have?

Solution:
She still has 50 – 12 = \$38

The question is "38 is what percent of 50?"

It is $\frac{38 \times 100}{50} = 76\%$

If x% of y is z, What is y?

In another type of question, the question provides the percentage of an amount, and then asks for the amount itself. General form of these questions is "If x% of y is z, what is y?"

Solution: $\frac{x \cdot y}{100} = z \Rightarrow y = \frac{100z}{x}$

Examples:

1. (Easy)
22% of b is 60. What is b?

Solution:

$$\frac{22b}{100} = 60 \Rightarrow b = \frac{60 \times 100}{22} = 272.73$$

2. (Easy)
105% of c is 55. What is c?

Solution:

$$\frac{55 \times 100}{105} = 52.38$$

3. (Medium)
In Maryland there is 5% sales tax. If you buy a sweater and pay \$35 including the tax, how much is the sweater without the tax?

Solution:
If there is 5% sales tax, \$35 is 100 + 5 = 105% of the sweater's price, with tax. Only the sweater , without tax is

$$\frac{35 \times 100}{105} = \$33.33$$

Percent of a Percent

x% of y% of a number is (xy)/100 percent of the same number.

Examples:

1. (Easy)
20 percent of 80% of k is $\frac{20 \times 80}{100}$ = 16% of k.

2. (Medium)
In a 3-day camping trip, the Brown family consumed 35% of the food in the first day. They consumed 55% of the remaining food on the 2nd day. What percent of the food remained for the 3rd day?

Solution:
Remaining food after the first day =
100 – 35 = 65%

2nd day's consumption = $\frac{65 \times 55}{100}$ = 35.75%

3rd day's consumption =
100 – 35 – 35.75 = 29.25%

Practice Exercises

1. (Easy)
What is 7% of 70?

2. (Easy)
What is 107% of 70?

3. (Easy)
What is 100% of 70?

4. (Easy)
5 is what percent of 25?

5. (Easy)
125 is what percent of 25?

6. (Easy)
25 is what percent of 25?

7. (Medium)
If 2% of the number x is 9, what is x?

8. (Medium)
If 120% of the number x is 36, what is x?

9. (Medium)
If 100% of the number x is 96, what is x?

10. (Medium)
What is 20% of 30% of 60?

11. (Medium)
If 10% of 45% of the number x is 5, what is x?

12. (Medium)
If 45% of 10% of the number x is 5, what is x?

13. (Medium)
Jane is traveling between two towns 60 miles apart. In the first 5 minutes, she traveled 7% of the road. In the next 5 minutes, she traveled 10% of the remaining distance. How far is she from her destination?

Answers:
1. 4.9; **2.** 74.9; **3.** 70; **4.** 20; **5.** 500;
6. 100%; **7.** 450; **8.** 30; **9.** 96; **10.** 3.6; **11.** 111.111;
12. 111.111; **13.** 50.22 miles

Fraction, Percentage Mixes

Sometimes percentages and fractions are mixed in a question. The solution usually requires converting fractions to percentages and vice versa. Here is what you need to know:

Fraction - Percentage Conversions

Fraction equivalent of x%

Fraction equivalent of x% is x/100.

Example: (Easy)

71% of 90 is $\frac{71}{100}$ th of 90

Percentage equivalent of x/y

Percentage equivalent of x/y of z is 100x/y percent of z.

Example: (Easy)

3/4 th of n is $\frac{100 \times 3}{4}$ = 75% of n

Practice Exercises:

a. (Medium)
2/3rd of 90 is what percent of 90?

b. (Medium)
4/3rd of 90 is what percent of 90?

c. (Medium)
20% of 50 is what fraction of 50?

d. (Medium)
120% of 50 is what fraction of 50?

Answers: a. 66.7%; **b.** 133.3%; **c.** 1/5; **d.** 6/5

Powers

Most of the questions about powers are at Medium to Hard level. But occasionally, you can find some Easy questions as well.

Integer Powers

Positive Integer Powers of a Number

Positive mth power of a number, a, is written as a^m where m is a positive integer. It is equal to the multiplication of "a" by itself "m" times. "m" is called the **exponent** and "a" is called the **base**.

Examples:

1. (Easy)

$3^4 = 3 \times 3 \times 3 \times 3 = 81$

All powers of positive numbers are positive.

2. (Easy)

$(-3)^4 = (-3) \times (-3) \times (-3) \times (-3) = 81$

Even powers of negative numbers are positive.

3. (Easy)

$(-5)^3 = (-5) \times (-5) \times (-5) = -125$

Odd powers of negative numbers are negative.

4. (Easy)

$0^4 = 0 \times 0 \times 0 \times 0 = 0$

Positive powers of zero is zero.

5. (Easy)

$2.5^1 = 2.5$

First power of all the numbers equals the number itself.

6. (Medium)

$0.6^2 = 0.6 \times 0.6 = 0.36$

$0.4^3 = 0.4 \times 0.4 \times 0.4 = 0.064$

Second or more powers of positive numbers less than 1 are less than the number itself.
In the above examples, 0.36 < 0.6 and 0.064 < 0.4

7. (Medium)

$0.2^2 = 0.04$

$0.2^3 = 0.008$

$0.2^4 = 0.0016$

Higher positive powers of positive numbers less than 1 get smaller as the exponent increases.

In the above examples $0.2^4 < 0.2^3 < 0.2^2$

Practice Exercises:

1. Without using your calculator, evaluate the following expressions.

- **a.** (Easy) $2^4 = ?$
- **b.** (Easy) $(-7)^2 = ?$
- **c.** (Easy) $-(-7)^2 = ?$
- **d.** (Easy) $(-4)^3 = ?$
- **e.** (Easy) $-(-4)^3 = ?$
- **f.** (Easy) $\left(\frac{1}{2}\right)^4 = ?$
- **g.** (Easy) $0.2^5 = ?$

2. (Medium) Put the following numbers in ascending order.

$0.78, 0.78^3, 0.78^2, 0.78^5$

Answers: 1. a. 16; **1. b.** 49; **1. c.** -49; **1. d.** -64; **1. e.** 64; **1. f.** 1/16; **1. g.** 0.00032; **2.** 0.78^5, 0.78^3, 0.78^2, 0.78

Zeroth Power of a Number

Zeroth power of any non-zero number is one.

By definition, $a^0 = 1$ for all the non-zero values of a.

0^0 *is undefined.*

Examples:

1. (Easy) $3^0 = 1$

2. (Easy) $(-2)^0 = 1$

3. (Easy) $1.7^0 = 1$

Negative Integer Powers of a Number

Negative mth power of a non-zero number, a, is written as a^{-m} where m is a positive integer.

$$a^{-m} = \frac{1}{a^m}$$

Examples:

1. (Easy) $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

2. (Easy) $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$.

3. (Easy) $(-5)^{-3} = \frac{1}{(-5)^3} = -\frac{1}{125}$

Practice Exercises:

Don’t use your calculator. Give your answers as fractions.

1. (Easy) $2^{-5} = ?$

2. (Easy) $(-2)^{-5} = ?$

3. (Easy) $-(-2)^{-5} = ?$

4. (Easy) $\left(\frac{1}{3}\right)^4 = ?$

5. (Easy) $3^{-4} = ?$

6. (Medium) $\left(\frac{1}{3}\right)^{-4} = ?$

7. (Medium) $\left(-\frac{1}{5}\right)^{-2} = ?$

Answers:

1. 1/32; **2.** -1/32; **3.** 1/32; **4.** 1/81; **5.** 1/81; **6.** 81; **7.** 25

Multiplication and Division of Terms With the Same Base

When two terms with the same base are multiplied, the exponents are added.

$$s^n \times s^k = s^{(n+k)} \text{ and } \frac{s^n}{s^k} = s^n \times s^{-k} = s^{n-k}$$

Examples:

1. (Medium)

$3^2 \times 3^3 = 3^5$

Here 5 = 2 + 3 is the addition of the exponents.

2. (Medium)

$3^2 \times 3^{-3} = 3^{-1} = \frac{1}{3}$

Here -1= 2 - 3 is the addition of the exponents 2 and -3. As you can see the exponents can also be negative. The rule does not change.

3. (Medium)

$\frac{15^8}{15^9} = 15^{(8-9)} = 15^{-1} = \frac{1}{15}$

Distribution of Powers for Multiplication

$(a \times b)^m = a^m \times b^m$

Examples:

1. (Medium) $(2 \times 8)^2 = 2^2 \times 8^2 = 4 \times 64 = 256$

2. (Medium) $(3.7x)^7 = 3.7^7x^7 = 9493.19x^7$

3. (Medium)

$(3.7x)^{-7} = 3.7^{-7}x^{-7} = \frac{1}{3.7^7x^7} = \frac{1}{9493.19x^\circ}$

Note: The base can be any number. It does not have to be an integer.

4. (Medium)

$(-2)^3 \times 3.5^3 = (-2 \times 3.5)^3 = (-7)^3 = -343$

Note: The base can be negative as well.

Distribution of Powers for Division

$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

Examples:

1. (Medium) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

2. (Medium) $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$

Power of a Power

$(a^k)^s = a^{(k \times s)}$

Examples:

1. (Easy) $(2^3)^2 = 2^6 = 64$

2. (Easy) $(2^3)^{-2} = 2^{-6} = \frac{1}{64}$

3. (Easy) $(2^{-3})^{-2} = 2^6 = 64$

4. (Easy) $((-2)^{-3})^{-2} = (-2)^6 = 64$

Practice Exercises:

Don't use your calculator.

1. (Easy) $2^5 \times 2^2 = ?$

2. (Easy) $8^9 \times 8^5 \times 8^{-12} = ?$

3. (Easy)

$0.045^{16} \times 0.045^{17} \times 0.045^{-30} \times 0.045^{-3} = ?$

4. (Easy) $\frac{6^6}{6^4} = ?$

5. (Easy) $\frac{2^4}{z} = ?$

6. (Easy) $\left(\frac{2}{z}\right)^4 = ?$

7. (Easy) $(xyz)^3 = ?$

8. (Easy) $\left(\frac{2}{3} \times \frac{1}{a}\right)^4 = ?$

9. (Easy) $4 \times \left(\frac{1}{2}\right)^5 = ?$

10. (Easy) $(abc)^{-8} = ?$

Answers: 1. 128; **2.** 64 (No need to calculate the individual terms. Add the exponents first.); **3.** 1 (No need to calculate the individual terms. Add the exponents first.); **4.** 36; **5.** $\frac{16}{z}$; **6.** $\frac{16}{z^4}$; **7.** $x^3 \cdot y^3 \cdot z^3$; **8.** $\frac{16}{81a^4}$; **9.** 1/8; **10.** $\frac{1}{a^8 \cdot b^8 \cdot c^8}$

Order of Operations with Powers

Integer powers are actually multiplications. Therefore in calculating equations with powers, they come first. So, the order of operations is:

1	2	3
Powers	Multiplications & Divisions	Additions & Subtractions

From Left to Right

→

Note: If there are parentheses in the expression, calculate them first.

Examples:

1. (Easy)

$$\frac{2^3 \times 5}{10} - 7 + 8 = \frac{8 \times 5}{10} - 7 + 8 =$$

$$\frac{40}{10} - 7 + 8 = 4 - 7 + 8 = 12 - 7 = 5$$

2. (Medium)

$$\left(\frac{1}{4}\right)^{-2} + (3 + 2) \times (12 - 9)^2 - 18 =$$

$$\frac{1}{4^{-2}} + 5 \times 3^2 - 18 = 4^2 + 5 \times 3^2 - 18 =$$

$$16 + 5 \times 9 - 18 = 16 + 45 - 18 = 61 - 18 = 43$$

3. (Hard)

$$\frac{2^5 \times 6^3}{(5 - 1)^2 \times 8} = ?$$

Solution:

First you need to express all the terms as the powers of 2. Here is how:

$$6^3 = (2 \times 3)^3 = 2^3 \times 3^3$$

$$(5 - 1)^2 = 4^2 = (2^2)^2 = 2^4$$

$$8 = 2^3$$

Then substitute these terms into the expression:

$$\frac{2^5 \times 6^3}{(5 - 1)^2 \times 8} = \frac{2^5 \times 2^3 \times 3^3}{2^4 \times 2^3} =$$

$$2^{(5 + 3 - 4 - 3)} \times 3^3 = 2 \times 3^3 = 54$$

Practice Exercises:
Don't use your calculator.

1. (Medium)

If x = -1, $\frac{(3^3)^{-x}}{3} = ?$

2. (Medium)
If a = 1, $\left(\frac{23 \times 18}{76 \div 9}\right)^{(a^2 - 2a + 1)} \times 441 = ?$

3. (Hard)

$$\frac{2^{-5} \times 6^3 \times 8}{(5 - 1)^2} = ?$$

Answer: 1. 9; **2.** 441; **3.** 27/8

Square Root

Square Root

Definition

In the previous section, all thepowers were integers. Powers can also be rational numbers. All the properties of the powers described in the previous section are valid for rational powers as well. One such rational power is 1/2. It is defined as follows:

Let $a = b^2$. Then take the (1/2)nd power of both sides of the equation,

$$a^{1/2} = (b^2)^{\frac{1}{2}} = b^{2 \cdot \frac{1}{2}} = b$$

Because a is the square of the real number b, a cannot be negative. However, b can be positive or negative or zero. Remember that $b^2 = (-b)^2$.

In general, (1/2)nd power of a number, $a^{1/2}$, has two values. One is positive and the other one is negative. The positive one is called the **square root**. $\sqrt{a}$ is used to describe the positive root of "a".

Square root of numbers greater than 1 are smaller than the number itself.

If a > 1, then $a > \sqrt{a}$

Examples:

1. (Medium) $4 > \sqrt{4} = 2$

2. (Medium) $25 > \sqrt{25} = 5$

3. (Medium) $81 > \sqrt{81} = 9$

4. (Medium) $1.44 > \sqrt{1.44} = 1.2$

Notice that non-integer numbers also have square roots. You can use your calculator to calculate square root if necessary.

Square root of numbers less than 1 are bigger than the number itself.

Examples:

1. (Medium) $0.04 < \sqrt{0.04} = 0.2$

2. (Medium) $0.01 < \sqrt{0.01} = 0.1$

3. (Medium) $0.09 < \sqrt{0.09} = 0.3$

Square roots of zero and one are equal to themselves.

$\sqrt{0} = 0$ and $\sqrt{1} = 1$

Practice Exercises:

1. (Easy)
Put the numbers below in order, from the smallest to the biggest.

$\sqrt{4}$, $2\sqrt{2}$, $\sqrt{2} + 2$, 4

2. (Easy)
Put the numbers below in order, from the smallest to the biggest.

$\sqrt{0.4}$, $2\sqrt{0.2}$, $\sqrt{0}$, $\sqrt{1}$, 0.4

Answers: 1. $\sqrt{4}$, $2\sqrt{2}$, $\sqrt{2} + 2$, 4;
2. $\sqrt{0}$, 0.4, $\sqrt{0.4}$, $2\sqrt{0.2}$, $\sqrt{1}$

The square roots of negative numbers are imaginary and not the subject of SAT.

Basic Operations With Square Root

Since square root is a fractional power, the distribution rules for powers apply to square roots as well. Below is the summary of these rules.

Square Root of Multiplication:

$\sqrt{s \times h} = \sqrt{s} \times \sqrt{h}$

Examples:

1. (Easy)

$$\sqrt{4 \times 36} = \sqrt{4} \times \sqrt{36} = 2 \times 6 = 12$$

2. (Easy)

$$3 \cdot \sqrt{7} = \sqrt{9} \cdot \sqrt{7} = \sqrt{63}$$

3. (Medium)

$$2 \times \sqrt{\frac{25 \times 0.16}{0.09 \times 4}} = 2 \times \frac{\sqrt{25} \times \sqrt{0.16}}{\sqrt{0.09} \times \sqrt{4}}$$

$$2 \times \frac{5 \times 0.4}{0.3 \times 2} = \frac{2}{0.3} = \frac{20}{3}$$

Square Root of Division:
You can take the square root of fractions as well. In this case, you can apply the distribution of powers for the division rule.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Examples:

1. (Easy)

$$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

2. (Easy)

$$\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$$

3. (Medium)

$$\frac{3\sqrt{10}}{\sqrt{3}} = \frac{\sqrt{9} \cdot \sqrt{10}}{\sqrt{3}} = \sqrt{\frac{9 \cdot 10}{3}} = \sqrt{30}$$

Square Root of Addition and Subtraction
You <u>cannot</u> apply the distribution rule to addition and subtraction:

$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$ and $\sqrt{a - b} \neq \sqrt{a} - \sqrt{b}$

Examples:

1. (Easy)

$\sqrt{16 + 9} = \sqrt{25} = 5$
On the other hand $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$

2. (Easy)

$\sqrt{25 - 9} = \sqrt{16} = 4$

On the other hand

$\sqrt{25} - \sqrt{9} = 5 - 3 = 2$

When you see addition and subtraction inside of a square root sign, first calculate the expression under the square root, and then perform the square root operation.

Power of a Square Root:

$$(\sqrt{a})^b = \sqrt{a^b}$$

Examples:

1. (Easy)
$(\sqrt{4})^3 = 2^3 = 8$ or $(\sqrt{4})^3 = \sqrt{4^3} = \sqrt{64} = 8$

2. (Easy) $(\sqrt{71})^2 = \sqrt{71^2} = 71$

3. (Easy) $(\sqrt{3})^4 = \sqrt{3^4} = \sqrt{81} = 9$

Square Root of a Square Root

When there is a square root of a root, you start from the inner most square root and proceed toward the outer most square root.

Examples:

1. (Easy) $\sqrt{\sqrt{16}} = \sqrt{4} = 2$

2. (Easy) $\sqrt{\sqrt{\sqrt{\sqrt{256}}}} = \sqrt{\sqrt{\sqrt{16}}} = \sqrt{\sqrt{4}} = \sqrt{2}$

Order of Operations with Square Root

As mentioned earlier, square root is a special power operation. If there are multiple arithmetic operations in an expression with no parentheses, the expression is calculated as follows:

1	2	3
Powers & Square Root	Multiplications & Divisions	Additions & Subtractions

From Left to Right

$\longrightarrow$

Note: If there are parentheses in the expression, calculate them first.

Practice Exercises:

Don't use your calculator.

1. (Easy)
a = 13 and b = 36. What is $\sqrt{a + b}$?

2. (Easy)

$$\frac{\sqrt{45}}{\sqrt{5}} = ?$$

3. (Easy)

$$\sqrt{2} \times \sqrt{18} = ?$$

4. (Medium)

$$\frac{\sqrt{2} \times \sqrt{18}}{3} = ?$$

5. (Medium)

$$\left(\sqrt{\frac{3}{2}}\right)^2 = ?$$

6. (Medium)

$$\sqrt{\sqrt{\sqrt{\sqrt{256}}}} \cdot \sqrt{\sqrt{4}} = ?$$

7. (Medium)

$$(2^3 - \sqrt{64}) \times \sqrt{27 + 2} + 5 = ?$$

8. (Medium)

$$\left(\sqrt{\frac{a^2}{2}}\right)^4 - \frac{a^4}{4} + 1 = ?$$

9. (Medium)

$$3^{\left(\left(\sqrt{\frac{a^2}{2}}\right)^4 - \frac{a^4}{4} + 1\right)} = ?$$

Answers: 1. 7; **2**. 3; **3.** 6; **4.** 2; **5.** 1.5; **6.** 2; **7.** 0; **8.** 1; **9.** 3

Fractional Powers

Rational powers of real numbers are not limited to the square root. Exponents can be other rational numbers.

n^{th} Root

$a^{\frac{1}{n}}$ is called the nth root of a, where n is a positive integer.

Example: (Medium)

5th root of 2 is $2^{\frac{1}{5}}$

$a^{\frac{1}{n}}$ can also be expressed as $\sqrt[n]{a}$

Example: (Medium)

$$2^{\frac{1}{5}} = \sqrt[5]{2}$$

Finding the Base of the nth Root of a Number:

If $y = x^{\frac{1}{n}}$, then $x = y^n$

Explanation:

If $y = x^{\frac{1}{n}}$, then taking nth power of both sides of the equation:

$$y^n = \left(x^{\frac{1}{n}}\right)^n = x^{\frac{1}{n} \cdot n} = x$$

Examples:

1. (Easy)

$p^{\frac{1}{3}} = 9$ ➔ $p = 9^3 = 729$

2. (Easy)
If $b^2 = 49$, b = 7 or b = -7

3. (Medium)
$\sqrt[3]{x} = 4 \Rightarrow \sqrt[3]{x} = x^{\frac{1}{3}} = 4 \Rightarrow x = 4^3 = 64$

Note that the answer is not $x = 4^{\frac{1}{3}}$

Practice Exercises:

1. (Easy)
If $p^{\frac{1}{4}} = 5$, then p = ?

2. (Medium)
If $\sqrt[3]{y} = -4$, then y = ?

3. (Medium)
If $z^{\frac{-1}{4}} = -5$, then z = ?

4. (Medium)
If $(1/z)^{\frac{-1}{4}} = -5$, then z = ?

Answers: 1. 625; **2.** -64; **3.** 1/625; **4.** 625

n

n^{th} root of a positive number x is a real number.

Examples:

1. (Medium)
Let b be the second root of 3 ➔ $b = 3^{\frac{1}{2}}$
In this expression b is a real number. In fact
$b = \sqrt{3} \cong 1.73$ or $b = -\sqrt{3} \cong -1.73$

2. (Medium)
Let b be the third root of 2 ➔ $b = 2^{\frac{1}{3}}$
In this expression b is a real number. In fact $b \cong 1.26$

n^{th} root of a negative number x is a real number only if the root, n, is odd.

Examples:

1. (Medium)
Let b be the third root of -8 ➔ $b = (-8)^{\frac{1}{3}}$
In this expression b is a negative real number. In fact b = -2

2. (Medium)
Let b be the second root of -3 ➔ $b = (-3)^{\frac{1}{2}}$,
In this expression b is not a real number. You can convince yourself by taking the square of each side of the equation. This will yield $b^2 = -3$. Since the square of real numbers can never be negative, you can conclude that b is not a real number.

$(m/n)^{th}$ Power:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m = \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$$

Example: (Medium)

$2^{\frac{2}{3}} = \left(2^{\frac{1}{3}}\right)^2 = (\sqrt[3]{2})^2 = (1.26)^2 = 1.587$ and

$2^{\frac{2}{3}} = (2^2)^{\frac{1}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4} = 1.587$

Use your calculator to verify these results.

Practice Exercises:
Don’t use your calculator.

1. (Easy)
$4^{\frac{1}{2}} \times 4^2 = ?$

2. (Medium)
$8^{\frac{3}{5}} \times 8^5 \times 8^{-\frac{12}{3}} = ?$

3. (Medium)
$0.045^{\frac{1}{2}} \times 0.045^{\frac{2}{3}} \times 0.045^{-3} \times 0.045^{-\frac{3}{2}} = ?$

4. (Medium)
$\dfrac{6^{\frac{6}{5}}}{6^{\frac{1}{5}}} = ?$

5. (Medium)
$\sqrt[4]{16} = ?$

6. (Medium)
$8^{\frac{2}{3}} = ?$

7. (Medium)
$(xyz)^{\frac{3}{2}} = ?$

8. (Medium)
$\left(\frac{2}{3} + \frac{1}{3}\right)^{-\frac{1}{4}} = ?$

9. (Medium)
$\sqrt[5]{35-3} = ?$

10. (Medium)
$(abc)^{-\frac{8}{7}} = ?$

11. (Medium)
If $z^3 = -8$, then z = ?

12. (Medium)
If $\sqrt[3]{a} = 3$, then a = ?

Answers:
1. 32 or -32; **2.** $2^{\frac{24}{5}}$; **3.** $0.045^{-\frac{10}{3}}$; **4.** 6; **5.** 2;
6. 4; **7.** $x^{\frac{3}{2}} \cdot y^{\frac{3}{2}} \cdot z^{\frac{3}{2}}$; **8.** 1; **9.** 2; **10.** $\dfrac{1}{a^{\frac{8}{7}} \cdot b^{\frac{8}{7}} \cdot c^{\frac{8}{7}}}$; **11.** -2;
12. 27

Finding the Base of the (n/m)th Power of a Number

If $y = x^{\frac{n}{m}}$, then $x = y^{\frac{m}{n}}$

Examples:

1. (Easy)
$x^{\frac{1}{3}} = 2$ ➔ $x = 2^3 = 8$

2. (Medium)
$p^{\frac{2}{3}} = 9$ ➔ $p = 9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3 = 3^3 = 27$ or

$p = 9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3 = (-3)^3 = -27$

3. (Medium)
$x^3 = -8$ ➔ $x = (-8)^{\frac{1}{3}} = -2$

4. (Medium)
$x^{-\frac{3}{2}} = -8$ ➔

$x = (-8)^{-\frac{2}{3}} = \left((-8)^{\frac{1}{3}}\right)^{-2} =$

$(-2)^{-2} = \dfrac{1}{(-2)^2} = \dfrac{1}{4}$

Practice Exercises:

1. (Medium)
If $a^{\frac{2}{3}} = 4$, then a = ?

2. (Medium)
If $a^{\frac{4}{3}} = 16$, then a = ?

3. (Medium)
If $a^{\frac{3}{2}} = -8$, then a = ?

4. (Medium)
If $a^{-\frac{3}{2}} = -8$, then a = ?

Answers: 1. 8 or -8; **2.** 8 or -8; **3.** 4; **4.** 1/4

Positive fractional powers of zero is zero.

Example: (Easy) $0^{\frac{4}{3}} = 0$

Negative Numbers

There are some SAT questions in which negative numbers are involved. They are not difficult questions, but some students make careless mistakes because they don't check the signs carefully.

Sometimes negative numbers act differently than some people think. So in this section, we summarize the unusual behaviors of negative numbers.

(Negative) x (Negative) = Positive*:* $(-5) \cdot (-3) = 15$

(Negative) x (Positive) = Negative*:* $(-5) \cdot 3 = -15$

(Negative) /(Negative) = Positive*:* $(-5)/(-3) = 5/3$

(Negative) / (Positive) = Negative*:* $(-5)/3 = -5/3$

Odd powers of negative numbers are negative:

Example: (Medium) $(-2)^3 = -8$

Even powers of negative numbers are positive:

Example: (Medium) $(-2)^4 = 16$

Negative numbers decrease in value as they become more negative:

Example: (Easy) –50 is less than -3

Practice Exercises:

1. (Easy)
-58 + 7 = ?

2. (Easy)
-58 - 7 = ?

3. (Easy)
7 - 58 = ?

4. (Easy)
$-8 \times 7 = ?$

5. (Easy)
$-8 \times (-7) = ?$

6. (Easy)
$8 \times (-7) = ?$

7. (Easy)
$-7 \times 8 = ?$

8. (Easy)
-6 / 2 = ?

9. (Easy)
6 / (-2) = ?

10. (Easy)
(-6) / (-2) = ?

11. (Easy)
-2 / 6 = ?

12. (Easy)
2 / (-6) = ?

13. (Easy)
(-2) / (-6) = ?

14. (Medium)
$(-3)^4 = ?$

15. (Medium)
$(-3)^3 = ?$

16. (Medium)
$-3^{-3} = ?$

17. (Medium)
$(-3)^{-4} = ?$

18. Put the following numbers in order from small to big:
28, 1, 0.8, -500, 17, -3.5, -1/2, -2/3, -1, $(-3)^{-3}$

Answers: 1. -51; **2.** –65; **3.** –51; **4.** –56; **5.** 56; **6.** –56; **7.** –56; **8.** –3; **9.** –3; **10.** 3; **11.** –1/3; **12.** –1/3; **13.** 1/3; **14.** 81; **15.** -27; **16.** -1/27; **17.** 1/81;

18. -500, -3.5, -1, -2/3, -1/2, $(-3)^{-3}$, 0.8, 1, 17, 28

Numbers Between –1 and 1

There are a few questions that involve the numbers between -1 and 1. We have already covered these numbers in the previous sections. However, it is easy to forget them and make careless mistakes.

Numbers between -1 and 1 act differently than some people think. So in this section, we have summarized the unusual behaviors of such numbers.

Note that -1, 0 and 1 are excluded from this section.

Multiplication of two numbers between <u>0 and 1</u> is <u>less than</u> both of the numbers:

If 0 < a < 1 and 0 < b < 1 ➔ ab < a and ab < b

Examples:

1. (Easy)
$0.7 \times 0.9 = 0.63$
Both 0.7 and 0.9 are larger than 0.63

2. (Medium)
$\frac{1}{3} \times \frac{5}{6} = \frac{5}{18}$
Both 1/3 and 5/6 are larger than 5/18

Use your calculator to verify the result.

Multiplication of two numbers between <u>-1 and 0</u> is <u>more than</u> both of the numbers:

If -1 < a < 0 and -1 < b < 0 ➔ ab > a and ab > b, because ab is positive and both a and b are negative.

Example: (Medium)

$(-0.7) \times (-0.9) = 0.63$.

Both -0.7 and -0.9 are smaller than 0.63 because they are negative numbers.

Positive integer powers of a number between <u>0 and 1</u> <u>decreases</u> as the exponent increases:

If $0 < a < 1$ ➔ $a > a^2 > a^3 > a^4$...

Example: (Medium)

For $a = 0.9$
$a^2 = 0.9^2 = 0.81$
$a^3 = 0.9^3 = 0.729$
$a^4 = 0.9^4 = 0.6561$

$0.9 > 0.81 > 0.729 > 0.6561$... ➔

$0.9 > 0.9^2 > 0.9^3 > 0.9^4$

Negative integer powers of a number between <u>0 and 1</u> are more than 1, and <u>increases</u> as the exponent decreases, i.e., as it becomes more negative:

If $0 < a < 1$ ➔ $1 < a^{-1} < a^{-2} < a^{-3} < a^{-4}$...

Example: (Medium)

For $a = 0.3$
$a^{-1} = 0.3^{-1} = 3.33$
$a^{-2} = 0.3^{-2} = 11.11$
$a^{-3} = 0.3^{-3} = 37.04$
$a^{-4} = 0.3^{-4} = 123.46$

$1 < 3.33 < 11.11 < 37.04 < 123.46$... ➔
$1 < 0.3^{-1} < 0.3^{-2} < 0.3^{-3} < 0.3^{-4}$...

Negative even integer powers of a number between <u>-1 and 0</u> are more than 1, and <u>increases</u> as the exponent decreases, i.e., as it becomes more negative:

$-1 < a < 0$ ➔ $1 < a^{-2} < a^{-4}$...
Note that even powers of negative numbers are positive.

Example: (Medium)

For $a = -0.3$
$a^{-2} = (-0.3)^{-2} = 11.11$
$a^{-4} = (-0.3)^{-4} = 123.46$

$1 < 11.11 < 123.46$... ➔ $1 < (-0.3)^{-2} < (-0.3)^{-4}$...

Negative odd integer powers of a number between <u>-1 and 0</u> are less than -1, and <u>decreases</u> as the exponent decreases, i.e., as it becomes more negative:

If $-1 < a < 0$ ➔ $-1 > a^{-1} > a^{-3}$...
Note that negative numbers decrease in value as they become more negative.

Example: (Medium)

For $a = -0.3$
$a^{-1} = (-0.3)^{-1} = -3.333$
$a^{-3} = (-0.3)^{-3} = -37.04$

$-1 > -3.333 > -37.04$...

Below table summarizes the facts explained above about the powers of numbers between -1 and 0 and 0 to 1.

	$-1 < a < 0$	$0 < a < 1$
$n > 0$, even	$a < a^n$	$a > a^n$
$n > 0$, odd	$a < a^n$	$a > a^n$
$n < 0$, even	$a^n > 1$	$a^n > 1$
$n < 0$, odd	$a^n < -1$	$a^n > 1$

Square root of a number between 0 and 1 is larger than itself:

If $0 < a < 1$ ➔ $a < \sqrt{a}$

Example: (Medium)

For $a = 0.49$, $\sqrt{a} = \sqrt{0.49} = 0.7$
$0.7 > 0.49$

Practice Exercises:

1. (Easy)
If $0 < a < b < 1$, which is bigger, a^2 or b^2?

2. (Medium)
If $-1 < a < b < 0$, which is bigger, a^2 or b^2?

3. (Medium)
If $-1 < a < b < 0$, which is bigger, a^2 or b^3?

4. (Hard)
If $-1 < a < 0$ and $0 < b < 1$, put the following numbers in order from the smallest to the biggest:

$-1, 0, a, b, 1, ab, (ab)^4, (ab)^7, (ab)^{-4}, (ab)^{-7}$

Answers:

1. b^2; **2.** a^2; **3.** a^2;
4. $(ab)^{-7}, -1, a, ab, (ab)^7, 0, (ab)^4, b, 1, (ab)^{-4}$

Divisibility

What is Divisibility?

If an integer, x, can be divided by another integer, y, without any remainder, x is divisible by y.

If an integer x is divisible by another integer, y, then x = ny, where n is an integer. Both "y" and "n" are called the "divisors" or the "factors" of x.

Divisibility of a Number

Every Number is Divisible by 1 and by Itself

When you divide a number by 1, you get the same number itself.
Example: (Easy) 7/1 = 7

When you divide a number by itself, you get 1.
Example: (Easy) 5/5 = 1 or 123/123 = 1

Divisibility by 2

All the even numbers and only the even numbers are divisible by 2. So if the units digit of a number is even, it is an even number and it is divisible by 2.
Examples:

1. (Easy) -122, 6, 1,556,002 are all divisible by 2.

2. (Easy) -101, 321, 1,000,001 are NOT divisible by 2.

Integers divisible by 2 can be expressed by 2n, where n is an integer.
Examples:

1. (Easy)
$28 = 2 \times 14 = 2n, n = 14$
$2 = 2 \times 1 = 2n, n = 1$
$0 = 2 \times 0 = 2n, n = 0$
$-50 = 2 \times (-25) = 2n, n = -25$

are all divisible by 2.

2. (Medium)
If n is divisible by 2, what is the remainder of $\frac{n-7}{2}$?

Solution:
n is divisible by 2 ➔ n = 2k, where k is an integer. ➔
$\frac{n-7}{2} = \frac{2k-7}{2} = \frac{2k}{2} - \frac{7}{2} = k - 3 - \frac{1}{2}$ ➔
The remainder is 1.

Divisibility by 3

If the addition of all the digits of a number is divisible by 3, the number itself is divisible by 3.

Examples:

1. (Easy)
-33, 165, 1999110 are all divisible by 3.
For these numbers, additions of digits are
3 + 3 = 6,
1 + 6 + 5 = 12 and
1 + 9 + 9 + 9 + 1 + 1 + 0 = 30, respectively.

2. (Easy)
32, 166, 1999 are not divisible by 3 because, additions of the digits of these numbers are not divisible by 3. The additions are 5, 13 and 28 respectively.

Integers divisible by 3 can be expressed by 3n, where n is an integer.

Examples:

1. (Easy)
$27 = 3 \times 9 = 3n, n = 9$
$3 = 3 \times 1 = 3n, n = 1$
$0 = 3 \times 0 = 3n, n = 0$
$-54 = 3 \times (-18) = 3n, n = -18$

are all divisible by 3.

2. (Medium)
n is divisible by 3. What is the remainder of $\frac{2n-8}{3}$?

Solution:
n is divisible by 3 ➔ n = 3k, where k is an integer. ➔
$\frac{2n-8}{3} = \frac{6k-8}{3} = \frac{6k}{3} - \frac{8}{3} = 2k - 2 - \frac{2}{3}$ ➔
The remainder is 2.

Divisibility by 4

If the last two digits of a number is divisible by 4, then the number itself is divisible by 4.

Examples:

1. (Easy)
-56, 100567712, 211104 are all divisible by 4, because the last two digits of these numbers, 56, 12 and 4, are all divisible by 4.

2. (Easy)
54, 38, 334 are not divisible by 4, because their last two digits, 54, 38 and 34, are not divisible by 4.

Integers divisible by 4 can be expressed by 4n, where n is an integer.

Any number divisible by 4 is also divisible by 2.

Examples:

1. (Easy)

$28 = 4 \times 7 = 4n, n = 7$

$4 = 4 \times 1 = 4n, n = 1$

$0 = 4 \times 0 = 4n, n = 0$

$-60 = 4 \times (-15) = 4n, n = -15$

are all divisible by 4.

2. (Medium)
n is divisible by 4. What is the remainder of $\frac{2n+2}{4}$?

Solution:

n is divisible by 4 ➔ n = 4k, where k is an integer. ➔

$$\frac{2n+2}{4} = \frac{8k+2}{4} = \frac{8k}{4} + \frac{2}{4} = 2k - \frac{2}{4} \text{ ➔}$$

The remainder is 2.

Divisibility by 5

If the units digit of an integer is 0 or 5, the integer is divisible by 5.

Example:

1. (Easy) 10, 35, 3335, 78960 are all divisible by 5.

Integers divisible by 5 can be expressed by 5n, where n is an integer.

Examples:

1. (Easy)

$25 = 5 \times 5 = 5n, n = 5$

$5 = 5 \times 1 = 5n, n = 1$

$0 = 5 \times 0 = 5n, n = 0$

$-60 = 5 \times (-12) = 5n, n = -12$

are all divisible by 5.

2. (Medium)
3n is divisible by 5. What is the remainder of $\frac{6n+3}{5}$?

Solution:

3n is divisible by 5 ➔
3n = 5k, where k is an integer. ➔ $n = \frac{5k}{3}$ ➔

$$\frac{6n+3}{5} = \frac{6\left(\frac{5k}{3}\right)+3}{5} = \frac{10k}{5} + \frac{3}{5} = 2k + \frac{3}{5} \text{ ➔}$$

The remainder is 3.

Divisibility by 6

If an integer is divisible by both 2 and 3, it is divisible by 6.

Examples:

1. (Easy)
990, 186, 30012 are all divisible by 6, because they are even and the additions of their digits are divisible by 3.

2. (Easy)
333, 184, 30021 are not divisible by 6, because 333 and 30021 are not divisible by 2, and 184 is not divisible by 3.

Integers divisible by 6 can be expressed by 6n, where n is an integer.

Examples:

1. (Easy)

$24 = 6 \times 4 = 6n, n = 4$

$6 = 6 \times 1 = 6n, n = 1$

$0 = 6 \times 0 = 6n, n = 0$

$-60 = 6 \times (-10) = 6n, n = -10$

are all divisible by 6.

2. (Medium)
n is an even number and m = 123. What is the remainder of
$\frac{n \cdot m + 12}{6}$?

Solution:

n is an even number. ➔ n = 2k, where k is an integer.
m = 123 is divisible by 3: m = 3 × 41 ➔

$$\frac{n \cdot m + 12}{6} = \frac{2k \times 3 \times 41 + 12}{6} =$$

$$\frac{6 \times 41k}{6} + \frac{12}{6} = 41k + 2 \text{ ➔}$$

The remainder is 0.

Divisibility by 7

There is no easy rule to find out if a number is divisible by 7. You need to divide the number by 7 and see for yourself.

Integers divisible by 7 can be expressed by 7n, where n is an integer.

Example:

1. (Easy)

$28 = 7 \times 4 = 7n, n = 4$

$7 = 7 \times 1 = 7n, n = 1$

$0 = 7 \times 0 = 7n, n = 0$

$-56 = 7 \times (-8) = 7n, n = -8$

are all divisible by 7.

Divisibility by 8

If the last 3 digits of an integer is divisible by 8, the integer is divisible by 8.

Example:

1. (Easy)
2345832 is divisible by 8 because 832 is divisible by 8.

Integers divisible by 8 can be expressed by 8n, where n is an integer.

Any number divisible by 8 is also divisible by both 2 and 4.

Any number divisible by both 2 and 4 is also divisible by 8.

Examples:

1. (Easy)
$32 = 8 \times 4 = 8n, n = 4$
$8 = 8 \times 1 = 8n, n = 1$
$0 = 8 \times 0 = 8n, n = 0$
$-856 = 8 \times (-107) = 8n, n = -107$

are all divisible by 8.

2. (Easy)
Let m = 11172 and n = 4589028. Is nm divisible by 8.

Solution:
m is even, hence divisible by 2.
The last two digits, 28, of n is divisible by 4, hence n is divisible by 4.
Therefore mn is divisible by $2 \times 4 = 8$

Divisibility by 9

If the addition of the digits of an integer is divisible by 9, the integer itself is divisible by 9.

Example:

1. (Easy)
2709, 1881, 49572 are all divisible by 9.
For these numbers the addition of the digits are
$2 + 7 + 0 + 9 = 18$
$1+8+8+1 = 18$ and
$4 + 9 + 5 + 7 + 2 = 27$, respectively.

Integers divisible by 9 can be expressed by 9n, where n is an integer.

Any number divisible by 9 is also divisible by 3.

Example:

1. (Easy)
$27 = 9 \times 3 = 9n, n = 3$
$9 = 9 \times 1 = 9n, n = 1$
$0 = 9 \times 0 = 9n, n = 0$
$-72 = 9 \times (-8) = 9n, n = -8$

are all divisible by 9.

Divisibility by 10

An integer is divisible by 10 if the units digit of the integer is 0.

Example:

1. (Easy)
220, 17490. 890 are all divisible by 10.

Integers divisible by 10 can be expressed by 10n, where n is an integer.

Any number divisible by 10 is also divisible by both 2 and 5.

Example:

1. (Easy)
$20 = 10 \times 2 = 10n, n = 2,$
$10 = 10 \times 1 = 10n, n = 1,$
$0 = 10 \times 0 = 10n, n = 0,$
$-50 = 10 \times (-5) = 10n, n = -5$

are all divisible by 10.

Prime Numbers

If a positive integer is divisible only by 1 and itself it is a **prime number**. A prime number always has two distinct factors, 1 and itself. Note that 1 is not a prime number because 1 has only one factor.

First 10 prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Divisibility by Multiple Integers

If an integer, n, is divisible by both m and k, and if m and k are not divisible by each other, n is also divisible by $m \times k$ as long as $m \cdot k \le n$

Example:

1. (Easy)
132 is divisible by both 2 and 3. Also 2 and 3 are not divisible with each other. Hence, 132 is divisible by $2 \times 3 = 6$

If an integer, s, divisible by another integer, k, it is also divisible by all the factors of k.

Example:

1. (Easy)
1880 is divisible by 10. The factors of 10 are 2 and 5. 1880 is also divisible by both 2 and 5.

Divisibility of Multiplication

Multiple of 2 integers is divisible by:

a. All the numbers that each individual integer is divisible by.

b. The multiplication of each of the divisors as long as the multiplication of the divisors does not exceed the multiple of the original integers.

Examples:

1. (Hard)
Let $m = n \times s$. m is divisible by n, s, 1, $n \times s$, all the numbers that n is divisible by and all the numbers that s is divisible by.

2. (Hard)
$1476 = 12 \times 123$

1476 is divisible by:

- **a.** 12, 123
- **b.** 1, 1476 (All integers are divisible by 1 and themselves.)
- **c.** 2, 3, 4, 6 (Because 12 is divisible by these numbers)
- **d.** 3, 41 (Because 123 is divisible by 3 and 41)
- **e.** $2 \times 3 = 6$ (already included in c)
  $3 \times 3 = 9$
  $4 \times 3 = 12$ (already included in a)
  $6 \times 3 = 18$
  $2 \times 41 = 82$
  $3 \times 41 = 123$ (already included in a)
  $4 \times 41 = 164$
  $6 \times 41 = 246$
  because these are obtained by multiplying the factors of 12 (2, 3, 4, 6) and the factors of 123 (3, 41)

If an integer is a power of another integer, n = s^k**, it is divisible by:**

- **a.** 1
- **b.** All the positive powers (up to k) of s: ($s, s^2, s^3, \ldots s^k$)
- **c.** All the numbers that s is divisible by, i.e., factors of s.
- **d.** All the powers and multiples of the factors of s as long as the total power of these multiples do not exceed k.

Examples:

1. (Easy)

$9 = 3^2$

9 is divisible by 1, 3, and $3^2 = 9$

2. (Medium)
7 is a prime number and it is divisible by 1 and 7 only.

$2401 = 7^4$ is divisible by

1, 7, $7^2 = 49$, $7^3 = 343$ and $7^4 = 2401$

3. (Hard)

$216 = 6^3$

216 is divisible by (1 and the powers of 6):
1, 6, 36, 216
and the factors of 6 between 1 and 6:
2, 3
and the powers of the factors of 6:

$2^2 = 4$, $2^3 = 8$, $3^2 = 9$, $3^3 = 27$

and the multiples of the powers of the factors of 6:

$2 \times 3 = 6$ (already included)
$2^2 \times 3 = 12$
$2^3 \times 3 = 18$
$2 \times 3^2 = 18$
$2^2 \times 3^2 = 36$ (already included)
$2 \times 3^3 = 54$
$2^3 \times 3^2 = 72$
$2^2 \times 3^3 = 108$
$2^3 \times 3^3 = 216$ (already included)

Practice Exercises:

1. (Easy)
If n is divisible by 3, and, m is divisible by 2,
$n \cdot m$ is divisible by which of the following:

(A) 1
(B) 2
(C) 3
(D) 6
(E) All of the above

2. (Medium)
n is divisible by 3, and, m is divisible by 2. Is n + m divisible by 5?

3. (Medium)
If a is divisible by 3, what is the remainder of (a + 2)/3?

4. (Medium)
If a is divisible by 3, what is the remainder of (a + 3)/3?

5. (Medium)
If a is divisible by 7, and k is an integer.
What is the remainder of $\frac{a + 7k}{7}$?

6. (Medium)
If n is divisible by 3, m is divisible by 2. Can
$n \cdot m + 11$ be divisible by 2?

Hint: Since m is divisible by 2, it is even. Hence n.m is even.

7. (Hard)
Which of the following can possibly be a prime number?

- **(A)** Multiplication of two positive consecutive integers.
- **(B)** 15958725
- **(C)** Square root of a number divisible by 9.
- **(D)** 1284692
- **(E)** Addition of two positive consecutive integers.

Answers:
1. (E); **2.** No, not always.; **3.** 2; **4.** 0; **5.** 0; **6.** No; **7.** (E)

Even & Odd Numbers

There are a few SAT questions involving even and odd numbers. We already covered some of them in the Divisibility section. Here is what you also need to know about them:

Zero is an even number.

All integers with even unit digits are even.
Example: (Easy) 3889374 is even, because 4 is even.

All integers with odd unit digits are odd.
Example: (Easy) -1889373 is odd, because 3 is odd.

All even numbers can be written as 2n, where n is an integer.
Example: (Easy) $6 = 2 \times 3$

All odd numbers can be written as 2n + 1, where n is an integer.
Example: (Easy) $7 = 2 \times 3 + 1$

Even + Even = Even
Example: (Easy) 6 + 8 = 14 is even.

Even – Even = Even
Example: (Easy) 6 - 8 = -2 is even.

Odd – Odd = Even
Example: (Easy) 7 - 9 = -2 is even.

Odd + Odd = Even
Example: (Easy) 7 + 9 = 16 is even.

Even + Odd = Odd
Example: (Easy) 6 + 7 = 13 is odd.

Even – Odd = Odd
Example: (Easy) 6 - 7 = -1 is odd.

Even x *Even = Even*
Example: (Easy) $8 \times 128 = 1024$ is even.

Even x *Odd = Even*
Example: (Easy) $8 \times 127 = 1016$ is even.

Odd x *Odd = Odd*
Example: (Easy) $9 \times 127 = 1143$ is odd.

All the positive integer powers of even numbers are even.
Example: (Easy) $2^3 = 8$ which is even, because 2 is even and 3 is positive.

Negative powers of integers are not integers, so they can not be even or odd.
Example: (Easy) $2^{-3} = \frac{1}{8}$

All the non-negative integer powers of odd numbers are odd.
Example: (Medium) $3^5 = 243$, $3^4 = 81$, and $3^0 = 1$ are all odd, because the base, 3, is odd and the powers, 5, 4 and 0 are non-negative.

One of the 2 consecutive integers is even and the other one is odd.
Example: (Easy) 33 is odd and 34 is even.

Multiplication of two or more consecutive integers is always even because one of these integers is even.
Example: (Easy) $3 \times 4 \times 5 = 60$ is even, because 4 is even.

In a series of multiplications, if there is an even number, the result is even.
Example: (Medium) $7 \times 5^3 \times 2 \times 27 \times 89$ is even, because 2 is even.

Absolute Value

Before you take new SAT, you should be familiar with both the concept and the notation of absolute value. In this section, we will only introduce the concept and simple applications. In Chapter 7, Algebra, more complex applications of absolute value will be discussed.

Definition:

Absolute value operation makes all the negative numbers positive. In this operation, only the sign of the number changes from negative to positive.

Absolute value of non-negative numbers equal to themselves.

The absolute value of the number x is denoted by **|x|**.

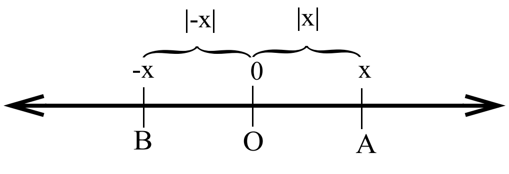
Below the same information is itemized and expressed in mathematical notation:

- The absolute value of zero is zero: $|0| = 0$
- The absolute value of non-negative numbers are equal to themselves: $|x| = x$ for $x \geq 0$
  For example $|12| = 12$
- The absolute value of negative numbers are equal to the negative of the numbers: $|x| = -x$ for $x < 0$.
  For example $|-12| = -(-12) = 12$
- The absolute value of a number are equal to the absolute value of the negative of the number:
  $|q| = |-q|$
  For example $|15| = |-15| = 15$

Another Definition of Absolute Value as Distance:

Let A, with it's coordinate x, be a point on the number line as shown in the below figure. The absolute value of the number x is the distance between point A and the origin, O.

If point B's coordinate is -x, then the absolute value of -x is the distance between point B and the origin, O.



Since distance is always positive, in this definition, you can see that absolute value operation flips the sign of the negative numbers to positive.

For example:
If A and B are two points with coordinates 5 and -5 on a number line, then the distances of point A and point B from the origin is equal to $|5| = 5$ and $|-5| = 5$, respectively.

Absolute value of multiplication:
$|a \cdot b| = |a| \cdot |b|$

Example: (Easy)

$$|2 \cdot (-3)| = |2| \cdot |-3| = 2 \cdot 3 = 6$$

Absolute value of division:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \ (b \neq 0)$$

Example: (Easy)

$$\left|\frac{-2}{-3}\right| = \frac{|-2|}{|-3|} = \frac{2}{3}$$

Absolute value of powers:
$|a^n| = |a|^n$

Examples:

1. (Easy)
$|3^2| = |3|^2 = 9$

2. (Easy)
$|-2^3| = |-2|^3 = 2^3 = 8$

Absolute value of addition:
$|a + b| \leq |a| + |b|$

Examples:

1. (Medium)
$|4 + 5| = |9| = 9 = |4| + |5|$

2. (Medium)
$|4 + (- 5)| = |-1| = 1 < |4| + |-5| = 9$

Order of Operations with Absolute Value

The absolute value is like a parenthesis. Therefore calculate the expressions inside of the absolute value sign, starting with the innermost absolute value sign, in the order given in previous sections. Then, remove the absolute value signs according to the rules given above.

Note: If there are parentheses in the expression, calculate them first.

Examples:

1. (Easy)

$$\left|\left|\frac{2^3 \times 5}{10}\right| - |-7| + 8\right| = \left|\frac{8 \times 5}{10} - 7 + 8\right| = ?$$

Solution:

$$\left|\left|\frac{2^3 \times 5}{10}\right| - |-7| + 8\right| = \left|\frac{8 \times 5}{10} - 7 + 8\right| =$$

$$\left|\left|\frac{40}{10}\right| - 7 + 8\right| = |4 - 7 + 8| = |12 - 7| = 5$$

2. (Medium)

Evaluate $\left|\left(\frac{1}{4}\right)^{-2} - (3+2)\times(12-9)^2 - 18\right|$

Solution:

$$\left|\left(\frac{1}{4}\right)^{-2} - (3+2)\times(12-9)^2 - 18\right| =$$

$$\left|\left(\frac{1}{4}\right)^{-2} - 5\times 3^2 - 18\right| = |16 - 5\times 9 - 18| =$$

$$|16 - 45 - 18| = |-47| = 47$$

3. (Medium) Don’t use your calculator.

Evaluate $\left|\frac{\left|-2^5\times 6^3\right|}{(5-1)^2\times 8}\right|$.

Solution:

First you need to express all the terms as of powers of 2. Here is how:

$$6^3 = (2\times 3)^3 = 2^3\times 3^3$$

$$(5-1)^2 = 4^2 = (2^2)^2 = 2^4$$

$$8 = 2^3$$

Then substitute these terms into the expression:

$$\left|\frac{\left|(-2)^5\times 6^3\right|}{(5-1)^2\times 8}\right| = \left|\frac{\left|(-2)^5\times 2^3\times 3^3\right|}{2^4\times 2^3}\right| =$$

$$\frac{|(-2)|^5\times|2|^3\times|3|^3}{|2|^4\times|2|^3} = \frac{2^5\times 2^3\times 3^3}{2^4\times 2^3}$$

$$2^{(5+3-4-3)}\times 3^3 = 2\times 3^3 = 54$$

Practice Exercises:

1. (Easy) $|7| = ?$

2. (Easy) $|-7.1| = ?$

3. (Easy) $|-7/2| = ?$

4. (Easy) $|(-7)^2| = ?$

5. (Easy) $|(-3)^3| = ?$

6. (Easy) $|-7|^2 - |(-3)^3| = ?$

7. (Easy) $\left|\sqrt{25}\right| - \sqrt{|-25|} = ?$

8. (Medium) $\frac{|3\cdot(-8)|^{-1}}{2^{-3}} - \frac{2}{3} = ?$

Answers:

1. 7; **2.** 7.1; **3.** 3.5; **4.** 49; **5.** 27; **6.** 22; **7.** 0; **8.** -1/3

Exercises

Basic Arithmetic - Addition, Subtraction, Multiplication, Division

1. (Easy)
Kathy is 8 years old. Her brother, John, is one year younger than half of her age. How old is John?

2. (Easy)
Julia's daily allowance is always one-quarter of her age. For example, at age 8, she was getting $2 every day. If she is 12 years old, what is her weekly income?

3. (Easy)
If m = 320/12, what is 1/m?

(A) 0.03
(B) 0.04
(C) 2/80
(D) 3/80
(E) 3/90

4. (Easy)
If m = 320/12, what is $m \times \frac{12}{320}$?

5. (Medium)
David drives from home to school. He travels at 50 miles/hr. for half an hour, then increases his speed to 60 miles/hr. for 15 minutes and reaches the school. He then starts his return trip home by using the same route. For 10 minutes he travels at the speed of 30 miles/hr. At the end of 10 minutes, how many more miles does he need to travel to reach home?

6. (Medium)
Jack studied for the SAT exam for 4 consecutive days. First day he studied for one hour and doubled the study time each day. How many total hours did he study at the end of the fourth day?

Decimals, Fractions, Ratios and Percentages

Decimals, Fractions and Ratios

1. (Easy)
Which of the following has three equal numbers?

(A) 1/5, 0.5, 3/15
(B) 1/5, 0.5, 2/10
(C) 1/3, 0.3, 3/9
(D) 3/2, 6/4, 1.5
(E) 4/2, 2/4, 2

2. (Easy)
4/5 of the students in a class are taking the music class. If there are 25 students in the class, how many students are NOT taking the music class?

3. (Easy)
Exactly 1/8 of all the books in Mary's home belong to her. Which of the following could possibly be the total number of books in her home?

(A) 40
(B) 45
(C) 50
(D) 55
(E) 60

4. (Easy)
x, y and x/y are positive integers. Which of the following is <u>never</u> correct?

I. x > y
II. y > x
III. x = y

(A) I only
(B) II only
(C) I and III
(D) II and III
(E) None

Hint: If x is less than y, then x/y is always less than one, and it is not an integer.

5. (Easy)
x/y is 1/10 th of z/y. What is x/z?

- **(A)** 1/10
- **(B)** 1
- **(C)** 1/100
- **(D)** 100
- **(E)** 10

6. (Easy)
It takes 2 lbs. of tomatoes to make spaghetti sauce for 6. How many pounds of tomatoes are required to make spaghetti sauce for 18 people?

7. (Easy)
Sue has 36 books. 1/4 of her books are about history. If the ratio of her math books to history books is 1/3, how many math books does she have?

8. (Medium)
If 2/3 rd of $x/2$ is 5, what is 1/5 th of $3x$?

- **(A)** 1
- **(B)** 3
- **(C)** 5
- **(D)** 7
- **(E)** 9

9. (Medium)
Three of Don's shirts are white and one of his shirts is blue. Half of his remaining shirts are green. If his white shirts are 3/10 of all his shirts, how many green shirts does he have?

Hint: Calculate the total number of shirts first.

10. (Medium)
On a street, 1/3 rd of all the houses are two-story houses. Among them, 4/5 th is older than 20 years. What is the ratio of the 2-story houses that are 20 years or younger to the total number of houses?

11. (Medium)
On a street, 1/3 rd of all the houses are 2-story houses. Among them, 4/5 th are older than 20 years. What is the ratio of 2-story houses that are 20 years or younger to the homes which are not 2-story?

12. (Hard)
In a 25 player soccer team, the ratio of freshmen to seniors is 2 to 5, and the ratio of juniors to seniors is 3 to 5. If there are 4 freshmen in the team, how many players are sophomores?

13. (Hard)
Kate eats 3 meals a day. At breakfast, she eats 3/8 th of the total daily calories. For lunch she eats 50 calories less than 3/4 th of what she eats for breakfast. For dinner, she eats 710 calories. How much is her total daily calorie intake?

Hint: Let her daily calorie intake be x. Calculate the dinner calories in terms of x.

Percentages

1. (Easy)
88% of the students attend a field trip. If there are 50 students in the school, how many are in the field trip?

2. (Easy)
I spend \$2.00 for lunch. You spend \$3.00 for lunch. You spend what percent of the amount that I spend?

3. (Easy)
Joe gave 15% of his money to his friend. If he gave \$12, how much money he has left with?

Hint: 15% of his money is \$12.

4. (Easy)
Joe makes 20% less than Karen. If Joe makes \$4000 a month, how much does Karen make?

5. (Medium)
For the last two consecutive years, Susan's income increased by 4% each year. If she is making \$60,000 a year now, what was her yearly income two years ago?

Hint: Assume that her income 2 years ago was x. Find an expression for today's income in terms of x. Then solve for x.

6. (Medium)
Bob paid \$46.50 in a restaurant for the food and the tip. How much is the tip if it is 15% of the cost of the food?

Hint: Food plus tip add up to 115% of the food price.

7. (Hard)
A store owner has 90% markup on all the items in the store. During a sale, all the items are sold with 20% discount. If he sells an item during the sale, what is the percentage markup?

Hint: Let the cost of an item to the owner be x. The normal sals price without the discount is 190% of the purchase price.

8. (Hard)
Monthly electric bill of a school during summer vacation is 40% of the electric bill of the winter months. If there are 2.5 months of summer vacation in a year and the total electric bill is \$10,000 for the whole year, what is the electric bill for each summer month?

Hint: Let the winter months bills be x. Find the total bill for the whole year in terms of x.

Fraction, Ratio, Percentage Mixes

1. (Easy)
m is 20% of n. What is n/m?

2. (Medium)
In a math class 1/6 of the students are English major, and the students with engineering major are 200% of the English major students. What fraction of the students has engineering major?

3. (Medium)
m/n = 7, n is what percent of m?

4. (Medium)
m/n = 7, m is what percent of n?

5. (Medium)
m is 15% of n, k is 10% of n. What is k/m?

6. (Medium)
m is 15% of n, k is 10% of n. k is what percent of m?

7. (Medium)
In a classroom, female/male ratio is 2/3. Females are what percent of all the students?

Hint: Out of 5 students, 2 are female and 3 are male.

8. (Hard)
m/n = 2/5, k/n = 1/3. k is what percent of m?

Hint: Find k/m

9. (Hard)
m/n = 2/5, k/n = 1/3. k is what percent of 6m?

Hint: Find k/6m

10. (Hard)
m is 15% of n, k is 10% of n. k is what percent of 10m?

11. (Hard)
m/n = 3. m is what percent of m + n?

Hint: Add 1 to n/m to find m/(m + n)

12. (Hard)
20% of Jane's outfits is black. 1/4 th of her remaining outfits is blue. Half of the rest of her outfits is white and the other half is red. Her red outfits is what percentage of her black outfits?

13. (Hard)
n is a positive number. x is obtained by increasing n 35% and decreasing the result by 35%. What is the ratio of n to x?

14. (Hard)
In a high school 5/9 th of the students are female. If 1/3 rd of all the female students are under 16, and at least half of the male students are under 16, what is the minimum percentage of the students under 16?

Powers and Square Root

1. (Easy)
If 2x + 2 = 0. What is 2^x ?

2. (Easy)
If $\sqrt{x} = 7$, what is x - 1?

3. (Easy)
If $x^2 = 64$, what is 2x + 1?

4. (Medium)
Which of the following statements is true for all positive values of x?

(A) $\sqrt{x} - \sqrt{2x} > 0$
(B) $\sqrt{x} - \sqrt{2x} < 0$
(C) $\sqrt{x} - \sqrt{2x} = 0$
(D) $\sqrt{x} - \sqrt{2x} > 1 - \sqrt{2}$
(E) $\sqrt{x} - \sqrt{2x} < 1 - \sqrt{2}$

5. (Medium)
If s and k are two positive integers, and $n = k^s$. Which of the following can <u>not</u> be n - 1?

I. -1
II. 0
III. 1

(A) I only
(B) II only
(C) III only
(D) I and II
(E) II and III

6. (Medium)
If c is an integer, and $r = (-1)^c + (1)^c$, what values r can take?

7. (Medium)
b is a non-zero integer. $\frac{b^{2a}}{b^3} = \frac{b^2}{b^{6a}}$, what is the value of a?

8. (Medium)
The volume of a cube with side length a is a^3. What is the volume of a cube with side length a/2?

9. (Medium)
If $a^{\frac{2}{3}} = a^{\frac{1}{2}}$, then a = ?

10. (Medium)
If $\overset{\underline{3}}{\sqrt[4]{a}} = 2a^{\frac{4}{3}} - 2^{\frac{1}{3}}$, then a = ?

(A) $\overset{\underline{1}}{\sqrt[4]{2}}$
(B) $\sqrt[4]{2}$
(C) $-\overset{\underline{1}}{\sqrt[4]{2}}$
(D) $\overset{\underline{-1}}{\sqrt[4]{2}}$
(E) $-\sqrt[4]{2}$

11. (Medium)
Which of the following expressions below is <u>not</u> equal to $a^{\left(\frac{3}{2}\right)}$?

(A) $\sqrt[3]{a}^{\,2}$

(B) $\sqrt[2]{a}^{\,3}$

(C) $\left(a^{\frac{1}{2}}\right)^3$

(D) $(a^3)^{\frac{1}{2}}$

(E) $\sqrt{\dfrac{1}{a^{-3}}}$

12. (Medium)
If x > 0, which of the following statements may possibly be true?

I. $x - \sqrt{x} > 0$
II. $x - \sqrt{x} < 0$
III. $x - \sqrt{x} = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

13. (Medium)
If x > 0, which of the following statements may possibly be true?

I. $\sqrt{x} - \sqrt{\frac{1}{x}} > 0$
II. $\sqrt{x} - \sqrt{\frac{1}{x}} < 0$
III. $\sqrt{x} - \sqrt{\frac{1}{x}} = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

14. (Medium)
If x > 0, which of the following statements must be true?

I. $(\sqrt{x})^2 - x > 0$
II. $(\sqrt{x})^2 - x < 0$
III. $(\sqrt{x})^2 - x = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

15. (Medium)
If x > 0, which of the following statements may possibly be true?

I. $(\sqrt{x})^2 - x^2 > 0$
II. $(\sqrt{x})^2 - x^2 < 0$
III. $(\sqrt{x})^2 - x^2 = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

16. (Medium)
If x < 0, which of the following statements must be true?

I. $(\sqrt{-x})^2 - x > 0$
II. $(\sqrt{-x})^2 - x < 0$
III. $(\sqrt{-x})^2 - x = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

17. (Medium)
$\sqrt{32x} = ?$

(A) $4\sqrt{8x}$
(B) $8\sqrt{4x}$
(C) $4x\sqrt{2}$
(D) $4\sqrt{2x}$
(E) $4\sqrt{4x}$

18. (Hard)
If x > 0, y < 0, and $\frac{x^6}{y^3} = \frac{3^3}{(3y)^{-3}}$, what is x/y?

19. (Hard)
b is a non-positive integer, and $t = (2)^b - 1$. What are the minimum and the maximum values for t?

20. (Hard)
If $(x^2 - 2x + 1)^{(\sqrt{8}x)} = 1$, then x = ?

21. (Hard)
If $a^{-\frac{3}{2}}$ is a negative real number, which of the following is true?

- **(A)** "a" is a positive real number.
- **(B)** "a" is a negative real number.
- **(C)** a = 0
- **(D)** "a" is a non-zero real number.
- **(E)** "a" can not be a real number.

22. (Hard)
-1 < x < 0. Put the following terms in order, from the smallest to the biggest:

x, $\sqrt{-x}$, $\sqrt{x^2}$, x^2

Negative Numbers

1. (Easy)

a. a = 3, b = -5. What is $(a - b)^2$?

b. a = 3, b = -5. What is $\frac{(a - b)^2}{b - a}$?

c. a = 3, b = 5. What is $\frac{(a - b)^2}{(b - a)^{-3}}$?

d. a = 3, b = 5. What is $(-a + b)^2 \cdot (a - b)^3$?

2. (Easy)

$x^2 = 4$. Which of the following may be x – 1?

- **(A)** –5
- **(B)** –4
- **(C)** –3
- **(D)** 2
- **(E)** None of the above.

3. (Easy)
If x – 13 is a negative odd integer, then x could be which of the following?

- **(A)** –5
- **(B)** 5
- **(C)** 10
- **(D)** 15
- **(E)** 20

4. (Medium)
x, y are non-zero integers and x/y > 1. x - y may be which of the following?

- **(A)** Positive only.
- **(B)** Negative only.
- **(C)** Zero only.
- **(D)** Non-negative only.
- **(E)** Any non-zero integer.

5. (Medium)
A, B, C and D are points on a number line. The coordinate of A is -5.5 and the coordinate of B is 7 more than A. C is at equal distance from A and B. The coordinate of D is 5 less than the coordinate of C. What is the distance between A and D?

6. (Medium)
x is a real number. Which of the following is always true?

- **(A)** $x^2 > x$
- **(B)** $\sqrt{x} < x$
- **(C)** x^2 is positive
- **(D)** x^2 is non-negative
- **(E)** All of the above.

7. (Medium)
$(x - y) \cdot (y - x)^2 \cdot (x - y)^3$ may be

I. negative
II. positive
III. zero

(A) I only.
(B) II only.
(C) I and III only.
(D) II and III only.
(E) I and II and III

8. (Medium)
$(y - x) \cdot (x - y)^3$ may be

I. negative
II. positive
III. zero

(A) I only.
(B) II only.
(C) I and III only.
(D) II and III only.
(E) I and II and III

9. (Medium)
$(y - x)^4 \div (x - y)^3$ may be

I. negative
II. positive
III. zero

(A) I only.
(B) II only.
(C) I and III only.
(D) II and III only.
(E) I and II and III

10. (Medium)
If $s^2 < c < d^3$, then cd may be:

(A) Positive only.
(B) Negative only.
(C) Zero only.
(D) Non-negative only.
(E) Any number, positive, negative or zero.

11. (Hard)
$1 < s^2 < c < d^3$. (scd - 1) can not be:

(A) Positive.
(B) Negative.
(C) Between -1 and 1. (-1 and 1 are included)
(D) Between -2 and 0. (-2 and 0 are included)
(E) Even.

12. (Hard)
If $a^2 > a$, which of the following must always be true?

I. $a > 1$
II. $a < -1$
III. $a^3 > a$

(A) I only
(B) II only
(C) III only
(D) I or II
(E) I and III

Hint: Consider the negative values of "a" as well as its positive values.

Numbers Between -1 and 1

1. (Medium)
If x is a positive number, which is smaller, x^2 or x^3?

2. (Medium)
If x is a negative number, which is smaller, x^2 or x^3?

3. (Medium)
If x is a positive number, which is smaller, x or $\sqrt{x}$?

4. (Medium)
x is an integer. Which of the following may be correct?

I. $x^2 > x$
II. $x^2 < x$
III. $x^2 = x$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) II and III

5. (Medium)
x is a non-negative integer. Which of the following may be correct?

I. $x > \sqrt{x}$
II. $x < \sqrt{x}$
III. $x = \sqrt{x}$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) II and III

6. (Medium)
x is an integer. Which of the following may be correct?

I. $x^3 > x^2$
II. $x^3 < x^2$
III. $x^3 = x^2$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I, II and III

7. (Hard)
If $-1 < x < 0$, $0 < y < 1$ and $z = xy + 1$, which of the following statements is true?

(A) $z < -1$
(B) $z = -1$
(C) $-1 < z < 0$
(D) $0 < z < 1$
(E) $z > 1$

8. (Hard)
If $-1 < x < 0$, rearrange the following terms from the smallest to the biggest: x, x^2, x^3, $x+2$, $1/x$.

Divisibility

1. (Medium)
If a is divisible by 6, what is the remainder of $(a + 12)/3$?

Hint: If a is divisible by 6, it is also divisible by 3.

2. (Medium)
n is divisible by 3, m is divisible by 2. Is $n \cdot m + 12$ divisible by 2?

Hint: Since m is divisible by 2, it is even. Hence n.m is even.

3. (Medium)
When a is divided by 4, the remainder is 3. What is the remainder of $(7a + 3)/4$?

Hint: $a = 4m + 3$, where m is an integer.

4. (Medium)
a is divisible by 4.
What is the remainder of $(3a^2 + 5)/16$?

Hint: If a is divisible by 4, a^2 is divisible by 16.

5. (Medium)
If a is a positive, even number, what is the remainder of $((a + 3)^5)/2$?

6. (Medium)
The remainder is 2 when an integer a is divided by 3. Which of the following numbers is divisible by 3?

I. 3a
II. a + 1
III. a + 3

(A) I only
(B) II only
(C) III only
(D) I & II
(E) I & III

Hint: The remainder of a/3 is 2 $\rightarrow$ $a/3 = n + 2/3$, where n is an integer.

7. (Hard)
a is the addition of 255 and a positive integer, k. What is the lowest possible value of k for a to be divisible by:

a. 4
b. 8
c. 9
d. 10

8. (Hard)
a is the addition of 255 and a 2-digit integer, k. What is the highest possible value of k for a to be divisible by:

a. 2

b. 3

c. 5

d. 6

e. 7

9. (Hard)
a and b are both positive and odd numbers, divisible by 3. $a + b + ab$ is divisible by:

I. 2
II. 3
III. 6

(A) I only
(B) II only
(C) III only
(D) I & II
(E) I & II & III

Hint: The addition of two odd numbers is even. The multiplication of two odd numbers is odd.

10. (Hard)
a is divisible by 4.
What is the remainder of $\frac{2(a+1)^2}{16}$?

Hint: If a is divisible by 4, a^2 and 4a are both divisible by 16.

11. (Hard)
If a is an even number, what is the remainder of $\frac{(a-3)^2}{4}$?

Hint: „a is even" means $a = 2n$, where n is an integer.

12. (Hard)
a is a positive, even integer, divisible by 3.
What is the minimum positive value of x, so that $(a+x)^2$ is divisible by 9 for all values of a?

Hint: „a is even and divisible by 3" means $a = 2 \times 3n$, where n is an integer.

13. (Hard)
If a is an even number, which of the following can never be true?

(A) a is divisible by 3.
(B) The remainder of a/4 is 1
(C) The remainder of a/4 is 2
(D) a is divisible by 7
(E) a^2 is divisible by 3.

14. (Hard)
a and b are both positive and odd numbers, divisible by 3. $a + b + ab + 3$ is divisible by:

I. 2
II. 3
III. 6

(A) I only
(B) II only
(C) III only
(D) I & II only
(E) I & II & III

Hint: $a = 3(2n+1)$, where n is a non-negative integer. $b = 3(2m+1)$, where m is an non-negative integer.

15. (Hard)
a, b, c and d are positive integers. $a + b + c = 18$ and $c = 2d$. If we form a 3-digit number, abc, with unit, tens, and hundreds digits being c, b, and a respectively, which of the following statements is <u>not</u> correct for all values of a, b, c and d?

(A) abc is divisible by 1
(B) abc is divisible by 2
(C) abc is divisible by 3
(D) abc is divisible by 4
(E) abc is divisible by 6

Even & Odd Numbers

1. (Easy)
If n is an integer, $6n - 7$ must be:

(A) Even.
(B) Odd.
(C) Prime.
(D) Divisible by 6.
(E) Divisible by 7.

2. (Easy)
In a set of integers from 1 to 10,

a. How many of them are both prime and even?

b. How many of them are both prime and odd?

3. (Medium)
If n^4 is even, then n must be:

- **(A)** Positive and even.
- **(B)** Positive and odd.
- **(C)** Negative and odd.
- **(D)** Odd.
- **(E)** Even.

4. (Medium)
If $5 - n^2 = 5$, n must be:

- **(A)** Even.
- **(B)** Odd.
- **(C)** Positive.
- **(D)** Negative.
- **(E)** None of the above.

5. (Medium)
Addition of the squares of two consecutive integers is:

- **(A)** Odd and negative.
- **(B)** Even and negative.
- **(C)** Odd and positive.
- **(D)** Even and positive.
- **(E)** Not enough information is provided to make a decision.

6. (Hard)
Which of the following numbers can be an integer added to its square?

- **(A)** –2
- **(B)** 1
- **(C)** 2
- **(D)** 9
- **(E)** 11

7. (Hard)
Which of the following can be the multiplication of 3 consecutive integers?

- **(A)** 137
- **(B)** 0
- **(C)** 1377
- **(D)** –2
- **(E)** 57

8. (Hard)
Which of the following can be the addition of 4 consecutive integers?

- **(A)** 877
- **(B)** –485
- **(C)** 1
- **(D)** 1453
- **(E)** 1686

Absolute Value

1. (Easy)
On a number line, p and q are the coordinates of two points. Which of the following is the (Distance of p from the origin) + (Distance of q from the origin)?

- **(A)** $|p + q|$
- **(B)** $|p| + |q|$
- **(C)** $|p - q|$
- **(D)** $|p| - |q|$
- **(E)** None of the above.

2. (Easy)
What is the value of x at which $|2x - 8|$ is at its minimum?

3. (Easy)
If $x = -5$, then $|x - |x - 4|| - |4 - x| = ?$

4. (Easy)
If $a < b < 0$, $|a + b| = ?$

5. (Easy)
If $a < b$, $|a - b| = ?$

6. (Medium)
If $a < b < 0$, $|3a - b| = ?$

7. (Medium)
If $x < 0$, then $|x - 3| - x^3 - (3x)^{-3}$ is

- **(A)** Negative
- **(B)** Positive
- **(C)** 0
- **(D)** Less than $|x - 3|$
- **(E)** Non-negative

8. (Medium)
On a number line, p and q are the coordinates of two points. Which of the following is the distance between p and q?

(A) (Distance of p from the origin) + (Distance of q from the origin)

(B) (Distance of p from the origin) - (Distance of q from the origin)

(C) $p - q$

(D) $|p - q|$

(E) $|p| - |q|$

9. (Medium)
On a number line, p and q are the coordinates of two points. Which of the following statements is not always true?

(A) $|p - q| \le |p| + |q|$

(B) $|p - q| \le |p| + |-q|$

(C) $|p - q| \le |p| - |q|$

(D) $|p - q|^2 \le (|p| + |q|)^2$

(E) None of the above.

Answers

Addition, Subtraction, Multiplication, Division

1. 3 years old
2. $21
3. (D)
4. 1
5. 35 miles
6. 15

Decimals, Fractions, Ratios and Percentages

Decimals, Fractions and Ratios

1. (D)
2. 5
3. (A)
4. (B)
5. (A)
6. 6 lbs.
7. 3
8. (E)
9. 3
10. 1/15
11. 1/10
12. 5
13. 1920 calories.

Percentages

1. 44
2. 150%
3. $68
4. $5000
5. $55,473.37
6. $6.07
7. 52%
8. $380.95/mnt

Fraction, Ratio, Percentage Mixes

1. 5
2. 1/3
3. 14.29%
4. 700%
5. 2/3
6. 66.7%
7. 40%
8. 83.3%
9. 13.9%
10. 6.7%
11. 75%
12. 150%
13. 1.14
14. 40.74%

Powers and Square Root

1. 1/2 or 0.5
2. 48
3. 17 or -15
4. (B)
5. (A)
6. 0 or 2
7. 5/8
8. $\frac{a^3}{8}$
9. 0 or 1
10. (B)
11. (B)
12. (E)
13. (E)
14. (C)
15. (E)
16. (A)
17. (D)
18. -3
19. minimum = -1, maximum = 0
20. 0 or 2
21. (A)
22. $x, x^2, \sqrt{x^2}, \sqrt{-x}$

Negative Numbers

1. **a.** 64; **b.** –8; **c.** 32; **d.** –32
2. (C)
3. (C)
4. (E)
5. 1.5
6. (D)
7. (D)
8. (C)
9. (E)
10. (A)
11. (D)
12. (D)

Numbers Between -1 and 1

1. Not enough information.
2. x^3
3. Not enough information.
4. (D)
5. (D)
6. (E)
7. (D)
8. $1/x, x, x^3, x^2, x + 2$

Divisibility

1. 0
2. Yes.
3. 0
4. 5
5. 1
6. (D)
7. **a.** 1; **b.** 1; **c.** 6; **d.** 5
8. **a.** 99; **b.** 99; **c.** 95; **d.** 99; **e.** 95
9. (B)
10. 2
11. 1
12. 3
13. (C)
14. (E)
15. (D)

Even and Odd Numbers

1. (B)
2. **a.** 1; **b.** 3
3. (E)
4. (A)
5. (C)
6. (C)
7. (B)
8. (E)

Absolute Value

1. (B)
2. 4
3. 5
4. -a - b
5. b - a
6. -3a + b
7. (B)
8. (D)
9. (C)

Solutions

Basic Arithmetic - Addition, Subtraction, Multiplication, Division

1. Answer: 3 years old

John's age = $\frac{\text{Kathy's Age}}{2} - 1 = \frac{8}{2} - 1 = 3$

2. Answer: $21

Julia's daily income at age 12 = 12/4 = $3

Her weekly income = 3 x 7 = $21

3. Answer: (D)

$m = \frac{320}{12}$ ➔ $\frac{1}{m} = \frac{12}{320} = \frac{12 \div 4}{320 \div 4} = \frac{3}{80}$

The answer is (D).

4. Answer: 1

$m = \frac{320}{12}$ ➔ $m \times \frac{12}{320} = \frac{320}{12} \times \frac{12}{320} = 1$

5. Answer: 35 miles

David's first part of the travel = $50 \times \frac{1}{2} = 25$ miles.

David's second part of the travel = $60 \times \frac{15}{60} = 15$ miles.

Note that 15 minutes = 15/60 hour.

The distance between his home and school is 25 + 15 = 40 miles.

David's last part of the travel on his return trip is $30 \times \frac{10}{60} = 5$ miles.

Therefore his distance from home at the end is 40 - 5 = 35 miles.

6. Answer: 15 hours

Jack studied:

First day = 1 hour.

Second day = 1 x 2 = 2 hours.

Third day = 2 x 2 = 4 hours.

Fourth day = 4 x 2 = 8 hours.

Total = 1 + 2 + 4 + 8 = 15 hours.

Decimals, Fractions, Ratios, Percentages

Decimals, Fractions and Ratios

1. Answer: (D)

Let's consider each case one by one.

(A) $\frac{1}{5} = 0.2 \neq 0.5$ (Not the answer)

(B) $\frac{1}{5} = 0.2 \neq 0.5$ (Not the answer)

(C) $0.3 = \frac{3}{10} \neq \frac{3}{9}$ (Not the answer)

(D) $\frac{3}{2} = \frac{3 \times 2}{2 \times 2} = \frac{6}{4} = 1.5$

The answer is (D). If necessary, use your calculator to verify this result.

2. Answer: 5

$\frac{4}{5} \times 25 = 20$ students are taking the music class.

Therefore 25 - 20 = 5 students are not taking it.

3. Answer: (A)

Since Mary has exactly 1/8 of the books, the total number of books has to be divisible by 8. Among all the answers, only 40 is divisible by 8. Hence the answer is (A).

4. Answer: (B)

If y > x, x/y is less than 1. So it can never be an integer.

Hence the answer is (B).

5. Answer: (A)

x/y is 1/10 th of z/y ➔ $\frac{x}{y} = \frac{1}{10} \cdot \frac{z}{y}$ ➔ $x = \frac{z}{10}$ ➔ $\frac{x}{z} = \frac{1}{10}$

The answer is (A).

6. Answer: 6 lbs.

$\frac{2}{6} \times 18 = 6$ lbs. of tomatoes are required to make spaghetti for 18 people.

7. Answer: 3

History books: 36/4 = 9

Math books: $(1/3) \times 9 = 3$

8. Answer: (E)

$\frac{2}{3} \times \frac{x}{2} = 5$ ➔ $x = 15$ ➔ $\frac{1}{5} \times 3 \times 15 = 9$

The answer is (E).

9. Answer: 3

Let x be the total number of shirts.

Don has 3 white shirts and they are $3/10$ all of his shirts. ➔ The number of white shirts is

$\frac{3}{10} \cdot x = 3$ ➔ $x = 10$

Number of non-blue and non-white shirts = 10 – 3 – 1 = 6

Green shirts are one-half of the remaining shirts.

Hence Green shirts = 6/2 = 3

10. Answer: 1/15

(5/5) - (4/5) = 1/5 th of two-story houses are 20 years or younger.

Since the 2-story houses are only 1/3 of all the houses, first you need to find 1/5 th of 1/3 rd to calculate the ratio of the two-story homes that are less than 20 years old to the total number of homes.

It is $\frac{1}{5}\cdot\frac{1}{3} = \frac{1}{15}$

11. Answer: 1/10

(5/5) - (4/5) = 1/5 th of 2-story houses are 20 years or younger.

Since the 2-story houses are only 1/3 rd of all the houses, first you need to find 1/5 th of 1/3 rd to calculate the ratio of the two-story homes that are less than 20 years old to the total number of homes.

As you found in the previous question, it is

$$\frac{1}{5}\cdot\frac{1}{3} = \frac{1}{15}$$

Since 1/3 of all the houses are two-story,
(3/3) - (1/3) = 2/3 of all the houses do not have two stories.

So the ratio of the two-story houses that are 20 years or younger to the homes which are not 2-story is $\frac{1}{15} \div \frac{2}{3} = \frac{1}{10}$

12. Answer: 5

Let f, j and s be the number of freshman, junior and senior students in the team. You can formulate the information given in the question as follows:

$$\left.\begin{array}{l} \frac{f}{s} = \frac{2}{5} \\ \frac{j}{s} = \frac{3}{5} \\ f = 4 \end{array}\right\} \Rightarrow \begin{array}{ll} \frac{f}{s} = \frac{4}{s} = \frac{2}{5} & \Rightarrow s = 10 \\ \frac{j}{s} = \frac{j}{10} = \frac{3}{5} & \Rightarrow j = 6 \end{array}$$

sophomores: 25 - 10 - 6 - 4 = 5

13. Answer: 1920 calories

Let x be is the total daily calorie intake.

Breakfast calories = 3x/8

Lunch calories =

$$\text{Breakfast calories} \times \frac{3}{4} - 50 = \left(\frac{3x}{8}\right)\cdot\frac{3}{4} - 50 = \frac{9x}{32} - 50$$

Dinner calories = Total calories - Breakfast calories - Lunch calories =

$$x - \frac{3x}{8} - \frac{9x}{32} + 50 = x\cdot\left(1 - \frac{21}{32}\right) + 50 =$$

$$\frac{11x}{32} + 50 = 710 \Rightarrow x = 660 \times \frac{32}{11} = 1920 \text{ calories}$$

Percentages

1. Answer: 44

88% of 50 = $\frac{88 \times 50}{100} = 44$

2. Answer: 150%

$$\frac{3 \times 100}{2} = 150\%$$

3. Answer: \$68

Let m be the Joe's original amount of money.

The amount that he gives to his friend:

$$\frac{m \times 15}{100} = 12 \Rightarrow m = \frac{12 \times 100}{15} = \$80$$

What is left for him is 80 – 12 = \$68

4. Answer: \$5000

Let J and K be the Joe's and Karen's monthly income, respectively.

$$J = K - \frac{20}{100}K = \frac{100 - 20}{100}K = \frac{4}{5}K = 4000 \Rightarrow$$

K = \$5000

5. Answer: \$55,473.37

Let x and y be Susan's income two years ago and a year ago, respectively.

After the 1st year, her income =
y = x + (4x)/100 = 1.04x.

This year's income = y + (4y)/100 =

$$1.04x + \frac{4 \times 1.04x}{100} = (1.04 + 0.0416)x =$$

1.0816x = 60000 ➔
x = 60000/1.081 = \$55,473.37

6. Answer: \$6.07

Let F and T be the money paid for the food and the tip respectively.

T + F = 46.50 ➔ F = 46.50 - T

Since T is 15% of F,

$$T = F\cdot\frac{15}{100} = (46.50 - T)\cdot\frac{15}{100} \Rightarrow$$

$$T + \frac{15}{100}T = \frac{115}{100}T = 46.50\cdot\frac{15}{100} \Rightarrow$$

$$T = 46.50\cdot\frac{15}{115} = \$6.07$$

7. Answer: 52%

Let x be the cost of an item to the store owner.

Sales price without the discount is $\frac{190x}{100} = 1.9x$

Sales price with 20% discount is 100 - 20 = 80% of 1.9x, which is $\frac{1.9 \times 80x}{100} = 1.52x$

The dollar amount of the markup is 1.52x – x.
Now the question is "The markup, 1.52x – x, is what percent of purchase price, x?".

It is $\frac{(1.52x - x)\cdot 100}{x} = 52\,\%$

8. Answer: \$380.95/mnt.

Number of winter months = 12 - 2.5 = 9.5 months

Let x be the winter month's bill. Then, total bill = (Number of winter months) x (monthly winter bill) + (Number of summer months) x (monthly summer bill) =

$$9.5x + \frac{2.5\cdot 40x}{100} = 10.5x = 10000 \Rightarrow$$

Monthly winter bill = 10000/10.5 = \$952.38

Monthly summer bill = $\frac{952.38 \times 40}{100} = \$380.95$

Fraction, Ratio, Percentage Mixes

1. Answer: 5
$m = (20n)/100 \rightarrow n/m = 100/20 = 5$

2. Answer: 1/3
Let s be the total number of students.
Number of English major students: $\frac{s}{6}$ ➔
Number of Engineering major students:
200% of 1/6 of students = $\frac{200}{100} \cdot \frac{s}{6} = \frac{2s}{6} = \frac{s}{3}$ ➔
1/3 rd of the students are Engineering major.

3. Answer: 14.29%
$\frac{m}{n} = 7 \rightarrow n = \frac{m}{7} = \frac{100m}{100 \cdot 7} = \frac{14.29m}{100} \rightarrow$
n is 14.29% of m.

4. Answer: 700%
$\frac{m}{n} = 7 \rightarrow m = 7n = \frac{100 \times 7n}{100} = \frac{700n}{100} \rightarrow$
m is 700% of n.

5. Answer: 2/3
$m = \frac{15n}{100}$ and $k = \frac{10n}{100} \rightarrow$
$\frac{k}{m} = \frac{10n}{100} \div \frac{15n}{100} = \frac{10n}{100} \cdot \frac{100}{15n} = \frac{10}{15} = \frac{2}{3}$

6. Answer: 66.7%2/3
$m = \frac{15n}{100}$ and $k = \frac{10n}{100} \rightarrow$
$\frac{k}{m} = \frac{10n}{100} \div \frac{15n}{100} = \frac{10n}{100} \cdot \frac{100}{15n} = \frac{10}{15} = \frac{2}{3} \rightarrow$
$k = \frac{2 \times 100m}{3}$ = 66.7% of m.

7. Answer: 40%
Female/Male ratio = 2/3 means out of every 5 students, 2 of them are female. ➔
female/total ratio is 2/5 ➔
Percentage of female students = $\frac{2 \times 100}{5} = 40\%$

8. Answer: 83.3%
$k/m = \frac{k}{n} \div \frac{m}{n} = \frac{1}{3} \div \frac{2}{5} = 5/6 \rightarrow$ k is $\frac{5 \times 100}{6} =$ 83.3% of m.

9. Answer: 13.9%
$k/m = \frac{k}{n} \div \frac{m}{n} = \frac{1}{3} \div \frac{2}{5} = 5/6 \rightarrow k/6m = 5/36 \rightarrow$
k is $\frac{5 \times 100}{36}$ = 13.9% of 6m.

10. Answer: 6.7%
$m = \frac{15n}{100}$ and $k = \frac{10n}{100} \rightarrow$
$\frac{k}{m} = \frac{10n}{100} \div \frac{15n}{100} = \frac{10}{15} = \frac{2}{3} \rightarrow$
$\frac{k}{10m} = \frac{2}{30} = \frac{1}{15} \rightarrow$ k is 100/15 = 6.7% of 10m.

11. Answer: 75%
$(n/m) + 1 = (m + n)/m = 1/3 + 1 = 4/3 \rightarrow$
$m/(m + n) = 3/4 \rightarrow$ m is $\frac{3 \times 100}{4}$ = 75% of m + n.

12. Answer: 150%
20% of Jane's outfits is black ➔
20/100 = 1/5 th of her outfits is black. ➔
The ratio of outfits that are not black =
5/5 – 1/5 = 4/5 ➔
1/4 th of 4/5 th of her outfits are blue ➔
Her blue outfits = $\frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$ th of her outfits. ➔
The fraction of her non-black and non-blue outfits = 5/5 – 1/5 – 1/5 = 3/5 ➔
The fraction of her red outfits = (3/5) / 2 = 3/10 ➔
The ratio of her red outfits to her black outfits =
$\frac{3}{10} \div \frac{1}{5} = \frac{3}{2} \rightarrow$
Jane's red outfits is $\frac{3 \times 100}{2}$ = 150% of her black outfits.

13. Answer: 1.14
Increase n by 35%:
Amount of increase = 35n/100 = 0.35n ➔
The value after the increase: n + 0.35n = 1.35n

Decrease 1.35n by 35%:
Amount of decrease: $\frac{1.35 \times 35 \times n}{100} = 0.4725n$
Value after decrease:
x = 1.35n - 0.4725n = 0.8775n ➔
n/x = 1/0.8775 = 1.14

14. Answer: 40.74%
Male students are $\frac{9}{9} - \frac{5}{9} = \frac{4}{9}$ th of the students.
Females under 16: $\frac{5}{9} \cdot \frac{1}{3} = \frac{5}{27}$
Males under 16: $\frac{4}{9} \cdot \frac{1}{2} = \frac{4}{18} = \frac{2}{9}$ or more.
Male and Female students under 16:
5/27 + 2/9 = 11/27 =
1100/27 percent = 40.74% or more.

Powers and Square Root

1. Answer: 1/2 or 0.5
$2x + 2 = 0 \rightarrow x = -1 \rightarrow 2^x = 2^{-1} = \frac{1}{2} = 0.5$

2. Answer: 48
$\sqrt{x} = 7 \rightarrow x = 7^2 = 49 \rightarrow x - 1 = 49 - 1 = 48$

3. Answer: 17 or -15
$x^2 = 64 \rightarrow x = 8$ or $x = -8$
$x = 8 \rightarrow 2x + 1 = 17$ and $x = -8 \rightarrow 2x + 1 = -15$

4. Answer: (B)
$\sqrt{x} - \sqrt{2x} = \sqrt{x}(1 - \sqrt{2}) = (1 - 1.414)\sqrt{x} = -0.1414\sqrt{x}$, which is always negative for all x > 0. The answer is (B).

5. Answer: (A)
k = 1 ➔ $n = 1^s = 1$ ➔ n - 1 = 0 (Case II)
$k \geq 2$ ➔ $n \geq 2$ ➔ $n - 1 \geq 1$ (Case III)

In fact, the minimum value of n is 1. So, n - 1 is never negative. The answer is (A).

6. Answer: 0 or 2
You need to consider 5 different values of c.
1) c = 0: then $r = (-1)^c + (1)^c = (-1)^0 + (1)^0 = 1 + 1 = 2$
2) c > 0 and odd: $r = (-1)^c + (1)^c = -1 + 1 = 0$
3) c > 0 and even: $r = (-1)^c + (1)^c = 1 + 1 = 2$
4) c < 0 and odd: $r = (-1)^c + (1)^c = -1 + 1 = 0$
5) c < 0 and even: $r = (-1)^c + (1)^c = 1 + 1 = 2$
The answer is 0 or 2.
Note that in this case, all you need to consider is even or odd values of c.
Cases 1, 3 and 5 give the same result because in all these 3 cases c is even.
Cases 2 and 4 give the same result because in both of these cases c is odd.

7. Answer: 5/8

$\frac{b^{2a}}{b^3} = \frac{b^2}{b^{6a}}$ ➔ $b^5 = b^{8a}$ ➔ 5 = 8a ➔ a = 5/8

8. Answer: $a^3/8$
The volume of a cube with side $\left(\frac{a}{2}\right)^3 = \frac{a^3}{2^3} = \frac{a^3}{8}$

9. Answer: 0 or 1
$a^{\frac{2}{3}} = a^{\frac{1}{2}}$ ➔ a = 0 or a = 1

10. Answer: (B)
$\sqrt[\frac{3}{4}]{a} = 2a^{\frac{4}{3}} - 2^{\frac{1}{3}}$ ➔ $a^{\frac{4}{3}} = 2a^{\frac{4}{3}} - 2^{\frac{1}{3}}$ ➔

$a^{\frac{4}{3}} = 2^{\frac{1}{3}}$ ➔ $(a^4)^{\frac{1}{3}} = 2^{\frac{1}{3}}$ ➔ $a^4 = 2$ ➔

$a = 2^{\frac{1}{4}} = \sqrt[4]{2}$

The answer is (B).

11. Answer: (B)
$\sqrt[\frac{3}{2}]{a} = a^{\frac{2}{3}} \neq a^{\frac{3}{2}}$

The answer is (B).

12. Answer: (E)
x > 1 ➔ $x - \sqrt{x} > 0$ Case I is true.
x < 1 ➔ $x - \sqrt{x} < 0$ Case II is true.
x = 1 ➔ $x - \sqrt{x} = 0$ Case III is true.
The answer is (E).

13. Answer: (E)

x > 1 ➔ $\sqrt{x} - (\sqrt{1/x} > 0)$ Case I is true.

x < 1 ➔ $\sqrt{x} - \sqrt{1/x} < 0$ Case II is true.

x = 1 ➔ $\sqrt{x} - \sqrt{1/x} = 0$ Case III is true.
The answer is (E).

14. Answer: (C)
$(\sqrt{x})^2 = x$ for all non-negative values of x.
Hence $(\sqrt{x})^2 - x = 0$ for all x > 0.
The answer is (C).

15. Answer: (E)
$\sqrt{x^2} - x^2 = x - x^2 = x(1 - x)$ for all x > 0.
x < 1 ➔ 1 - x > 0 ➔ $\sqrt{x^2} - x^2 > 0$ Case I is true.
x > 1 ➔ 1 - x < 0 ➔ $\sqrt{x^2} - x^2 < 0$ Case II is true.
x = 1 ➔ 1 - x = 0 ➔ $\sqrt{x^2} - x^2 = 0$ Case III is true.
The answer is (E).

16. Answer: (A)
$(\sqrt{-x})^2 - x = -x - x = -2x$ which is always positive for x < 0.

The answer is (A).

17. Answer: (D)
$\sqrt{32x} = \sqrt{16 \times 2x} = 4\sqrt{2x}$
The answer is (D).

18. Answer: -3
$\frac{x^6}{y^3} = \frac{3^3}{(3y)^{-3}}$ ➔ $\frac{x^6}{y^3} = 3^3 3^3 y^3$ ➔
$x^6 = 3^6 y^6$ ➔ $x = 3y$ or $x = -3y$ ➔
x/y = 3 or x/y = -3
Since x > 0 and y < 0 then x/y = -3

19. Answer: Minimum = -1, Maximum = 0
"b is non-positive integer" means it can be 0, -1, -2, -3, -4, -5, ...

b = 0 ➔ $2^b = 2^0 = 1$
b = -1 ➔ $2^b = 2^{-1} = 1/2 = 0.5$
b = -2 ➔ $2^b = 2^{-2} = 1/4 = 0.25$

:

b = -25➔ $2^b = 2^{-25} = 1/(2^{25}) = 0.00000003$

You can see above that as b decreases from its maximum value of 0 and becomes more negative, 2^b decreases and approaches to 0. ➔

b = 0 ➔ $(2)^b = 1$ ➔ $t = (2)^b - 1 = 0$
b is infinitely negative ➔ $(2)^b = 0$ ➔ $t = (2)^b - 1 = -1$

Hence t is in between -1 and 0.

20. Answer: 0 and 2
$(x^2 - 2x + 1)^{(\sqrt{8}x)} = 1$ ➔
Either $x^2 - 2x + 1 = 1$ (All powers of 1 are 1) or
$\sqrt{8}x = 0$ (0th powers of non-zero numbers are 1)
$x^2 - 2x + 1 = 1$ ➔ $(x - 1)^2 = 1$ ➔
x - 1 = 1 ➔ x = 2 or x - 1 = -1 ➔ x = 0
$\sqrt{8}x = 0$ ➔ x = 0
The answer is x = 2 or x = 0

21. Answer: (A)

$a^{-\frac{3}{2}} = \left(a^{\frac{1}{2}}\right)^{-3}$

Since $a^{\frac{1}{2}}$ can not be real for any negative value of a, a must be zero or positive.

However, for a = 0, $a^{\frac{1}{2}} = 0$ and

$\left(a^{\frac{1}{2}}\right)^{-3} = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, which is not defined.

Hence a can not be zero. The answer is (A).

22. Answer: $x < x^2 < \sqrt{x^2} < \sqrt{-x}$

Since x is negative, it is the smallest of all the terms.

Among the remaining 3 terms, $\sqrt{x^2} = -x$
(Note that -x is positive.)

Since $-1 < x < 0$, then $x^2 < -x < \sqrt{-x}$ ➔
$x^2 < \sqrt{x^2} < \sqrt{-x}$
Therefore $x < x^2 < \sqrt{x^2} < \sqrt{-x}$

For x = -0.04, calculate the above terms and verify that the order is correct.

Negative Numbers

1.

a. Answer: 64
$(a - b)^2 = (3 + 5)^2 = 8^2 = 64$

b. Answer: -8
$(b - a)^2 = (a - b)^2$ ➔
$\frac{(a - b)^2}{b - a} = \frac{(b - a)^2}{b - a} = b - a = -5 - 3 = -8$

c. Answer: 32
$(a - b)^2 = (b - a)^2$ ➔
$\frac{(a - b)^2}{(b - a)^{-3}} = (b - a)^2 \cdot (b - a)^3 =$
$(b - a)^5 = (5 - 3)^5 = (-2)^5 = 32$

d. Answer: -32
$(b - a)^2 = (a - b)^2$ ➔
$- a + b)^2 \cdot (a - b)^3 = (a - b)^{239} \cdot (a - b) =$
$(-2)^5 = -32$

2. Answer: (C)
$x^2 = 4$ ➔ x is 2 or –2 ➔ for x = 2, x - 1 = 1
For x = -2, x - 1 = –3
The answer is (C).

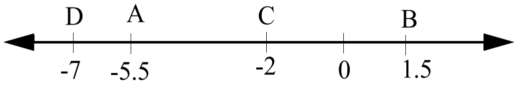
3. Answer: (C)
To make x - 13 odd, x has to be even. To make x - 13 negative x has to be less than 13. Hence the answer is (C).

4. Answer: (E)
$x/y > 1$ ➔ $x \neq y$ ➔ $x - y \neq 0$ and
x and y are either both positive or both negative.
If they are both positive, $x/y > 1$ ➔ $x > y$ ➔ $x - y > 0$
If they are both negative, $x/y > 1$ ➔ $x < y$ ➔ $x - y < 0$

Depending on the values of n and y, x - y can be negative or positive, but not zero.

For example,
x = 5, y = 2 ➔ x - y = 5 - 2 = 3 (positive)
x = -5, y = -2 ➔ x - y = -5 - (-2) = -3 (negative)
The answer is (E).

5. Answer: 1.5
Let a, b, c and d be the coordinates of the points A, B, C and D on the number line.
a = -5.5
The coordinate of B is 7 more than A ➔
the distance between A and B is 7. (b = -5.5 + 7 = 1.5)
C is at equal distance from A and B ➔
c = -5.5 + 7/2 = -2
d is 5 less than the c ➔ d = -2 - 5 = -7
Points A, B, C and D are shown in the below figure.



The distance between A and D = -5.5 - (-7) = 1.5

6. Answer: (D)
(A) is not true for $0 \le x \le 1$
(B) is not true for $0 \le x \le 1$
(C) is not true for x = 0
(D) is true because non-negative includes zero and positive.
The answer is (D).

7. Answer: (D)
$(y - x)^2 = (x - y)^2$ ➔
$(x - y) \cdot (y - x)^2 \cdot (x - y)^3 =$
$(x - y) \cdot (x - y)^2 \cdot (x - y)^3 = (x - y)^6$
For x = y, the expression is zero, otherwise it is positive.
Since the expression is an even power of x - y, it can never be negative.
The answer is (D).

8. Answer: (C)
y - x = -(x - y) ➔ $(y - x) \cdot (x - y)^3 =$
$-(x - y)^4$, which is always negative or zero.
The answer is (C).

9. Answer: (E)
$(y - x)^4 = (x - y)^4$ ➔

$(y - x)^4 \div (x - y)^3 = x - y$
Depending on the values of x and y, x - y can be positive, negative or zero. The answer is (E).

10. Answer: (A)
Since s^2 is always positive or zero ➔ c and d are always positive. ➔ $cd > 0$
The answer is (A).

11. Answer: (D)
$s^2 > 1$ ➔ $s < -1$ or $s > 1$
$1 < s^2 < c < d^3$ ➔ $c > 1$ and $d^3 > 1$ ➔ $d > 1$
For $s < -1$, $scd < -1$ ➔ $scd - 1 < -2$
For $s > 1$, $scd > 1$ ➔ $scd - 1 > 0$
So scd - 1 can not be between -2 and 0.
The answer is (D).

12. Answer: (D)
$a^2 > a$ ➔ $a < -1$ or $a > 1$
For $a < -1$, III is not true. The answer is (D).

Numbers Between -1 and 1

1. Answer: There is not enough information to answer the question.
$x < 1$ ➔ x^3 is smaller.
$x = 1$ ➔ they are equal.
$x > 1$ ➔ x^2 is smaller.

2. Answer: x^3
x^3 is always smaller, because x^3 is always negative and x^2 is always positive.

3. Answer: There is not enough information to answer the question.
$x < 1$ ➔ x is smaller.
$x = 1$ ➔ they are equal.
$x > 1$ ➔ $\sqrt{x}$ is smaller.

4. Answer: (D)
It is tempting to say that the answer is (A). Most of the time, the square of an integer is bigger than the integer itself. However, for $x = 0$ and $x = 1$, $x^2 = x$
The answer is (D).

5. Answer: (D)
It is tempting to say that the answer is (A). Most of the time, the square root of an integer is smaller than the integer itself. However, for $x = 0$ and $x = 1$,
$x = \sqrt{x}$.
The answer is (D).

6. Answer: (E)
It is tempting to say that the answer is (A). Most of the time, the cube of an integer is bigger than the square of the integer. However, for $x < 0$, $x^3 < x^2$ and for $x = 0$ or $x = 1$, $x^3 = x^2$
The answer is (E).

7. Answer: (D)
If $-1 < x < 0$ and $0 < y < 1$ ➔ $-1 < xy < 0$
Add 1 to all sides of the inequality, you will get
$-1 + 1 < xy + 1 < 0 + 1$ ➔
$0 < xy + 1 < 1$
The answer is (D).

8. Answer: $1/x < x < x^3 < x^2 < x + 2$
Since x is negative, 1/x, x and x^3 are negative numbers.
Because $-1 < x < 0$, 1/x is less than x. And x is less than x^3. Therefore $1/x < x < x^3$
On the other hand x^2 and $x + 2$ are positive numbers.
Because $-1< x < 0$, $0 < x^2 < 1$ and $x + 2 > 1$
Hence $x^2 < x + 2$
Combining the two results: $1/x < x < x^3 < x^2 < x + 2$

Note: You can understand this result best with an example. Choose a value for x between -1 and 0, such as -0.2. Calculate the above terms and confirm the above result.

Divisibility

1. Answer: 0
If a is divisible by 6, it is also divisible by 3.
$(a + 12)/3 = k + 12/3 = k + 4$, where k is an integer.
➔ the remainder is 0.

2. Answer: Yes
Yes.
n is divisible by 2. ➔ m is even. ➔ $n \cdot m$ is even,
$n \cdot m + 2$ is even and divisible by 2.

3. Answer: 0
When a is divided by 4, the remainder is 3 ➔
$a = 4m + 3$, where m is an integer. ➔
$(7a + 3)/4 = (7(4m + 3) + 3)/4 = 28m/4 + 24/4 = 7m + 6$
Hence the remainder is 0.

4. Answer: 5
If a is divisible by 4, a^2 is divisible by 16.
Hence the remainder of $(3a^2 + 5)/16$ is 5.

5. Answer: 1
If a is even, a + 3 is odd and $(a + 3)^5$ is also odd. So the remainder of $((a + 3)^5)/2$ is 1.

6. Answer: (D)
The remainder is 2 when an integer a is divided by 3 ➔ $a/3 = n + 2/3$ where n is an integer.

I. $3a/3 = a$ ➔ 3a is divisible by 3.

II. $(a + 1)/3 = a/3 + 1/3 = n + 2/3 + 1/3 = n + 3/3 = n + 1$ ➔
a + 1 is divisible by 3.

III. $(a + 3)/3 = a/3 + 1 = n + 1 + 2/3$ ➔
a + 3 is not divisible by 3.

The answer is (D).

7.

a. Answer: 1
An integer is divisible by 4 if the last two digits are divisible by 4. The minimum number we can add to 255 to make the last 2 digits divisible by 4 is 1. (56/4 = 14)

b. Answer: 1
An integer is divisible by 8 if the last three digits are divisible by 8. The minimum number we can add to 255 to make the last 3 digits divisible by 8 is 1. (256/8 = 32)

c. Answer: 6
If a is divisible by 9, a = 9s where s is an integer. ➔ 255 + k = 9s ➔ k = 9s - 255
For s < 29, k is negative.
The lowest positive value of k is obtained when s = 29
$k = 9 \times 29 - 255 = 261 - 255 = 6$

((255 + 6)/9 = 29)

d. Answer: 5
For a to be divisible by 10, its last digit must be 0. The minimum positive integer that we can add to make the unit digit 0, is 5.

((255 + 5)/10 = 26)

8.

a. Answer: 99
Since 255 is odd, k has to be odd as well to make the addition even. Highest possible 2 digit odd integer is 99.

b. Answer: 99
255 is divisible by 3. (255/3 = 85)
To make the addition divisible by 3, k has to be divisible by 3 as well. The highest possible 2 digit integer divisible by 3 is 99.

c. Answer: 95
255 is divisible by 5 (the units digit is 5). To make the addition divisible by 5, k has to be divisible by 5 as well. The highest possible 2 digit integer divisible by 5 is 95.

d. Answer: 99
If a number is divisible by 6, it is also divisible by 2 and 3. 255 is divisible by 3, but not by 2. To make the addition divisible by both 3 and 2, k has to be divisible by 3 and has to be odd. The highest possible 2 digit odd integer divisible by 3 is 99.

e. Answer: 95
The remainder of 255/7 is 3. ➔
For 255 + k to be divisible by 7, $k + 3$ must be divisible by 7.
The highest possible 2-digit integer satisfying this condition is 95.

9. Answer: (B)
If a and b are odd ➔ a + b is even and ab is odd. ➔ a + b + ab is odd.
An odd number is not divisible by even numbers. Hence, a + b + ab is not divisible by neither 2 nor 6.
On the other hand, all 3 terms in a + b + ab are divisible by 3, hence a + b + ab is divisible by 3. The answer is (B).

10. Answer: 2

$$\frac{2(a+1)^2}{16} = \frac{2(a+1)(a+1)}{16} = \frac{2(a^2+2a+1)}{16} =$$

$$\frac{2a^2}{16} + \frac{4a}{16} + \frac{2}{16}$$

a is divisible by 4 ➔ a = 4n, where n is an integer. ➔ $2a^2 = 2 \times 16n^2$ and $4a = 16n$. ➔ Both $2a^2$ and 4a are divisible by 16. Hence the remainders of the first and second terms are zero. Therefore the answer is 2.

11. Answer: 1
If a is an even number, a = 2n, where n is an integer. ➔

$$\frac{(a-3)^2}{4} = \frac{(2n-3)^2}{4} = \frac{4n^2-12n+9}{4} =$$

$$n^2 - 3n + 2 + \frac{1}{4}$$

So the remainder is 1.

12. Answer: 3
Step 1: If a is an even integer divisible by 3 ➔
$a = 2 \cdot 3 \cdot n = 6n$, where n is an integer.
Step 2: Substitute a into the expression:
$(a+x)^2 = (6n+x)^2 = 36n^2 + 12nx + x^2$
Step 3: For this expression to be divisible by 9 for all values of n, all the terms must be divisible by 9.
Step 4: First term, $36n^2$, is already divisible by 9, because $36 = 4 \cdot 9$ ➔
Step 5: If the last term, x^2, is divisible by 9,
$x^2 = 9m$, where m is an integer. ➔
Step 6: Minimum value of x is obtained when m = 1 ➔ $x = \sqrt{9} = 3$.
Step 7: To accept x = 3 as an answer, we need to make sure that, for x = 3, the second term, 12nx, is also divisible by 9. Substitute 3 for x: 12nx = 36n, which is divisible by 9. The answer is 3.

13. Answer: (C)
(A), (D) and (E) can sometimes be true. For example, if a = 6, (A) and (E) is true. If a is 14, (D) is true.

Let's look at (B) and (C).
a is even ➔ a = 2n, where n is an integer.
Then a/4 = 2n/4 = n/2. Here n can be odd or even.
If n is odd, the remainder will be 1. If n is even, the remainder will be 0. ➔
The remainder can never be 2.

The answer is (C).

14. Answer: (E)
If a is divisible by 3 and odd, then $a = 3(2n + 1)$, where n is a non-negative integer.
Similarly, if b is divisible by 3 and is odd, then $b = 3(2m + 1)$, where m is a non-negative integer.
Therefore $a + b + ab + 3 =$
$3(2n + 1) + 3(2m + 1) + 9(2n + 1)(2m + 1) + 3 =$
$36mn + 24m + 24n + 18$
Each term in this expression is divisible by 2, 3, and 6. Therefore the answer is (E).

15. Answer: (D)
$a + b + c = 18$ means abc is divisible by 3, because 18 is divisible by 3.
$c = 2d$ means abc is even, because its unit digit, c, is even. Hence it is divisible by 2.
If a number is divisible by 2 and 3, it is also divisible by 6.
Since all the numbers are divisible by 1, abc is also divisible by 1.
For abc to be divisible by 4, the number, bc must be divisible by 4. This number may be divisible by 4 for some values of b and c, but not all the possible values. The answer is (D).

Even & Odd Numbers

1. Answer: (B)
6n is even and 7 is odd. The subtraction of an odd integer from an even integer is odd. The answer is (B).

2.

- **a.** Answer: 1
  2 is both prime and even. In fact 2 is the only even prime number.
- **b.** Answer: 3
  3, 5 and 7 are both prime and odd. The answer is 3.

3. Answer: (E)
n can be either positive or negative. So (A) and (B) and (C) cannot always be correct.

Since n^4 is even, n must be even because multiplication of odd numbers is always odd. The answer is (E).

4. Answer: (A)
$n = 0$. Zero is an even integer. The answer is (A).

5. Answer: (C)
One of the integers is even and the other one is odd.
The square of the even integer is even and non-negative.
The square of the odd integer is odd and positive.
The addition of the squares of an even and an odd integer is odd and positive. The answer is (C).

6. Answer: (C)
The integer is either odd or even. If it is even, its square is even. If it is odd, its square is odd. The addition of two even or two odd integers is always even. Therefore the answer has to be even. So it can be either (A) or (C).
It is easy to see that the integer, $n = 1$.

Let's check. $n + n^2 = 1 + 1 = 2$
The answer is (C).

7. Answer: (B)
The multiplication of consecutive integers is always even because one of the integers has to be even. So the answer is either (B) or (D). Now it is easy to see that it cannot be (D) because there are no three consecutive integers the multiplication of which is -2.
If one of the integers is 0, the multiplication is zero.
These three consecutive integers can be
$\{-1, 0, 1\}$ or $\{-2, -1, 0\}$ or $\{0, 1, 2\}$.
The answer is (B).

8. Answer: (E)
The addition of 4 consecutive integers is the addition of 2 even and 2 odd integers, which is always even.
1684 is the only even answer. The answer is (E).
These integers are 420, 421, 422 and 423.

Absolute Value

1. Answer: (B)
By definition, the distance of a point from the origin is the absolute value of the coordinate of the point. Hence the answer is $|p| + |q|$, (B).

2. Answer: 4
Since the minimum value of any absolute value is 0, $|2x - 8| = 0$ at its minimum. It means $2x - 8 = 0$, then $x = 4$

3. Answer: 5
If $x = -5$, then $|x - |x - 4|| - |4 - x| =$
$|-5 - |-5 - 4|| - |4 - (-5)| = |-5 - |-9|| - |9| = |-5 - 9| - |9| =$
$14 - 9 = 5$

4. Answer: -a - b
Since both a and b are negative, $a + b$ is negative.
Hence $|a + b| = -a - b$

5. Answer: b - a
$a < b$ ➔ $a - b < 0$ (negative) ➔ $|a - b| = b - a$

6. Answer: -3a + b
If $a < b < 0$, $|3a - b| = |2a + (a - b)|$
$a < b$ ➔ $a - b < 0$ (negative) and $a < 0$ ➔
$2a < 0$ (negative)
Hence $2a + (a - b) < 0$ (negative) ➔
$|2a + (a - b)| = -2a - a + b = -3a + b$

7. Answer: (B)
For all values of $x < 0$, $|x - 3|$ is always positive.
Since $x < 0$, $x^3 < 0$ ➔ $-x^3 > 0$ and since $x < 0$ ➔
$(3x)^{-3} < 0$ ➔ $-(3x)^{-3} > 0$
Since all 3 terms of $|x - 3| - x^3 - (3x)^{-3}$ are positive, expression itself is always positive. The answer is (B).

8. Answer: (D)
The distance between 2 points on a number line is: $p - q$ if $p > q$ and it is $q - p$ if $q > p$. Therefore the answer is $|p - q|$, (D).

9. Answer: (C)
Let's examine each case:

(A) $|p - q| \le |p| + |q|$ is true for all the values of p and q.
Example: $p = 3$, $q = -2$ ➔ $|p - q| = 5$ and $|p| + |q| = 3 + 2 = 5$ ➔ $|p - q| = |p| + |q|$

(B) $|p - q| \le |p| + |-q|$
Same as (A), because $|q| = |-q|$

(C) $|p - q| \le |p| - |q|$
This expression is false when $q > p$
Example: $p = 1$, $q = 4$ ➔ $|p - q| = 3$ and $|p| - |q| = 1 - 4 = -3$ ➔ $|p - q| > |p| - |q|$
In fact, for $q > p$, $|p| - |q| < 0$. On the other hand $|p - q|$ is always positive and hence always bigger then $|p| - |q|$.
The answer is (C)

(D) $|p - q|^2 \le (|p| + |q|)^2$ If you take the square of both sides of the inequality in case (A), you will get this expression. Since case (A) is always true, this case is always true as well.

6

GEOMETRY

Geometry is an important math subject in the SAT. How important? About one-quarter of the questions are on geometry.

Geometry questions in the SAT are at all three levels. It is important that you study this chapter if you want to score more than 400 in the SAT.

Here we have used the "Learn by Example" method: the basic principle is simply stated and then explained by examples.

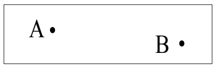
We have also provided Practice Exercises for most of the sections. These exercises are developed to help you practice your basic knowledge of the subject.

SAT questions are designed to measure your reasoning skills. We have provided these kind of exercises at the end of the chapter. Answers and the solutions to these questions are also provided at the end.

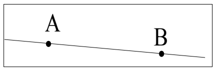
Points, Lines and Angles

Points

- A point is what you get when you touch the paper by the tip of your pen.
- A point has zero length, width and height.
- A point is represented by a "**.**" and with a single letter.
  Two points, A and B, are shown in the figure.



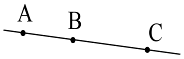
- Any two points always lie on a straight line as shown in the figure.



Examples:

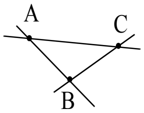
1. (Easy)
What is the minimum number of lines that can pass through 3 distinct points on a plane?

Solution:
The answer is 1. When all 3 points align, one line passes through all of them as shown in the figure.



2. (Easy)
What is the maximum number of lines that can pass through 3 distinct points on a plane?

Solution:
The answer is 3. When all 3 points are not aligned, 3 lines pass through them as shown in the figure.



Practice Exercises:

1. (Easy)
What is the minimum number of lines that can pass through 10 distinct points on a plane?

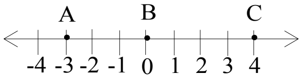
2. (Easy)
What is the maximum number of lines that can pass through 4 distinct points on a plane?

Answers: 1. 1; **2.** 6

Number Line

A number line is a line on which each point is represented by a number. The number that represents the point is called the coordinate of the point. Points can be identified on a number line if their coordinates are known.

For example in the figure, the coordinates of points A, B and C are -3, 0 and 4, respectively.



The notation P(x) is used to define point P's coordinate, x. For example points A(-3), B(0), and C(4) are displayed on the number line above.

The distance between two points on a number line is the absolute value of the difference of their coordinates.

For example, in the above figure, the distance between
A and B = AB = |-3 - 0| = 3
A and C = AC = |4 - (-3)| = 7
B and C = BC = |0 - 4| = 4

Practice Exercises:

1. (Easy)
What is the distance between

a. A(7/4) and B(0)?

b. A(-7) and B(0)?

c. A(-7/2) and B(3)?

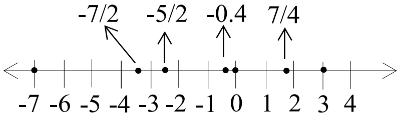
d. A(-0.4) and B(-5/2)?

2. (Easy)
Place the points in Exercise 1 on a number line.

3. (Easy)
On a number line, how many points are there at a distance 3 from the origin? What are their coordinates?

4. (Medium)
On a number line, how many points are there at a distance 5 from point A, if the coordinate of A is -1? What are their coordinates?

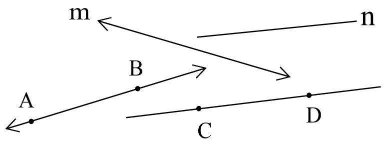
Answers:
1. a. 1.75; **b.** 7; **c.** 6.5; **d.** 2.1;
2.



3. Two points, the coordinates are: -3 and 3;
4. Two points, the coordinates are: -6 and 4

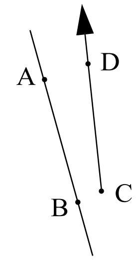
Lines

- **A line** is a collection or a set of aligned points.
- The below figure shows four lines, represented in different forms. They are represented by $\overleftrightarrow{m}$, $\overleftrightarrow{n}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$



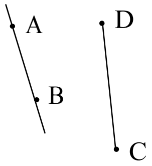
- A line continues in both directions indefinitely.
- When a line starts from a definite point and continues in only one direction indefinitely, it is called **a ray**.

  In the figure, two rays, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are displayed. In this representation, the points A and C are the starting points of the two rays respectively.



- As shown in the figure, **a line segment** is a segment between two points on the line.

  $\overline{AB}$ is a line segment, where A is the beginning of the line segment and B is the end of the line segment.



- The length of a line segment $\overline{AB}$ is the distance between point A and point B. It is represented by AB.

  For example, AB = 4 means the length of $\overline{AB}$ is 4 or the distance between A and B is 4.
- In general, if two geometric shapes are identical in shape and size, they are congruent.

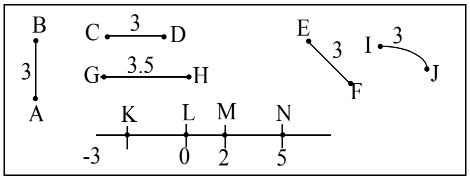
  A line segment has only one property, its length. So **two line segments are congruent** if they have the same length.

  The symbol used to describe the congruency is "≅" For example if $\overline{AB}$ and $\overline{CD}$ are congruent, then $\overline{AB} \cong \overline{CD}$.

  In the below figure,

  $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{KL}$ and $\overline{NM}$ are congruent, because they all are line segments with the same length, 3.

  $\overline{GH}$, $\overset{\frown}{IJ}$ and $\overline{LM}$ are not congruent with the others, because $\overline{GH}$ is longer, $\overline{LM}$ is shorter and $\overset{\frown}{IJ}$ is a curve, not a line segment.



Practice Exercises:

1. (Easy)
Match the elements in the right and left columns:

Line	$\overline{AB}$
Ray	$\overleftrightarrow{AB}$
Line Segment	AB
Length	$\overrightarrow{AB}$

2. (Easy)
If $\overline{XY} \cong \overline{PQ}$, which of the following must be true?

(A) $\overline{XY}$ and $\overline{PQ}$ are one and the same.

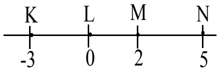
(B) $\overline{XY}$ and $\overline{PQ}$ can not cross each other.

(C) $\overline{XY}$ and $\overline{PQ}$ are parallel.

(D) XY = PQ

(E) All of the above.

3. (Medium)
In the figure, what are the 2 pairs of congruent line segments?

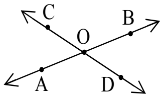


Answers:
1. Line: $\overleftrightarrow{AB}$, Ray: $\overrightarrow{AB}$, Line Segment: $\overline{AB}$, Length: AB;
2. (D); **3.** $\overline{KL} \cong \overline{MN}$ and $\overline{KM} \cong \overline{LN}$

Crossing Lines and Angles

When two lines or line segments or rays cross, the intersection is a point.

In the figure, the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is point O.



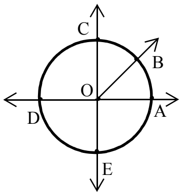
When two lines cross, two pairs of opposite angles are formed.
In the above figure, $\angle AOC$, $\angle BOD$ and $\angle BOC$, $\angle AOD$ are opposite angle pairs.

Each angle has two arms and one vertex.
For example, the two arms of $\angle BOD$ are $\overrightarrow{OB}$ and $\overrightarrow{OD}$ and the vertex is point O.

The opening of the two arms of an angle is the measure of the angle. Its unit is **degrees** and represented by "^{o}".

For example, 1^o, 90^o, 130^o, 180^o, 270^o, 350^o and 360^o are all measures of angles. Measure of angles can be between 0^o and 360^o.

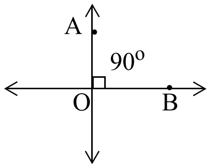
If we take a circle and divide it into 360 equal pie shapes, the measure of the angle for each pie is 1 degree (1^o). Below figure shows 0^o, 45^o, 90^o, 180^o, 270^o and 360^o angles.



$\angle AOA = 0°$	Identical arms
$\angle AOB = 45°$	$\overrightarrow{OB}$ bisects $\angle AOC$
$\angle AOC = 90°$	Perpendicular arms (see below)
$\angle AOD = 180°$	Arms in opposite directions
$\angle AOE = 270°$	Perpendicular arms (see below)
$\angle AOA = 360°$	Identical arms after a full circle.

Perpendicular Lines

When two lines cross, if the angle created is 90^o, then the two lines are said to be perpendicular to each other.



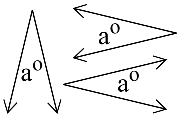
Perpendicular lines are represented by the symbol "$\perp$" For example, in the figure, $\angle AOB$ is 90^o.

Hence $\overleftrightarrow{AO} \perp \overleftrightarrow{BO}$.

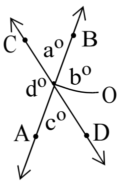
As shown in the figure, 90^o angle is also represented by "□ (square)" symbol.

Two angles are congruent if their measurements are the same.

For example all the angles shown in the figure are congruent.



When two lines cross:



a. The opposite angles created are equal. $b^o = d^o$ and $c^o = a^o$

b. Addition of all four angles around the intersection point is 360^o. $a^o + b^o + c^o + d^o = 360^o$

c. Addition of two adjacent angles is 180.

$a^o + b^o = c^o + b^o = c^o + d^o = a^o + d^o = 180^o$

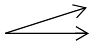
Example: (Easy)

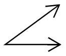
In the above figure, if $\angle AOC = 140°$, then $\angle BOD = 140°$ because $\angle AOC$ and $\angle BOD$ are opposite angles.

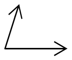
$\angle COB = \angle DOA = 180 - 140 = 40°$

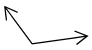
Practice Exercises:

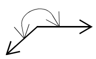
1. (Easy)
Which of the following is more likely to be 45^o?

(A) 

(B) 

(C) 

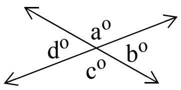
(D) 

(E) 

2. (Easy)
Make an approximate sketch of 30^o, 60^o, 100^o, 135^o, 225^o, 270^o and 359^o angles.

3. (Easy)

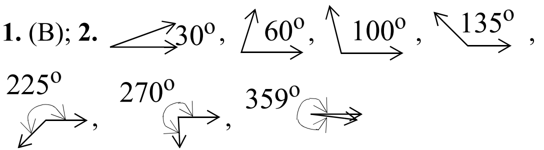
a. In the figure, if b = 33, what are the values of a, c and d?



b. If a = b = c = d = x, what is x?

Answers:

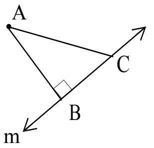
1. (B); **2.**



3. a. a = 147, c = 147, d = 33; **b.** 90

Distance Between a Point and a Line

The distance between a point, A, and a line, m, is the length of the line segment $\overline{AB}$ where B is a point on the line m and $\overline{AB} \perp \overleftrightarrow{m}$ as shown in the figure.



The distance between point A and line m is the minimum distance between point A and any point on the line m.

For example, in the figure above, B and C are two points on line m. Since $\overline{AB} \perp \overleftrightarrow{m}$, then AB < AC.

Example: (Medium)

Points A, B and line m are on the same plane. A and B are on the opposite side of line m. Q and R are two distinct points on line m.

If $\overline{AQ} \perp \overleftrightarrow{m}$ and $\overline{BR} \perp \overleftrightarrow{m}$, then which of the following statements must be true?

(A) $AB = AQ + BR$

(B) $AB > AQ + BR$

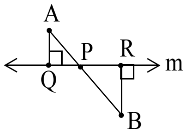
(C) $AB < AQ + BR$

(D) $AB \geq AQ + BR$

(E) $AB \leq AQ + BR$

Solution:

Half of the solution lies in drawing the figure correctly. In the figure, A and B are at opposite sides of line m.



$\overline{AQ} \perp \overleftrightarrow{m}$ and $\overline{BR} \perp \overleftrightarrow{m}$

AB is the distance between A and B. Let P be the cross section of $\overline{AB}$ and line m.

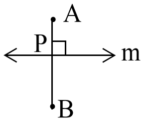
$\overline{AQ} \perp \overleftrightarrow{m}$ ➔ AQ is the distance between point A and line m ➔ $AQ < AP$

$\overline{BR} \perp \overleftrightarrow{m}$ ➔ BR is the distance between point B and line m ➔ $BR < BP$

If you add these two inequalities, you will have $AQ + BR < AP + BP$

Since $AP + BP = AB$, then $AQ + BR < AB$

At this point, you may conclude that the answer is (B). However, it is (D), because case (D) also considers a special situation in which Q, R and P are the same point and A, P and B are aligned as shown in the figure. In this case $AB = AQ + BR$



Practice Exercises:

1. (Easy)

P is a point outside of line $\overleftrightarrow{QR}$. If $\overline{PR} \perp \overleftrightarrow{QR}$, which of the following statements must be true?

(A) $PR < QR$

(B) $PR < PQ$

(C) $PR > QR$

(D) $PR > PQ$

(E) $PR = QR$

2. (Medium)

Points A, B and line m are on the same plane. A and B are on the same side of line m. P, Q and R are three distinct points on line m.

If $\overline{AQ} \perp \overleftrightarrow{m}$ and $\overline{BR} \perp \overleftrightarrow{m}$, then which of the following statements must be true?

(A) $AP + BP = AQ + BR$

(B) $AP + BP > AQ + BR$

(C) $AP + BP < AQ + BR$

(D) $AP + BP \geq AQ + BR$

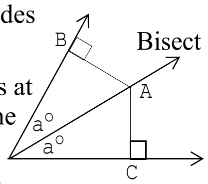
(E) $AP + BP \leq AQ + BR$

Answers: 1. (B); **2.** (B)

Bisect of an angle

Bisect of an angle is the line that divides the angle into two equal angles.

Any point on the bisect of an angle is at an equal distance from the arms of the angle.



For example, in the figure, AB = AC

Examples:

1. (Easy)

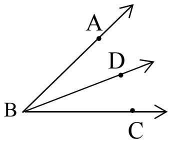
If $\angle ABC = 44°$ and $\overrightarrow{BD}$ is the bisect of $\angle ABC$, what is the measure of $\angle ABD$?

Solution:

In the figure, $\overrightarrow{BD}$ bisects $\angle ABC$.

$\angle ABC = 44°$ ➔

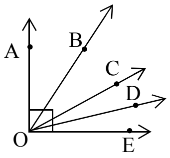
$\angle ABC = 44°/2 = 22°$



2. (Medium)

In the figure, $\overrightarrow{OB}$ bisects $\angle AOC$ and $\overrightarrow{OD}$ bisects $\angle COE$.

What is the measure of $\angle BOD$?



Solution:

In the figure, $\angle BOD = \angle BOC + \angle COD$

$\overrightarrow{OB}$ bisects $\angle AOC$ ➔ $\angle BOC = \frac{AOC}{2}$

$\overrightarrow{OD}$ bisects $\angle COE$ ➔ $COD = \frac{\angle COE}{2}$

Substituting the values of $\angle BOC$ and $\angle COD$ into the first equation: $\angle BOD = \frac{\angle AOC}{2} + \frac{\angle COE}{2} = \frac{\angle AOC + \angle COE}{2} = \frac{90}{2} = 45°$

Practice Exercises:

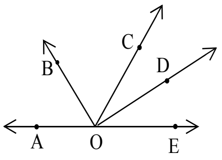
1. (Medium)
A, B, C and D are 4 distinct points on a plane. If $\angle BCA = 25°$ and $\angle BCD = 50°$, then which of the following must always be true?

- **(A)** $\overrightarrow{CB}$ is the bisect of $\angle ACD$
- **(B)** $\overrightarrow{AC}$ is the bisect of $\angle BCD$
- **(C)** $\overrightarrow{BD}$ is the bisect of $\angle ACD$
- **(D)** $\overrightarrow{CB}$ is the bisect of $\angle ABD$
- **(E)** None of the above.

Hint: Once you set one of the angles, there are 2 different ways of drawing the second angle.

2. (Medium)
In the figure, $\overrightarrow{OB}$ bisects $\angle AOC$ and $\overrightarrow{OD}$ bisects $\angle COE$.

What is the measure of $\angle BOD$?



Answer: 1. (E); **2.** 90^o

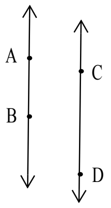
Parallel Lines

When two lines on the same plane never cross, they are said to be parallel.

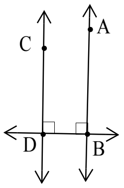
Two parallel lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are shown in the figure.

Parallel lines are represented with symbol "$\|$".

For example, in the figure, $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$

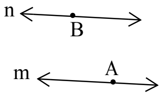


The distance between two parallel lines is the length of the perpendicular line segment that joins the two lines. In the figure, the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is BD.



Minimum distance between the two points, on each of the two parallel lines is the distance between these two lines. For example, in the above figure, BD < DA and BD < BC.

Example: (Easy)
m and n are two distinct parallel lines. A and B are points on line m and line n respectively. If the distance between n and m is d, which of the following statements is <u>wrong</u>?



- **(A)** The distance between A and B can be infinite.
- **(B)** AB can be equal to d - 1.
- **(C)** AB can be equal to d.
- **(D)** AB can be equal to d + 1.
- **(E)** AB is minimum when $\overline{AB}$ (not drawn in the figure) is perpendicular to line m.

<u>*Solution:*</u>
(A) is correct, because you can place points A and B as far apart as you want. Both lines extend indefinitely on both sides.

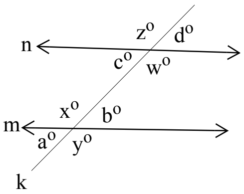
The answer is (B). Because the minimum distance between A and B is the distance between lines m and n, which is d. Since AB can not be less than d, it can not be equal to d - 1.

- When two parallel lines are crossed by a third line as shown in the figure,

$a = b = c = d$ and
$x = y = w = z$

$x + b = a + y =$
$b + y = a + x =$
$c + w = d + z =$
$d + w = c + z = 180$
$b + w = c + x =$
$d + y = a + z = 180$

$x + b + a + y =$
$c + w + d + z = 360$

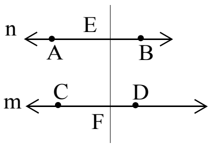


Examples:

1. (Easy)
If in the above figure, a = 50, what are the values of b, c, d, x, y, w and z?

<u>*Solution:*</u>
$b = c = d = 50$ and $x = y = w = z = 180 - 50 = 130$

2. (Easy)
Prove that if a line is perpendicular to one of the two parallel lines, it is perpendicular to the other line as well.



In the figure: $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$

$\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$ ➔ $\angle AEF = 90°$ ➔

$\angle EFC = \angle AEF = 90°$ ➔ $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$

Practice Exercises:

1. (Easy)

In the figure, are the lines m and n parallel?

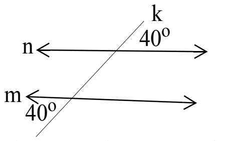


Figure not drawn to scale

2. (Easy)

In the figure, $n \parallel m$. If w + z = 260, then x - c = ?

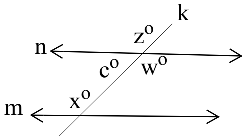
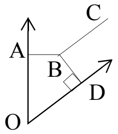


Figure not drawn to scale.

3. (Medium)

In the figure, $\overrightarrow{BC} \parallel \overrightarrow{OD}$ and $\angle ABC = 145°$.

What is the measure of $\angle ABD$?



Answers: 1. Yes; **2.** 80^o; **3.** 125^o

Polygons

General

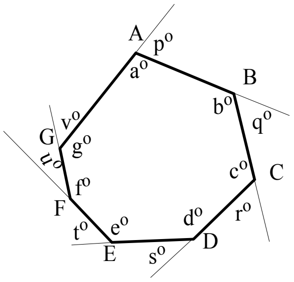
Definition

A polygon is a 2-dimensional object with 3 or more sides.

The figure displays a polygon with 7 sides.

The points, A, B, C, D, E, F and G are the 7 **vertices** or **corners** of the polygon.

$\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$ and $\overline{GA}$ are the 7 sides of the polygon.



A polygon is called **a regular polygon** if all the sides and/or the inner angles are equal.

The addition of the inner angles of a polygon is 180(n-2), where n is the number of sides or corners of the polygon.

Examples:

1. (Easy)

What is the addition of the inner angles of the 7-sided polygon displayed in the above figure?

<u>Solution:</u>

a^o, b^o, c^o, d^o, e^o, f^o and g^o are the measures of the 7 inner angles.

a + b + c + d + e + f + g = 180(7 - 2) = 900

2. (Medium)

The addition of the inner angles of a regular polygon is 540^o. What is the measure of one of the inner angles?

<u>Solution:</u>

540 = 180(n - 2), where n is the number of corners of the polygon. ➔ n = 540/180 + 2 = 3 + 2 = 5 ➔ one of the inner angles is 540/5 = 108^o

The addition of the outer angles of a polygon is independent of number of vertices and is always 360^o.

Examples:

1. (Easy)

Use the figure above to show that the addition of the outer angles of a 7-sided polygon is 360^o.

<u>Solution:</u>

In the above figure, p^o, q^o, r^o, s^o, t^o, u^o and v^o are the measures of the 7 outer angles.

Each of the outer angle is: 180^o - (adjacent inner angle)

p = 180 - a
q = 180 - b
r = 180 - c
s = 180 - d
t = 180 - e
u = 180 - f
v = 180 - g

The addition of the outer angles of this polygon is

p + q + r + s + t + u + v =

180 - a + 180 - b + 180 - c + 180 - d + 180 - e + 180 - f + 180 - g =

$(7 \times 180) - (a + b + c + d + e + f + g) =$

$1260 - 900 = 360$

2. (Easy)
What are the additions of the outer angles of a triangle, a rectangle and a pentagon?

Solution:

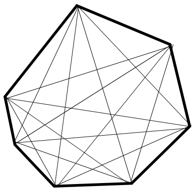
Since the addition of the outer angles of a polygon does not depend on the number of sides of the polygon, the answer is the 360^o for a triangle, a rectangle and a pentagon.

3. (Medium)
The addition of the inner angles of a regular polygon is 540^o. What is the measure of one of the outer angles?

Solution:

$540 = 180(n - 2)$, where n is the number of corners of the polygon. ➔ $n = 540/180 + 2 = 3 + 2 = 5$

The addition of the outer angles of a polygon is 360^o. Hence one of the outer angles is $360/5 = 72^o$



Diagonals of a polygon are the line segments from one vertex to another non-adjacent vertex.

Diagonals of a 7-sided polygon are shown in the figure.

The number of diagonals of an n-sided polygon is calculated in Chapter 8, "Combinations" section as an example.

This number is $\dfrac{n!}{2!(n-2)!} - n$.

The first term is the number of ways 2 corners can be selected out of n corners to form the diagonals. The second term subtracts the adjacent corners, since joining them forms the sides of the polygon, not the diagonals.

Examples:

1. (Medium)
How many diagonals does a 7-sided polygon have?

Solution:

$$\frac{7!}{2!(7-2)!} - 7 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{(1 \cdot 2) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)} - 7 =$$

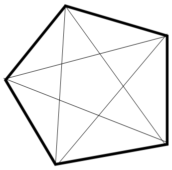
$$21 - 7 = 14$$

2. (Medium)
If a polygon has 5 diagonals, how many sides does it have?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

Solution:

We can eliminate (A) and (B) since triangle has no diagonals and rectangle has two. You know from the previous example, that 7-sided polygon has 14 diagonals. Hence the correct answer is either (C) or (D).



Let's draw the pentagon first and count the diagonals.

As you can see in the figure, the polygon has 5 sides. So the answer is (C).

You can also use the formula and calculate the number of diagonals for 5-sided polygon to get the answer:

$$\frac{5!}{2!(5-2)!} - 5 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2) \cdot (1 \cdot 2 \cdot 3)} - 5 =$$

$$10 - 5 = 5$$

Practice Exercises:

1. (Easy)
If the addition of all the inner angles of a polygon is 1080^o, how many vertices does it have?

2. (Easy)
The total of inner angles of a polygon is 1080^o. What is the total of the outer angles?

3. (Easy)
For a regular polygon the total of the inner angles is 1080^o, what is the measure of each inner angle?

4. (Medium)
For a regular polygon the value of an inner angle is 3 times the value of the adjacent outer angle, what is the measure of each inner angle?

5. (Medium)
The outer angle of a regular polygon is 36^o. How many diagonals does it have?

Answers: 1. 8; **2.** 360^o; **3.** 135^o, **4.** 135^o; **5.** 35

Special Polygons

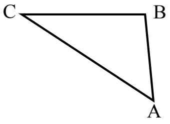
Most polygon questions in SAT are about special polygons. They are triangles, squares, rectangles, parallelograms and trapezoids.

Triangles: 3 - Sided Polygons

Triangles by far are the most popular polygons asked in the SAT. There are questions about triangles at every level. Here is what you need to know.

Definition:

A triangle is a polygon with 3 sides as shown in the figure. It is represented by ΔABC .

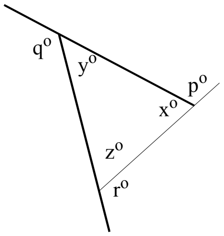


The Sum of the Inner Angles of a Triangle is 180^o

Since a triangle is a polygon, the sum of the inner angles of a triangle can be obtained by using the same formula given for the polygon.
This sum is $180(3 - 2) = 180^o$

Each outer angle of a triangle is the sum of the two non-adjacent inner angles.

In the figure:
$p = y + z$
$q = x + z$
$r = x + y$



Examples:

1. (Easy)
In the figure above, if $\angle A = 50°$, and $\angle B = 45°$, what is the measure of $\angle C$?

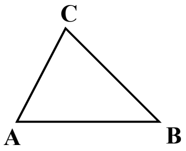


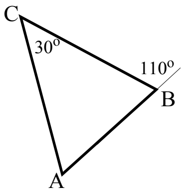
Figure is not drawn to scale.

Solution:
If $\angle A = 50°$, and $\angle B = 45°$, then $\angle C =$
$180^o - 50^o - 45^o = 85^o$

2. (Easy)
In the figure, A = ?

Solution:

$\angle A + 30° = 110°$ ➔
$\angle A = 110° - 30° = 80°$



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3. (Medium)
In the figure, x = ?

Solution:
$x + 4x = 110^o$ ➔
$x = 110^o/5 = 22^o$

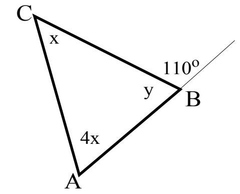


Figure is not drawn to scale.

Practice Exercises:

1. (Easy)
Two inner angles of a triangle are 20^o and 130^o. What is the measure of the third inner angle?

2. (Easy)
One of the outer angles of a triangle is 25^o and one of the inner angles is 15^o. What are the other two inner angles?

3. (Medium)
In the figure, what is the measure of $\angle C$?

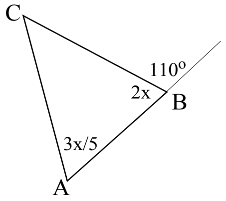


Figure is not drawn to scale.

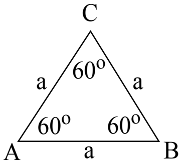
Answers: 1. 30^o; **2.** 10^o and 155^o; **3.** 89^o

Equilateral Triangle

Definition

A triangle is an equilateral triangle if all its three sides and/or three angles are equal.

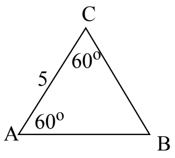
As shown in the figure, in an equilateral triangle, the measure of each angle is $\frac{180°}{3} = 60°$



Examples:

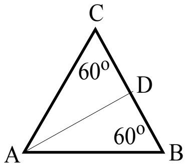
1. (Easy)
In the figure, what is AB + BC?

Solution:
If the measure of two inner angles of ΔABC is $60°$, ΔABC is an equilateral triangle.



$\angle B = 180° - 60° - 60° = 60°$ ➔
AB = BC = AC = 5 ➔ AB + BC = 5 + 5 = 10

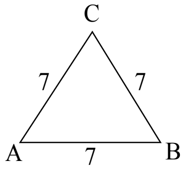
2. (Easy)
In the figure, if $\overline{DA}$ bisects $\angle A$, then $\angle ADB = ?$



Solution:
$\angle A = 180° - 60° - 60° = 60°$
$\overline{DA}$ bisects $\angle A$ ➔
$\angle DAB = 30°$ ➔
$\angle ADB = 180° - 60° - 30° = 90°$

Practice Exercises:

1. (Easy)
In the figure, what is the measure of $\angle B$?



2. (Easy)
In ΔABC, $\angle A = 60°$, $\angle B = \angle C$ and AC = 3
What is BC?

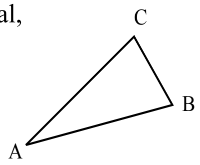
Answer: 1. $\angle B = 60°$; **2.** 3

Isosceles Triangle

If only two sides of a triangle are equal, it is called an isosceles triangle.

In the figure, you can see an isosceles triangle with AB = AC.

In an isosceles triangle, the two angles not formed by the crossing of the two equal sides are equal.
In the figure $\angle B = \angle C$



Example: (Medium)
If in ΔXYZ, XY = 3, $\angle X = 70°$ and $\angle Y = 55°$, then XZ = ?

Solution:
$\angle X = 70°$ and $\angle Y = 55°$ ➔
$\angle Z = 180° - 70° - 55° = 55°$
$\angle Y = \angle Z = 55°$ ➔
ΔXYZ is isosceles. ➔
XZ = XY = 3

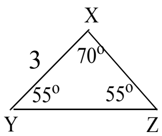
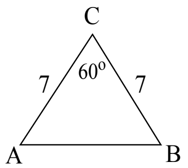


Figure is not drawn to scale.

Practice Exercises:

1. (Easy)
If in ΔXYZ, XY = YZ = 4, and $\angle Y = 100°$, then what is the measure of $\angle X$?

2. (Easy)
In the figure, what is the measure of $\angle A$, $\angle B$ and AB?

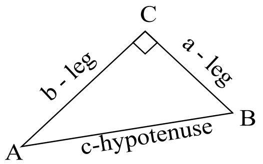


Answer: 1. 40°; **2.** $\angle A = \angle B = 60°$ and AB = 7

Right Triangle

A triangle is a right triangle if one of its inner angles is $90°$, as shown in the figure.

Hypotenuse is the longest side of a right triangle. It is the side opposite to the 90° angle.



In the figure, c is the hypotenuse of ΔABC.
The other two sides, a and b, are called the **legs** of the right triangle.

Pythagorean Theorem

In a right triangle, $a^2 + b^2 = c^2$, where a and b are the two legs and c is the hypotenuse of the triangle.

Example: (Easy)
In the figure above, if a = 3, b = 4, then c = ?

Solution:
$c = \sqrt{9 + 16} = 5$

Practice Exercise:

1. (Easy)
In a right triangle, if the hypotenuse is 6 and one of the legs is 4, what is the length of the other leg?

Answer: $2\sqrt{5}$

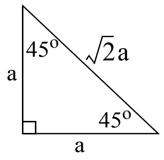
Special Right Triangles

There are two special right triangles.
They are $45°$-$45°$-$90°$ and $30°$-$60°$-$90°$ triangles.

Isosceles Right Triangle - 90° - 45° - 45°

General form of isosceles right triangle is shown in the figure.

If a right triangle is also an isosceles triangle, its angles are 90°, 45° and 45°, making the total 180°.



If the lengths of both of the legs of an isosceles right triangle are a, then the hypotenuse is $\sqrt{2}a$.

Examples:

1. (Easy)
If the length of one of the legs of an isosceles right triangle is 5, then the hypotenuse is $\sqrt{2} \cdot 5 \cong 7.07$

2. (Easy)
If the hypotenuse of an isosceles right triangle is 5, what are the lengths of its legs?

Solution:
Let the leg length be x. Then $\sqrt{2}x = 5$ ➔
$x = \frac{5}{\sqrt{2}} = 3.54$

Practice Exercises:

1. (Easy)
Hypotenuse of 90- 45- 45 triangle is $\sqrt{2}$, what are the lengths of its legs?

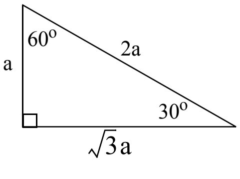
2. (Easy)
One angle of an isosceles triangle is 90^o, and the shortest side is $\sqrt{2}$. What are the other sides and angles?

3. (Easy)
Use the Pythagorean Theorem to prove that, if the shortest side length of an isosceles right triangle is a, then the hypotenuse is $\sqrt{2}a$.

4. (Easy)
Use the Pythagorean Theorem to prove that, if the hypotenuse of an isosceles right triangle is b, then the other side lengths are $b/(\sqrt{2})$.

Answers: 1. Both are 1; **2.** Side lengths are $\sqrt{2}$ and 2, angles are both 45^o; **3.** No answer is provided.; **4.** No answer is provided.

30^o - 60^o - 90^o Triangle



General form of 30^o - 60^o - 90^o triangle is shown in the figure.

If the length of the shortest leg of a 30^o - 60^o - 90^o triangle is a, then the length of the longer leg is $\sqrt{3}a$ and the length of the hypotenuse is 2a.

Note that the hypotenuse is twice as long as the shortest leg.

Examples:

1. (Easy)
If the shortest side length of a 30^o - 60^o - 90^o triangle is 5, then the other leg is $\sqrt{3} \cdot 5 \cong 8.66$ and the hypotenuse is 10.

2. (Easy)
If the hypotenuse of a 30^o - 60^o - 90^o is 6, what are the lengths of its legs?

Solution:
Let the shortest side length be x.
Then 6 = 2x ➔ x = 3
The length of the longer leg is $\sqrt{3}x = 3\sqrt{3} \cong 5.20$

Practice Exercises:

1. (Easy)
Hypotenuse of 30^o - 60^o - 90^o triangle is $\sqrt{3}$. What are the lengths of its legs?

2. (Easy)
One angle of a right triangle is 30^o, and the shortest side is $\sqrt{3}$. What are the other side lengths and the angles?

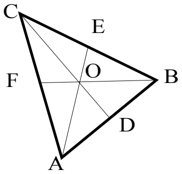
3. (Easy)
Use the Pythagorean Theorem to prove that, if the shortest side length of a 30^o - 60^o - 90^o triangle is a, then the hypotenuse is $2a$.

4. (Easy)
Use the Pythagorean Theorem to prove that, if the hypotenuse of a 30^o - 60^o - 90^o triangle is b, then the lengths of its legs are $b/2$ and $\sqrt{3}b/2$

Answers: 1. $\sqrt{3}/2$, 3/2; **2.** Side lengths: 3, $2\sqrt{3}$ and Angles: 60^o, 90^o; **3.** No answer is provided.; **4.** No answer is provided.

Medians and Centroid of a Triangle

In the figure, ABC is a triangle. D bisects $\overline{AB}$, E bisects $\overline{BC}$ and F bisects $\overline{AC}$.

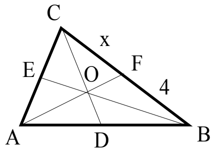


$\overline{AE}$, $\overline{CD}$ and $\overline{BF}$ are called **medians** of ΔABC.

Three medians intersect each other on one point, O. This intersection is called the **centroid** of ΔABC.

Example: (Medium)

In the figure, if AE = EC and AD = DB, then x = ?



Solution:

AE = EC ➔ $\overline{EB}$ is a median and

AD = DB ➔ $\overline{CD}$ is a median.

Intersection, O, of $\overline{EB}$ and $\overline{CD}$ is the centroid of ΔABC.

Since $\overline{AF}$ is passing through centroid O, $\overline{AF}$ is the third median of ΔABC, bisecting $\overline{BC}$. Therefore,

x = CF = FB = 4

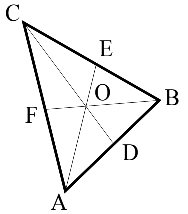
Practice Exercise:

1. (Medium)
   In ΔABC, $\overline{CD}$ is a median, CD = AD, $\angle B = 20°$ and $\overline{CE} \perp \overline{AB}$. What is the measure of $\angle ECD$?

Answer: 1. 50^o

Angle Bisectors and Incenter of a Triangle

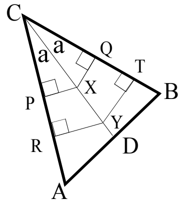
In the figure, $\overline{CD}$ bisects $\angle ACB$, $\overline{AE}$ bisects $\angle CAB$ and $\overline{BF}$ bisects $\angle CBA$.



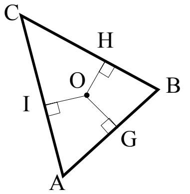
$\overline{AE}$, $\overline{CD}$ and $\overline{BF}$ are called **angle bisectors** of ΔABC.

Each angle bisector is a collection of points that are equal distance from its arms.

In the figure, $\overline{CD}$ is the angle bisector of $\angle ACB$. XP = XQ and YR = YT



Three bisectors intersect each other on one point, O. This intersection is called the **incenter** of ΔABC



Since incenter O lies on all three angle bisectors, it is at equal distance to the three sides of the triangle.

In the figure, OG = OH = OI

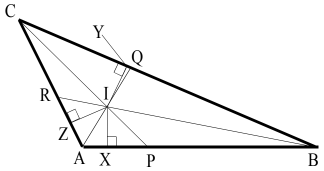
Note that $\overline{AH}$, $\overline{BI}$ and $\overline{CG}$ are <u>not</u> the angle bisectors of ΔABC.

Examples:

1. (Easy)
   Draw a triangle with two vertices 120^o and 20^o. Draw its angle bisectors and show the three distances of the incenter from the three sides of the triangle.

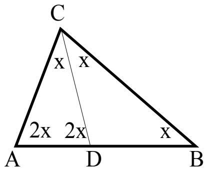
Solution:

In the figure,
$\angle A = 120°$,
$\angle B = 20°$
$\overline{AQ}$, $\overline{BR}$ and $\overline{CP}$ are angle bisectors.
IX = IY = IZ is the distance of incenter of ΔABC from its sides.



2. (Medium)
   In ΔABC, $\overline{CD}$ is an angle bisector. If AC = CD = DB, what is the measure of $\angle ACB$?

Solution:



In the figure, $x^o = \angle ACD$
$\overline{CD}$ is an angle bisector ➔
$\angle DCB = x^o$
CD = DB ➔ $\angle DBC = x^o$

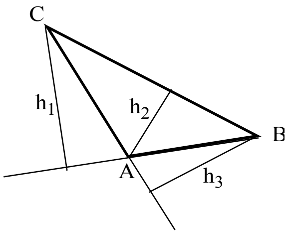
$\angle ADC$ is an outer angle of ΔCDB. ➔
$\angle ADC = \angle DCB + \angle DBC = 2x^o$
AC = CD ➔ $\angle CAD = \angle CDA = 2x^o$
Inner angles of $\Delta ADC = 180^o$ ➔ $5x^o = 180^o$ ➔
$x = 36^o$ ➔ $\angle ACB = 2x^o = 72^o$

The Base, Height and the Area of a Triangle

The area of a triangle is $\left(\text{base} \times \dfrac{\text{height}}{2}\right)$.

Base is the length of ANY one of the sides of the triangle. **Height** is the length of the line segment from the base to the opposite corner.

Base	Height	Area
AB	h_1	$\dfrac{AB \times h_1}{2}$
BC	h_2	$\dfrac{BC \times h_2}{2}$
AC	h_3	$\dfrac{AC \times h_3}{2}$



Above table provides the area of ΔABC by using three different set of bases and heights. All three areas displayed in this table are equal, because each one is the area of the same triangle ABC.

$$\frac{AB \times h_1}{2} = \frac{BC \times h_2}{2} = \frac{AC \times h_3}{2}$$

Also note that the height does not have to be inside the original triangle. In the above figure, only h_2 is inside ΔABC, h_1 and h_3 are both outside the triangle.

Example: (Medium)

In the above table, if AB = 2, AC = 3, $h_1 = 2.5$, then $h_3 = ?$

Solution:

$$\frac{AB \times h_1}{2} = \frac{AC \times h_3}{2} \Rightarrow 2 \times 2.5 = 3 \times h_3 \Rightarrow$$

$$h_3 = \frac{2 \times 2.5}{3} = \frac{5}{3}$$

Practice Exercises:

1. (Easy)

In the figure, if BC = 4 and AD = 3

What is the area of ΔABC ?

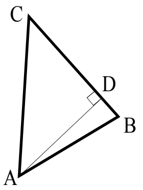


Figure is not drawn to scale.

2. (Medium)

In the figure, $\overline{CD}$ bisects $\overline{AB}$ and $\overline{CE}$ bisects $\angle ACB$.

Which of the following statements is true?

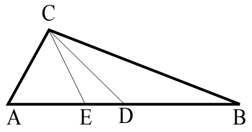


Figure is not drawn to scale.

I. AD = DB

II. Area of ΔADC = Area of ΔDBC

III. Area of ΔAEC = Area of ΔEBC

(A) I only

(B) II only

(C) III only

(D) I and II

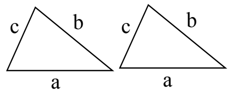
(E) I and III

Answers: 1. 6; **2.** (D)

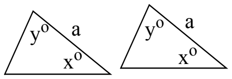
Congruent Triangles

Two triangles are congruent if they have the same shape and size. Below are a list of conditions that makes two triangles congruent, i.e. having the same shape and size.

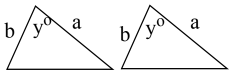
- Two triangles are congruent if their corresponding sides are equal in length.



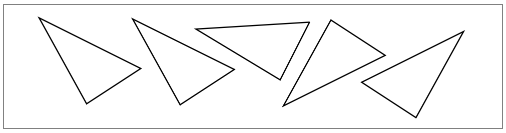
- Two triangles are congruent if one side and both of the neighboring angles are equal.



- Two triangles are congruent if the measure of one angle and the lengths of both neighboring sides are equal.



- You can obtain congruent triangles by moving, rotating or by taking the mirror images of any triangle. The triangles in the below figure are all congruent.

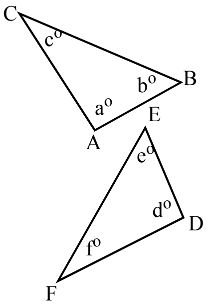


- The symbol ≅ represents congruency between the triangles. If $\Delta ABC \cong \Delta XYZ$, then AB = XY, AC = XZ, BC = YZ and $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$. It is important that you use the corresponding corners of the triangle to match the equal angles and sides. For example, if $\Delta ABC \cong \Delta XYZ$, then $\angle A = \angle X$, but $\angle A \neq \angle Y$ or $\angle A \neq \angle Z$.

Examples:

1. (Easy)

Figure shows two congruent triangles. Which angles and side lengths are equal?



Solution:

$a^o = d^o$, $b^o = e^o$, $c^o = f^o$

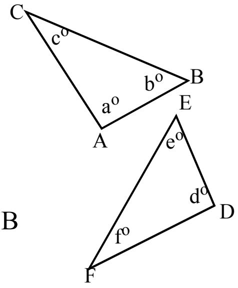
AB = DE, AC = DF, BC = EF

2. Consider the two triangles in the previous example. Which of the following is true?

I. $\Delta ABC \cong \Delta DEF$
II. $\Delta ACB \cong \Delta DEF$
III. $\Delta BCA \cong \Delta EFD$

- **(A)** I only
- **(B)** II only
- **(C)** I & II only
- **(D)** I & III only
- **(E)** I, II & III

Solution:



In the figure,
$a^o = d^o$, $b^o = e^o$, $c^o = f^o$
AB = DE, AC = DF, BC = EF ➔
$\Delta ABC \cong DEF$ or
$\Delta BCA \cong \Delta EFD$
In both of these cases
$\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$ and $\angle C$ corresponds to $\angle F$.
Hence Both I and III are correct.

In II, $\angle B$ corresponds to $\angle F$ and $\angle C$ corresponds to $\angle E$, which is wrong. Hence the answer is (D).

3. (Easy)
If $\Delta BAC \cong \Delta ZXY$, which angles and side lengths are equal?

Solution:

$\angle A = \angle X$, $\angle B = \angle Z$, $\angle C = \angle Y$ and

$AB = XZ$, $AC = XY$, $BC = ZY$

This exercise shows that you don't need to have the figure to decide which angles/sides are corresponding angles/sides.

4. (Easy)
In ΔABC, A = 72^o, B = 16^o and AB = 5
In ΔDEF, D = 72^o, E = 92^o and DF = 5

Is $\Delta ABC \cong \Delta DFE$?

Solution:

$$\left.\begin{array}{l} AB = DF = 5 \\ \angle A = \angle D = 72^o \\ \angle F = 180 - 72 - 92 = 16^o \\ \angle F = \angle B \end{array}\right\} \Rightarrow \Delta ABC \cong \Delta DFE$$

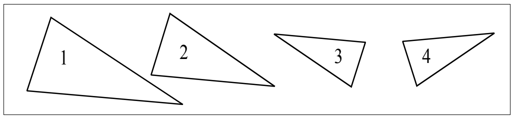
Since two corresponding angles and the sides in between these angles are equal, $\Delta ABC \cong \Delta DFE$

Practice Exercises:

1. (Easy)
If ΔOPQ and ΔXYZ are congruent, which of the following statements is <u>wrong</u>?

- **(A)** OQ = XZ
- **(B)** $\angle P = \angle Y$
- **(C)** OP = ZX
- **(D)** QP = YZ
- **(E)** $\Delta XYZ \cong \Delta OPQ$

2. (Easy)
Which of the following triangles are congruent?



- **(A)** 1 and 2
- **(B)** 2 and 3
- **(C)** 3 and 4
- **(D)** 1 and 2 and 3
- **(E)** 1 and 2 and 3 and 4

3. (Easy)
ΔABC is an equilateral triangle with AB = 3. ΔDEF is another equilateral triangle with EF = 3. Is $\Delta ABC \cong \Delta DEF$?

4. (Medium)
ΔABC is an isosceles triangle with A = 110^o and the length of its longest side is 6. ΔDEF is another isosceles triangle with D = 35 and the length of its longest side is 6. Which of the following can <u>never</u> be correct?

I. $\Delta ABC \cong \Delta DEF$
II. $\Delta CBA \cong \Delta DEF$
III. $\Delta BAC \cong \Delta DEF$

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I and II and III
- **(E)** None

5. (Medium)
$\overline{BD}$ bisects $\angle ADC$ and $\angle ABC$.

Is $\Delta ABD \cong \Delta CBD$?

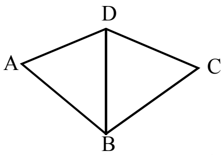


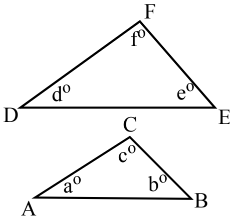
Figure is not drawn to scale

Answers: 1. (C); **2.** (C); **3.** Yes; **4.** (A); **5.** Yes

Similar Triangles

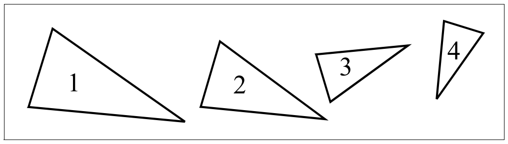
- **Two triangles are similar if they have the same corresponding angles.**

  For example, in the figure, ΔABC and ΔDEF are similar if $a = d$, $b = e$, $c = f$



- **Two triangles are similar if they have proportional sides.**
  For example, in the figure, ΔABC and ΔDEF are similar if $AB/DE = AC/DF = BC/EF$
- The symbol ~ represents similarity between the triangles.

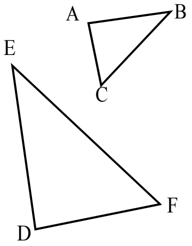
  For example, in the figure, $\Delta ABC \sim \Delta DEF$
- The corresponding sides of similar triangles can be made parallel by rotating or by taking mirror images of one of the triangles. In the below figure, all four triangles are similar.



The first two already have parallel sides. The third triangle's sides can be made parallel to the first two by turning it upside down. The fourth triangle's sides can be made parallel to the first two by rotating it 90^o counterclockwise.

Examples:

1. (Easy)
Two triangles in the figure are similar. Write down the relationship between them.



Solution:

$\Delta ABC \sim \Delta DEF$ and

$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and

$AB/DE = AC/DF = BC/EF$

2. (Medium)
Consider two triangles, ΔABC and ΔDEF. If $\angle A = \angle D$, $\angle B = \angle E$, $AC = 2$, $AB = 4$ and $FD = 1.5$, then $DE = ?$

Solution:

$\angle A = \angle D$ and $\angle B = \angle E \Rightarrow \Delta ABC \sim \Delta DEF$. Note that if the two angles of two triangles are equal, their third angle must also be equal because the sum of inner angles is 180^o for both triangles.

$\Delta ABC \sim \Delta DEF \Rightarrow AB/AC = DE/DF \Rightarrow 4/2 = DE/1.5 \Rightarrow DE = 3$

Practice Exercises:

1. (Easy)
If ΔOPQ and ΔXYZ are similar, which of the following statements must be true?

(A) $OQ = XZ$

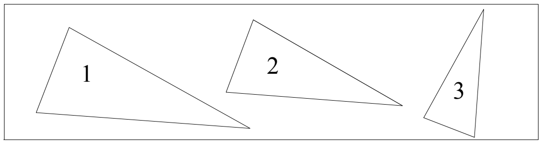
(B) $\angle P = \angle Y$

(C) $OP = ZX$

(D) $QP = YZ$

(E) $\Delta XYZ \cong \Delta OPQ$

2. (Easy)
To your best judgement, which of the following triangles are similar?



(A) 1 and 2

(B) 1 and 3

(C) 2 and 3

(D) 1 and 2 and 3

(E) None

3. (Medium)
ΔABC is an equilateral triangle with $AB = 3$
ΔDEF is another equilateral triangle with $EF = 11$.
Is $\Delta ABC \sim \Delta DEF$?

4. (Medium)
ΔABC is an isosceles triangle with $A = 110^o$ and length of longest side is 6. ΔDEF is another isosceles triangle with $D = 35$ and the length of its longest side is 17. Which of the following can <u>never</u> be correct?

(A) $\Delta ABC \sim \Delta DEF$

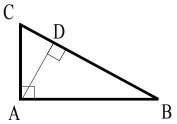
(B) $\Delta CBA \sim \Delta DEF$

(C) $\Delta BAC \sim \Delta DEF$

(D) $\Delta ABC \sim \Delta EFD$

(E) $\Delta CAB \sim \Delta FED$

5. (Hard)
In the figure, which of the following statements is <u>wrong</u>?



(A) $\Delta ADB \sim \Delta CAB$

(B) $\Delta ADC \sim \Delta ADB$

(C) $\Delta ADC \sim \Delta BDA$

(D) $\Delta ABC \sim \Delta DAC$

(E) $\Delta ABC \sim \Delta DBA \sim \Delta DAC$

Hint: Identify the three triangles in the figure. Write down their corresponding equal angles.

6. (Hard)
In the figure, calculate the area of ΔABC.

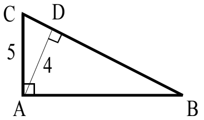
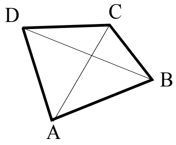


Figure is not drawn to scale.

Answers: 1. (B); **2.** (D); **3.** Yes; **4.** (A); **5.** (B); **6.** 50/3

Quadrangles: 4 - Sided Polygons

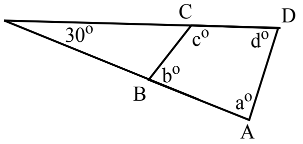


- A quadrangle is a 4-sided polygon that has 4 sides and 4 vertices as shown in the figure.
- $\overline{AC}$ and $\overline{BD}$ are the two diagonals.
- The inner angles of a 4-sided polygon is (4 - 2)180 = 360^o

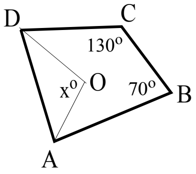
Practice Exercises:

1. (Easy)
Two inner angles of a quadrangle are both 90^o. The third angle is twice as large as the fourth angle. What are the other two angles of the quadrangle?

2. (Medium)
In the figure, what is c + b?



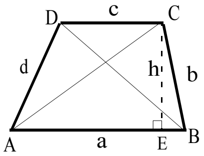
3. (Medium)
In the figure, if $\overline{AO}$ and $\overline{DO}$ are the angle bisectors of $\angle A$ and $\angle D$, respectively, what is x^o?



Answers: 1. 60^o and 120^o; **2.** 210^o; **3.** 100^o

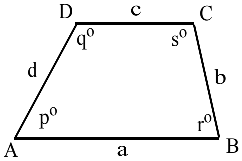
In SAT, there are four types of special quadrangles: Trapezoids, Parallelograms, Rectangles and Squares.

Trapezoids



- A trapezoid is a special quadrangle with two parallel and two non-parallel sides.

  In the figure, ABCD is a trapezoid, with $\overline{AB} \parallel \overline{CD}$.
- Height of a trapezoid is the distance between its parallel sides. In the figure, the height, h = EC.
- $\overline{AC}$ and $\overline{DB}$ are the two diagonals of the trapezoid.
- Area of a trapezoid =
  $\frac{1}{2}$ (Addition of the parallel sides)x(Height)
  In the above figure, the area = $\frac{1}{2}(a + c) \times h$
  For example, if a = 3 and c = 2 and EC = 6, then the area of the trapezoid is
  $\frac{1}{2}(3 + 2) \times 6 = 15$

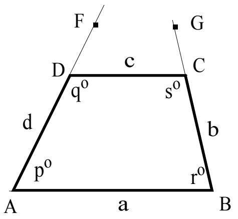


- In a trapezoid, the addition of the adjacent angles on each side of the <u>non-parallel</u> side is 180^o.

  In the figure,
  $p^o + q^o = r^o + s^o = 180^o$
  For example, if $\angle B = 66°$, then $\angle C = 180° - 66° = 114°$

Example: (Easy)
Prove that in a trapezoid, the addition of the adjacent angles on each side of the non-parallel side is 180^o.

<u>*Solution:*</u>



In the figure, $\overline{AB} \parallel \overline{CD}$ ➔
$p° = \angle CDF$

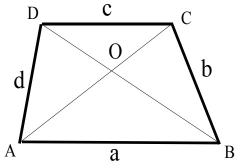
$q° + \angle CDF = 180°$ ➔
$p^o + q^o = 180^o$

Similarly, $\overline{AB} \parallel \overline{CD}$ ➔
$r° = \angle GCD$

$s° + \angle GCD = 180°$ ➔
$r^o + s^o = 180^o$

Practice Exercises:

1. (Medium)
Consider the trapezoid in the figure.
a, b, c and d are the four distinct lengths of the four sides of the trapezoid.



Which of the following statements is <u>wrong</u>?

I. $\Delta AOB \sim \Delta COD$
II. $\Delta ADO \sim \Delta BCO$
III. If $\angle DAB = 44°$, then $\angle DCB = 136°$

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I and III
- **(E)** II and III

2. (Medium)
ABCD is a trapezoid with EC = x and ED = 3x.

If the area of ABCD is 19.5, what is the area of the shaded region?

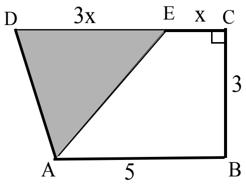


Figure is not drawn to scale.

Answer: 1. (E); **2.** 9

Parallelogram - A Special Trapezoid

- A parallelogram is a special trapezoid with **parallel opposite sides** as shown in the figure.

  $\overline{AD} \parallel \overline{BC}$ and

  $\overline{AB} \parallel \overline{DC}$

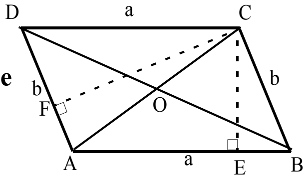
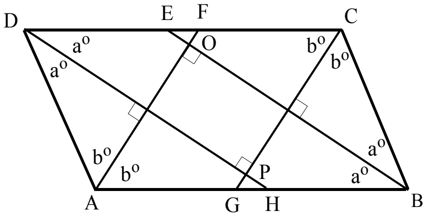


Figure A

- **Heights and Bases**
  The two heights of a parallelogram are the distances between its opposite parallel sides. In Figure A, the two heights are CE and CF.

  $\overline{AD}$ and $\overline{AB}$ are two bases for the heights CF and CE, respectively.
- **Diagonals**
  In Figure A, $\overline{AC}$ and $\overline{BD}$ are the diagonals of the parallelogram. Two diagonals of a parallelogram are <u>not</u> equal in length.
- **The opposite angles are equal**
  $\angle ADC = \angle ABC$ and $\angle DAB = \angle BCD$
- **Addition of the neighboring angles is 180°**
  $\angle ADC + \angle DAB = 180°$
  $\angle ADC + \angle DCB = 180°$
  $\angle DCB + \angle CBA = 180°$
  $\angle CBA + \angle BAD = 180°$
- **Two diagonals of a parallelogram bisect each other.**
  In Figure A, AO = CO and DO = BO
- **In a parallelogram, angle bisectors of the neighboring angles are perpendicular to each other.**



In the figure $\overline{BE} \perp \overline{CG}$, $\overline{DH} \perp \overline{AF}$, $\overline{BE} \perp \overline{AF}$ and $\overline{DH} \perp \overline{CG}$ where $\overline{BE}$, $\overline{AF}$, $\overline{DH}$ and $\overline{CG}$ are the angle bisectors.

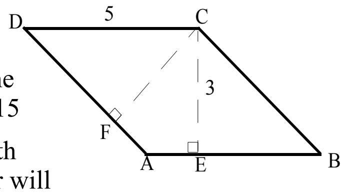
It is quite simple to prove the above statement. Here is how:

The inner angles of ΔAOB is 180^o ➔
$\angle AOB = 180^o - (b^o + a^o)$
The neighboring angles of ABCD is 180^o ➔
$2a^o + 2b^o = 180^o$ ➔ $a^o + b^o = 90^o$

Substituting the second equation into the first one:
$\angle AOB = 180 - 90 = 90°$

- **Area of a parallelogram = Base × Height**

For example, the area of the parallelogram in the figure is $3 \times 5 = 15$



Note that using both (Base, Height) pair will yield the same result.

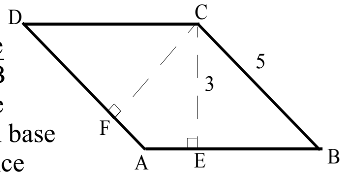
For example, in the figure,
$\text{Area} = AD \times CF = AB \times CE = 15$

Examples:

1. (Medium)
ABCD is a parallelogram with AB = 4, BC = 5 and the distance between C and $\overline{AB}$ is 3. What is the distance between C and $\overline{AD}$?

Solution:

As shown in the figure, the distance between C and $\overline{AB}$ is one height of the parallelogram with base $\overline{AB}$ and the distance between C to $\overline{AD}$ is the other height with base $\overline{AD}$.



The area of ABCD is $AB \times CE = 4 \times 3 = 12$
The same area is also $AD \times CF = 5 \times CF = 12$
➔ CF = 12/5 = 2.4

2. (Hard)
In the figure, ABCD is a parallelogram.

What is DF?

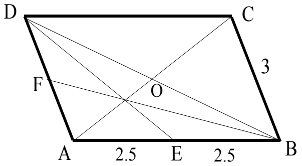


Figure is not drawn to scale.

Solution:

Because $\overline{AC}$ and $\overline{BD}$ are diagonals of ABCD, $\overline{AC}$ bisects $\overline{BD}$.

Hence $\overline{AO}$ is a median of ΔABD

Since AE = EB = 2.5, $\overline{DE}$ is another median of ΔABD.

Since the three medians of a triangle crosses at the same point, $\overline{BF}$ has to be the third median of ΔABD. Therefore DF = AF = 3/2 = 1.5

3.

In the figure, ABCD is a parallelogram.
$\angle B = 45°$

Calculate:

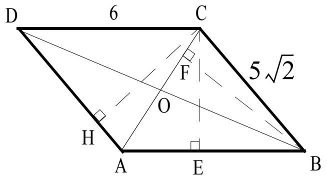


Figure is not drawn to scale

a. (Medium)
The height, EC.

b. (Hard)
The length of the diagonal $\overline{AC}$.

c. (Hard)
BF

d. (Hard)
The second height, CH

e. (Hard)
The length of the second diagonal $\overline{BD}$.

Solution:

a. $B = 45^o$ and $E = 90^o$ means $\angle ECB = 45°$ and ΔCEB is an isosceles right triangle with EC = EB.

Using the Pythagorean Theorem:

$(5\sqrt{2})^2 = EC^2 + EB^2$ ➔ $50 = 2EC^2$ ➔
EC = 5

b. AE = AB - EB ➔ AE = 6 - 5 = 1

Using the Pythagorean Theorem for ΔAEC:

$AC^2 = AE^2 + EC^2 = 1 + 25 = 26$ ➔
$AC = \sqrt{26}$

c. The area of

$$\Delta ABC = \frac{AB \times CE}{2} = \frac{6 \times 5}{2} = 15$$

The same area =

$$15 = \frac{AC \times BF}{2} = \frac{\sqrt{26} \times BF}{2} \text{ ➔}$$

$$BF = \frac{15 \times 2}{\sqrt{26}} = \frac{30}{\sqrt{26}}$$

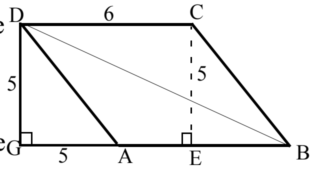
d. $\Delta ABC \cong \Delta CDA$ ➔

The area of ΔCDA =
The area of $\Delta ABC = 15$ ➔

$$\frac{DA \cdot HC}{2} = 15 \text{ ➔ } \frac{5\sqrt{2} \cdot HC}{2} = 15 \text{ ➔}$$

$$HC = \frac{6}{\sqrt{2}}$$

e. To calculate the second diagonal, let's draw the figure again with some additions.



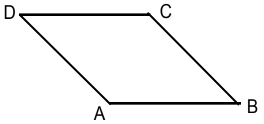
We have also removed some of the elements in the figure for clarity.

In the above figure, B, A and G are on the same line and $\overline{DG} \parallel \overline{EC}$.

$\Delta CEB \cong \Delta DGA$ ➔ ΔDGA is an isosceles right triangle with
GD = AG = EC = 5
GB = GA + AB = 5 + 6 = 11
Using the Pythagorean Theorem for ΔBDG:
$BD^2 = DG^2 + BG^2 = 5^2 + 11^2 = 25 + 121 = 146$ ➔ $BD = \sqrt{146}$

Practice Exercises:

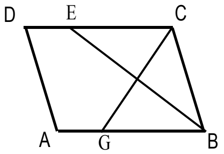
1. (Medium)
For the parallelogram in the figure, which of the following statements is true?



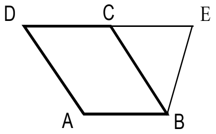
(A) Area of ΔABC - Area of $\Delta ABD > 0$
(B) Area of ΔABC - Area of $\Delta ABD < 0$
(C) Area of ΔABC - Area of $\Delta ABD = 0$
(D) Area of ΔABC - Area of $\Delta ABD \geq 0$
(E) Area of ΔABC - Area of $\Delta ABD \leq 0$

2. (Medium)
What are the two possible values for the side lengths of a parallelogram if the two heights are 2 and 3 and one side length is 4?

3. (Medium)
In the figure ABCD is a parallelogram. $\overline{CG}$ and $\overline{BE}$ are bisectors of $\angle C$ and $\angle B$ respectively.
Prove that $\overline{BE} \perp \overline{CG}$



4. (Medium)
In the figure, ABCD is a parallelogram, D, C and E are on the same line and $\overline{BE} \parallel \overline{AC}$. $\overline{AC}$ is not shown in the figure.



Which of the following statements is true?

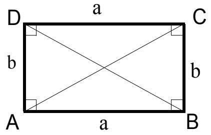
I. Area of ABED = Area of $\Delta ABD \times 4$
II. Area of ABED = $\frac{3}{2} \times$ Area of ABCD
III. Area of ΔABD - Area of $\Delta BCE = 0$

(A) I only
(B) II only
(C) III only
(D) I and III
(E) II and III

Answers: 1. (C); **2.** 8/3 and 6; **3.** See the section about the bisectors; **4.** (E)

Rectangle - A Special Parallelogram

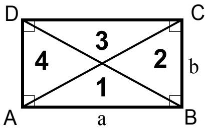
- A rectangle is a special parallelogram where all the inner angles are 90°, as shown in the figure.



- As it is true for parallelogram, in a rectangle, the opposite sides are equal in length and parallel to each other.
- In the above figure, a and b are the **length** and the **width** (or **height**) of the rectangle. AC and BD are the diagonals of the rectangle.
- Two **diagonals** of a rectangle are equal in length. For example, AC = BD in the above figure.
- The length of the diagonals is $AC = BD = \sqrt{a^2 + b^2}$ For example, if ABCD is a rectangle with AB = 4 and BC = 3, then the length of the diagonals of the rectangle is:
  $AC = BD = \sqrt{4^2 + 3^2} = 5$
- The **area** of a rectangle = Width × Length. In the above figure, the area of ABCD is ab.

Example:

Consider a rectangle with width 5 and length 6.
The area of this rectangle is $5 \times 6 = 30$.



- The areas of the 4 triangles in the figure are the same.
  The area of triangle 1 =
  The area of triangle 2 =
  The area of triangle 3 =
  The area of triangle 4 = (ab)/4

Practice Exercises:

1. (Easy)
What is the diagonal length of a rectangle with width 2 and length 3?

2. (Easy)
What is the diagonal length of a rectangle with width x and length y?

3. (Easy)
What is the diagonal length of a rectangle with width $\sqrt{3}$ and length 3?

4. (Easy)
What is the diagonal length of a rectangle with width w and length 3?

5. (Easy)
What is the area of a rectangle with width 2 and length 3?

6. (Easy)
What is the area of a rectangle with width x and length y?

7. (Easy)
What is the area of a rectangle with width $\sqrt{3}$ and length 3?

8. (Easy)
What is the area of a rectangle with width w and length 3?

9. (Easy)
What is the area of ΔAOB in the figure?

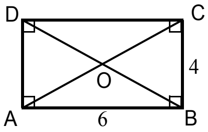
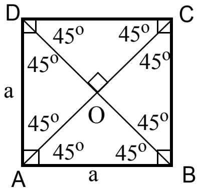


Figure is not drawn to scale.

Answers: 1. $\sqrt{13}$; **2.** $\sqrt{x^2 + y^2}$; **3.** $2\sqrt{3}$; **4.** $\sqrt{w^2 + 9}$; **5.** 6; **6.** xy; **7.** $3\sqrt{3}$, **8.** 3w; **9.** 6

Square: a Special Rectangle



- Square is special rectangle with width = length = a as shown in the figure.
- The area of ABCD is a^2.
- Diagonals of a square are perpendicular to each other.
- Diagonals of a square are also the angle bisectors.
- The diagonals of a square divides the square into 4 congruent triangles.

  In the above figure, they are ΔABO , ΔBCO , ΔCDO and ΔADO .
  The area of each triangle is $a^2/4$

Practice Exercises:

1. (Easy)
What is the diagonal length of a square with side length 3?

2. (Easy)
What is the diagonal length of a square with side length y?

3. (Easy)
What is the diagonal length of a square with side length $\sqrt{3}$?

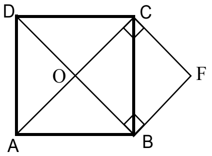
4. (Easy)
What is the area of a square with side length 3?

5. (Easy)
What is the area of a square with side length y?

6. (Easy)
What is the area of a square with side length $\sqrt{3}$?

7. (Medium)
ABCD is a square with area 12.

What is the area of OBFC?



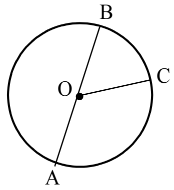
Hint: Don't calculate the lengths. Count the triangles.

Answers: 1. $3\sqrt{2}$; **2.** $\sqrt{2}y$; **3.** $\sqrt{6}$, **4.** 9; **5.** y^2; **6.** 3; **7.** 6

Circles

- A circle is a collection of points that are at equal distance from a fixed point called the **center of the circle**.

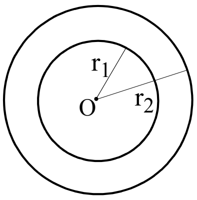
  In the figure the center of the circle O is the point O.



- The line segment inside a circle and passing through its center is called the **diameter** of the circle. In the figure, $\overline{AB}$ is the diameter of the circle O.
- Line segment between the center of the circle and any point on the circle is called the **radius** of the circle.

  In the figure above, $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ are all radii of the circle O. By definition OA = OB = OC.
- Diameter = $2 \times \text{Radius}$. For example if the diameter of a circle is 5, its radius of the circle is 2.5
- The **area of a circle** with radius r is πr^2 , where pi, $\pi \cong 3.1416$. All scientific calculators have π on them. You don't need to memorize the value of this constant. However, it may be helpful if you remember that it is a little bit more than 3.
  For example the area of a circle with radius 4 is approximately $\pi \cdot 4^2 \cong 48$. Here the symbol " $\cong$" is used to indicate an approximation. More accurate answer is 50.27.
- The **circumference** of a circle with radius r is $2\pi r$
  For example, if a circle has radius 4, its circumference is $8\pi \cong 25.13$
- Two circles are said to be concentric if they have the same center but different radii.

  Two circles in the figure are concentric.



Practice Exercises:

1. (Easy)
What is the area of a circle with radius 3?

2. (Easy)
What is the area of a circle with diameter 3?

3. (Easy)
What is the area of a circle with radius $\sqrt{7}$?

4. (Easy)
What is the area of a circle with diameter d?

5. (Easy)
What is the area of a circle with radius π ?

6. (Easy)
What is the circumference of a circle with radius 3?

7. (Easy)
What is the circumference of a circle with diameter 3?

8. (Easy)
What is the circumference of a circle with radius $\sqrt{7}$?

9. (Easy)
What is the circumference of a circle with diameter d?

10. (Easy)
What is the circumference of a circle with radius π ?

11. (Easy)
What is the circumference of a circle with area π ?

12. (Easy)
What is the area of a circle with circumference π ?

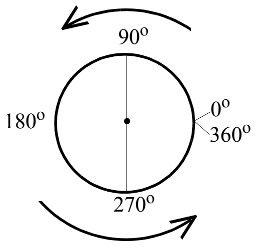
13. (Medium)
The ratio of the radii of the two concentric circles is 1/3. What is the ratio of their area?

Answers:

1. 9π ; **2.** $9\pi/4$; **3.** 7π ; **4.** $\pi d^2/4$; **5.** π^3 ; **6.** 6π ; **7.** 3π ; **8.** $2\sqrt{7}\pi$; **9.** πd ; **10.** $2\pi^2$; **11.** 2π ; **12.** $\frac{\pi}{4}$; **13.** 1/9

Arc of a Circle

- When you start from 0^o position, and travel counter clockwise you will pass through 90^o, 180^o and 270^o positions. When you travel the complete 360^o, you will reach the same position that you've started from.



Starting point and the direction of the turn is only for demonstration purposes. You can start from any point on the circle and go in either direction; after turning 360^o, you will reach the same point.

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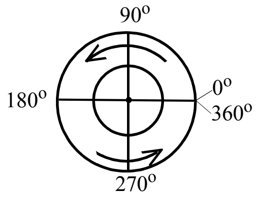
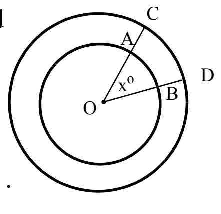
- Regardless of their sizes, all circles have 360^o around their center as shown in the figure. As you can see this is true for all circles, big or small.



- A segment of a circle is called the **arc of the circle**. For example in the figure, the curve AB is an arc of the circle O.

Arc AB is represented by $\overset{\frown}{AB}$.



In the figure,
the angle of $\overset{\frown}{AB}$ = the angle of $\overset{\frown}{CD}$ = x^o

- The length of an arc on a circle depends on both the radius of the circle and the angle traveled.

  The length of $\overset{\frown}{AB} = 2\pi r\,(x^o/360^o)$, where r is the radius of the circle and x^o is the angle traveled.

  For example, in the above figure, $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are the arcs of the two concentric circles with angle x^o. The length of the arc from A to B is smaller that the length of the arc from C to D even when the angle x^o is the same. This is because the radius of the inner circle is smaller than the radius of the outer circle. On the other hand, we can keep the radius the same and increase the angle traveled to increase the length of the arc.

- Three special arcs of a circle with radius r are

  a. Full circle itself (the longest arc of the circle) with length $2\pi r$ and $x^o = 360^o$

  b. Half circle with length πr and $x^o = 180^o$

  c. Quarter circle with length $\pi\frac{r}{2}$ and $x^o = 90^o$

Examples:

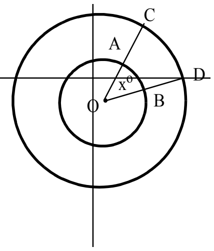
1. (Easy)
If the radius of a circle is 6 and the angle travelled from B to C on the circle is 30^o, what is the length of arc BC?

Solution:

The length of the arc BC =

$$2\pi r\left(\frac{x^\circ}{360^\circ}\right) = 12\pi\left(\frac{30}{360}\right) = \pi \cong 3.14$$

2. (Easy)
Two concentric circles are shown in the figure. If the radius of the outer circle is twice as long as the radius of the inner circle, what is the ratio of the length of $\overset{\frown}{AB}$ to the length of $\overset{\frown}{CD}$?



Solution:

Let r be the radius of the inner circle.
(The length of the $\overset{\frown}{AB}$)/(the length of the $\overset{\frown}{CD}$) =

$$\left(2\pi r\frac{x^\circ}{360^\circ}\right)/\left(2\pi 2r\frac{x^\circ}{360^\circ}\right) = \frac{1}{2}$$

Practice Exercises:

1. (Easy)
If the diameter of a circle is 6 and the angle

traveled from B to C on the circle is 63°, what is the length of $\overset{\frown}{BC}$?

2. (Easy)
If the distance travelled on a circle from B to C is 5 and the angle traveled is 45°, what is the radius of the circle?

3. (Easy)
For two concentric circles, the radius of the outer circle is twice as long as the radius of the inner circle. If you travel the same distance on both circles, what is the ratio of the angle of travel for the inner circle to the angle of travel for the outer circle?

4. (Medium)
In the figure, if the area of the outer circle is twice as large as the area of the inner circle, what is the ratio of the length of $\overset{\frown}{AB}$ to the length of $\overset{\frown}{CD}$?

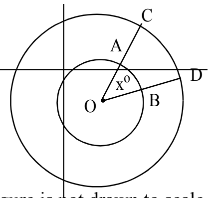
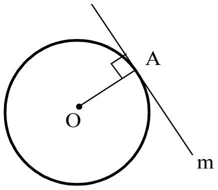


Figure is not drawn to scale.

Answers: 1. $\left(\frac{21}{20}\right)\pi$; **2.** $\frac{20}{\pi}$; **3.** 2; **4.** $\frac{1}{\sqrt{2}}$

Tangent to a Circle

- A line is tangent to a circle if it touches the circle at one and only one point. In the figure, line m is tangent to the circle O. It touches the circle at point A.

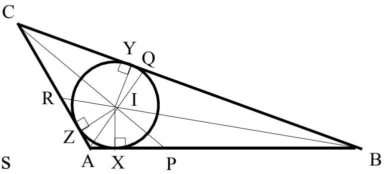


- A tangent line is always perpendicular to the radius of the circle at the point of contact.

  In the figure, $\overline{OA} \perp \overleftrightarrow{m}$

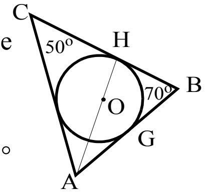
Circle Inscribed in an Triangle

- Since the incenter of ΔABC is at equal distance from the sides of ΔABC, it is the center of a circle inscribed in the triangle.



In the figure, the center of the circle, I, is the incenter of ΔABC. The 3 sides of ΔABC are tangent to the circle I at X, Y and Z.

Example: (Medium)
In the figure, circle O is inscribed in ΔABC. What is the measure of $\angle AHB$?



Solution:

$\angle A = 180° - 50° - 70° = 60°$

Since O is the center of a circle inscribed in ΔABC and

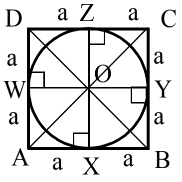
$\overline{AH}$ goes through O, $\overline{AH}$ has to be the angle bisector of ΔABC that divides $\angle A$ into two equal angles.

Hence $\angle HAB = \frac{60}{2} = 30° \Rightarrow$

$\angle AHB = 180 - 70 - 30 = 80^{o}$

Circle Inscribed in a Square

- In the figure, circle O is inscribed in square ABCD.
- The center of the circle, O, is the incenter of ABCD. The 4 sides of ABCD are tangent to the circle O at X, Y, Z and W.



- X, Y, Z and W bisect 4 sides of the square.
- The radius of the circle, a, is half of the side length, 2a.

Example: (Medium)
Circle O with area 25 is inscribed in square ABCD.

What is the area of ABCD?

Solution:

Let r be the radius of the circle.

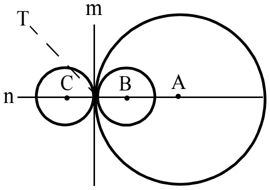
Area of the circle = $\pi r^2 = 25 \Rightarrow$

$r = \sqrt{\frac{25}{\pi}} = \frac{5}{\sqrt{\pi}} \Rightarrow$

Side length of ABCD = 2r $\Rightarrow$

Area of ABCD = $4r^2 = 4\left(\frac{5}{\sqrt{\pi}}\right)^2 = \frac{100}{\pi}$

Tangent Circles

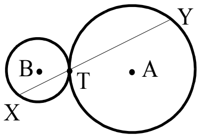


- Two circles are tangent to each other if they touch each other at one and only one point.

  Figure shows two different arrangements of tangent circles.

  Both circle B and circle C are tangent to circle A at point T.
- When two circles are tangent to each other, their centers and the point of contact, T, are aligned. In the figure A, B, C and T are all on line n.
- Line m, perpendicular to line n at point T, is tangent to all three tangent circles.

Example: (Medium)

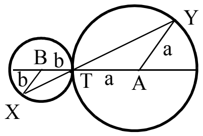


In the figure, circle A and circle B are tangent to each other at point T.

If $\angle TYA = 15°$, then

what is the measure of $\angle XBT$?

Solution:



ΔYAT is isosceles, because the length of 2 sides are both equal to the radius of circle A. ➔
$\angle YTA = \angle TYA = 15°$

$\angle XTB = \angle YTA = 15°$

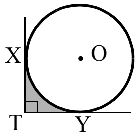
ΔXBT is isosceles, because the length of 2 sides are both equal to the radius of circle B. ➔
$\angle BXT = \angle XTB = 15°$ ➔
$\angle XBT = 180 - 15 - 15 = 150°$

Practice Exercises:

1. (Medium)
A circle is inscribed in an equilateral triangle. If the area of the circle is 9π, what is the area of the triangle?

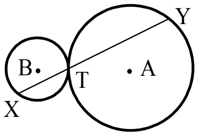
2. (Medium)
$\overline{TX}$ and $\overline{TY}$ are tangent to the circle O.

If the length of $\overset{\frown}{XY}$ is 5π, what is the area of the shaded region?



3. (Medium)
In the figure, circle A and circle B are tangent to each other at point T. $\angle TYA = 20°$ and BX = 3

What is the length of the arc $\overset{\frown}{XT}$?



Answers: 1. $27\sqrt{3}$; **2.** $100 - 25\pi$; **3.** $7\pi/3$

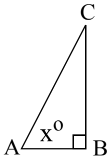
Trigonometry

In SAT, there will be questions about simple trigonometry. These questions can also be solved by using other methods as well. However, in many of these cases, using trigonometry can solve the problem faster and with less effort.

sin, cos, tan, cot

Definitions:

In a <u>right</u> triangle,



cos(x) =
(the Length of the Neighboring Leg) / (the Length of the Hypotenuse) = AB/AC

sin(x) =
(the Length of the Leg Across) /
(the Length of the Hypotenuse) = BC/AC

(tan(x) =
(the Length of the Leg Across) /
(the Length of the Neighboring Leg) = CB/AB

cot(x) =
(the Length of the Neighboring Leg) /
(the Length of the Leg Across) = 1/(tan(x) = AB/CB

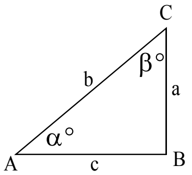
For example:

$$\cos(\alpha) = \frac{c}{b} \qquad \cos(\beta) = \frac{a}{b}$$

$$\sin(\alpha) = \frac{a}{b} \qquad \sin(\beta) = \frac{c}{b}$$

$$\tan(\alpha) = \frac{a}{c} \qquad \tan(\beta) = \frac{c}{a}$$

$$\cot(\alpha) = \frac{c}{a} \qquad \cot(\beta) = \frac{a}{c}$$



Note that:

- Tangent of an angle x is: tan(x) = sin(x)/cos(x) = 1/cot(x)
  cotangent of an angle x is: cot(x) = cos(x)/sin(x) = 1/tan(x)
- For $0° \le x \le 90°$ sin(x) and cos(x) are always between 0 and 1.
  sin(0) = 0 and sin(90) = 1
  cos(0) = 1 and cos(90) = 0
- For any given angle, x, $\sin^2(x) + \cos^2(x) = 1$
  Proof: Consider ΔABC in the above figure.
  $$\left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = \frac{BC^2 + AB^2}{AC^2} = \frac{AC^2}{AC^2} = 1$$
- Always remember that sin, cos, tan or cot of an angle does not depend on the side lengths of the right triangle, but they depend only on the measure of the angle.
- In a right triangle, the lengths of the sides do not matter, but their ratio does. In the above example, as long as the ratios, c/a, b/a, c/b etc., remain the same, sin, cos, tan and cot of α and β do not change.

Special Angles

In SAT, you will only deal with special angles. These angles are 45^o, 30^o and 60^o.

Here is what you need to know:

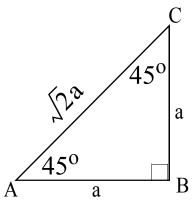
45^o

$$\cos(45°) = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin(45°) = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan(45°) = \frac{a}{a} = 1$$

$$\cot(45°) = \frac{a}{a} = 1$$



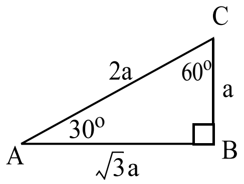
30^o

$$\cos(30°) = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\sin(30°) = \frac{a}{2a} = \frac{1}{2}$$

$$\tan(30°) = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

$$\cot(30°) = \frac{\sqrt{3}a}{a} = \sqrt{3}$$



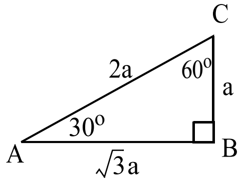
60^o

$$\cos(60°) = \frac{a}{2a} = \frac{1}{2}$$

$$\sin(60°) = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan(60°) = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\cot(60°) = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$



Examples:

1. (Easy)
If sin(x) = 0.8, what is cos(x)?

Solution:

$\sin^2(x) + \cos^2(x) = 1$ ➔ cos(x) =

$$\sqrt{1-0.8^2} = \sqrt{1-0.64} = \sqrt{0.36} = 0.6$$

2. (Medium)
What is the area of ΔABC ?

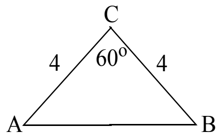
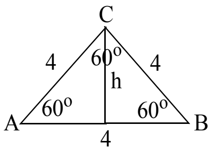


Figure is not drawn to sca



Solution:

Since $\angle C = 60°$ and AC = BC, ABC is an equilateral triangle with $\angle A = \angle B = 60°$ and AB = 4 as shown in the second figure.

The height h = 4sin(60) = $(4)(\sqrt{3}/2) = 2\sqrt{3}$ ➔

$$\text{Area} = \frac{4}{2} \times 2\sqrt{3} = 4\sqrt{3}$$

3. (Hard)
If tan(a) = 4, what is cos(a)?

Solution:

tan(a) = sin(a)/cos(a) = 4 ➔ $\sin^2(a)/\cos^2(a) = 16$ ➔

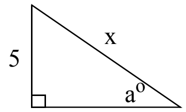
$\sin^2(a)/\cos^2(a) + 1 = 16 + 1 = 17$ ➔

$(\sin^2(a) + \cos^2(a))/\cos^2(a) = 17$ ➔

$1/\cos^2(a) = 17$ ➔ cos(a) = $1/(\sqrt{17})$

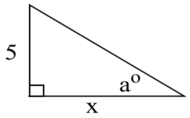
Practice Exercises:

1. (Easy)
What is x?



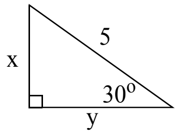
(A) 1/cos(a°)
(B) 5/cos(a°)
(C) sin(a°)
(D) 5sin(a°)
(E) 5/sin(a°)

2. (Easy)
What is x?

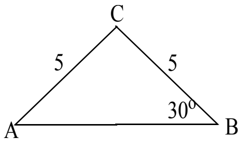


(A) 5cot(a°)
(B) 5tan(a°)
(C) 5cos(a°)
(D) 5sin(a°)
(E) 5/sin(a°)

3. (Easy)
What are x and y?



4. (Medium)
What is the area of ΔABC ?



(Figure is not drawn to scale.)

Answers: 1. (E); **2.** (A); **3.** x = 5/2, y = $5(\sqrt{3}/2)$;
4. $(25\sqrt{3})/4$

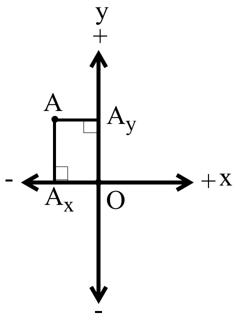
Coordinate Geometry

Points on the x-y Plane

Coordinates of a Point

Each point is represented by its x- and y- coordinates, (x,y) on the xy-plane.

x- and y- coordinates, A_x and A_y, of point A is shown in the figure. A_x is positive if it is on the right hand side of the origin O and it is negative if it is on the left hand side of O. A_y is positive if it is above the origin O and it is negative if it is below O, as indicated in the figure. In this case, A_x is negative and A_y is positive.

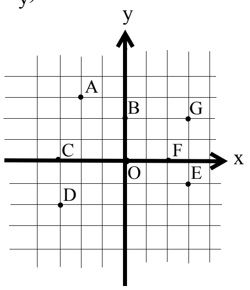


The absolute value of A_x and A_y depends on their distance from the origin, O.

Point A is represented as $A(A_x, A_y)$.

For example, in the figure, the coordinate representation of each point is as follows:

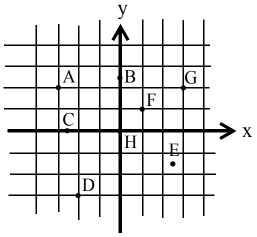
O(0,0); A(-2,3); B(0,2); C(-3,0); D(-3,-2); E(3,-1); F(2,0); G(3,2)



Each grid is one unit.

Practice Exercises:

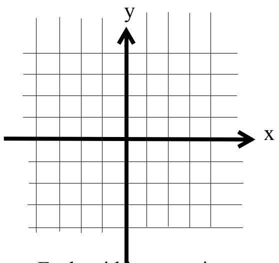
1. (Easy)
Give the coordinates of the points in the figure.



Each grid is one unit.

2. (Easy)
Place the below points on the xy-plane:

A(1/2,3); B(0,-2.5);
C(1/2,5/2); D(-2,-0.5);
E(0,0); G(-3,0)

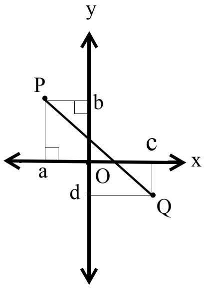


Each grid is one unit.

Answers: 1. A(-3, 2) B(0, 2.5) C(-2.5, 0) D(-2, -3) E(2.5, -1.5) F(1,1) G(3, 2), H(0,0);
2. No answer is provided.

Distance Between the Two Points

As shown in the figure, the distance between 2 points P(a, b) and Q(c, d) is
$PQ = \sqrt{(c-a)^2 + (d-b)^2}$



Examples:

1. (Medium)
What is the distance between A(1, 3) and B(2, 5)?

Solution:
$AB = \sqrt{(2-1)^2 + (5-3)^2} = \sqrt{1+4} = \sqrt{5}$

2. What is the distance between (-1, 3) and (-7, -2)

Solution:
It is:
$\sqrt{(-7-(-1))^2 + (-2-3)^2} = \sqrt{36+25} = \sqrt{61}$

Practice Exercises:

1. (Medium)
What is the distance between (-1, -2) and (-3, -4)?

2. (Medium)
If a distance between (3, -2) and (4, q) is 4, then q = ?

Answer: 1. $2\sqrt{2}$; **2.** $\sqrt{15} - 2$

Lines on x-y Plane

Slope of a Line

Slope of a line is the steepness of the line.

Positive and Negative Slopes

Slope of a line is positive if the line is going up (y is increasing as x increases), and it is negative if the line is coming down (y is decreasing as x increases).

In Figure A below, all the lines are going up, thus have positive slopes and in Figure B all the lines are going down, thus have negative slopes.

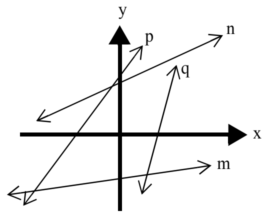


Figure A
Lines m, n, p and q all have positive slopes.

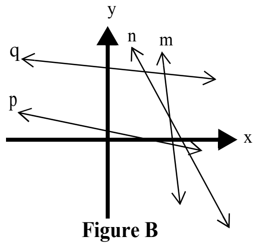


Figure B
Lines m, n, p and q all have negative slopes.

Comparing the Slopes

- Parallel lines have the same steepness, hence they have the same slope.
- Sliding a line without rotating it will create a line parallel to the original line, hence it will not change the line's slope. It is easier to compare the slopes if we slide them so that they all cross at one point. In the below figure, Figure B is obtained by sliding the lines in Figure A.

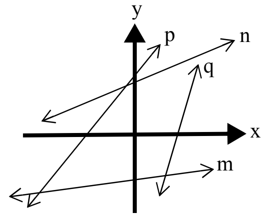


Figure A

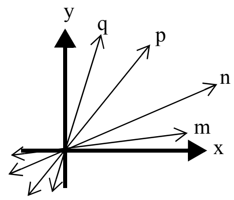


Figure B

- For positive slopes, steeper the line bigger the slope.

  For example, in the figure above, all the lines have positive slopes. Let S_m, S_n, S_p, S_q be the slopes of lines m, n, p, q respectively.

  Since line q is steeper than line p and line p is steeper than line n and line n is steeper than line m, the following relationship holds.

  $0 < S_m < S_n < S_p < S_q$
- As shown in the figure below, for negative slopes, if a line is steeper, it has smaller slope. This is because negative numbers decrease as they become more negative.

In the below figure, Figure B is obtained by sliding the lines in figure A.

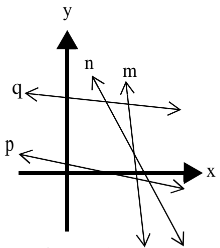


Figure A

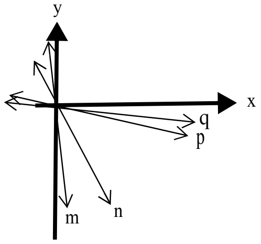


Figure B

Let S_m, S_n, S_p, S_q are the slopes of lines m, n, p, q, respectively.

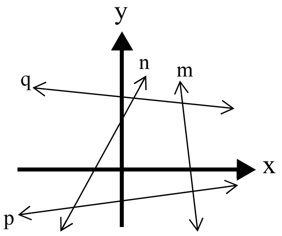
$S_m < S_n < S_p < S_q < 0$

Practice Exercise:

1. (Medium)
Graph lines m, n, p, q with slopes $S_m < S_n < 0 < S_p < S_q$

2. (Medium)
In the figure, S_m, S_n, S_p and S_q are the slopes of lines m, n, p and q respectively.

Put the slopes in order, from minimum to maximum. Indicate which ones have positive slopes and which ones have



Answers: 1. (No answer is provided.)
2. $S_m < S_q < 0 < S_p < S_n$

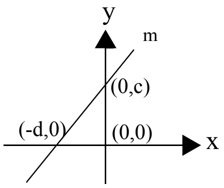
Equation of a Line

In the previous section, we have seen the general concepts about lines. In this section you will learn the equation of a line, how to calculate its slope and its intercepts quantitatively.

x and y Intercepts of a Line

x intercept is the value of x when y = 0. In the figure -d is the x intercept of line m.

y intercept is the value of y when x = 0. In the figure c is the y intercept of line m.



Slope of a Line

Let A(a,b) and P(p,q) be on the line m.

The slope of line m is $\frac{q-b}{p-a}$

For example, the points (-d,0) and (0, c) are on line m.

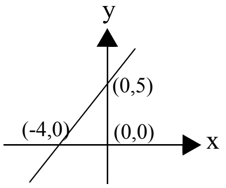
Hence the slope of the line m = $\frac{c-0}{0-(-d)} = \frac{c}{d}$

Equation of a Line with Slope and y-Intercept

Equation of a line is $y = sx + c$

s and c are the slope and the y-intercept of the line, respectively.

For example: In the figure, s = 5/4 and c = 5.



The equation of the line is:

$y = \frac{5}{4}x + 5$

Examples:

1. (Easy)
If y = 0.6x - 4, what are the slope, x- and y-intercepts?

Solution:

Slope = 0.6
y-intercept is -4
x-intercept is the value of x when y = 0 ➔
0 = 0.6x - 4 ➔ x = 0.6/4 = 0.15

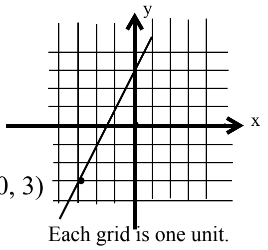
2. (Easy)
Equation of a line is given as y = 2x + 3. What is the x-coordinate of a point on the line if its y-coordinate is -3? Draw the line on the xy-plane and display the point.

Solution:

Equation: y = 2x + 3

Substitute y = -3 into the equation:
-3 = 2x + 3 ➔ x = -3

The line is shown in the figure, passing through (0, 3) and (-3, -3).



Each grid is one unit.

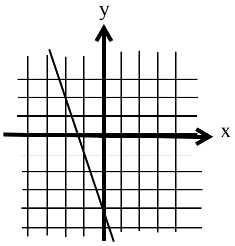
3. (Medium)
What is the equation and x-intercept of a line with slope -3 and y-intercept -4? Draw the line on the xy-plane.

Solution:

Equation: y = -3x - 4.

x-intercept is the value of x when y = 0.
0 = -3x - 4 ➔ x = -4/3

Figure shows the line passing through (-4/3, 0) and (0, -4).



Each grid is one unit.

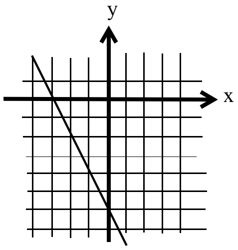
4. (Medium)
What is the y-intercept and the equation of a line with slope -2 and x-intercept -3? Draw the line on the xy-plane.

Solution:

Equation: y = -2x + c.

x-intercept = -3 ➔
0 = (-2)(-3) + c ➔
y-intercept = c = -6

The figure shows the line passing through (-3, 0) and (0, -6).



Practice Exercises:

1. (Easy)
If y = x - 4, what are the slope, x- and y-intercepts?

2. (Easy)
What is the slope of 2x + y = 1?

3. (Easy)
Write the equation of a line with slope 3 and y-intercept 5.

4. (Easy)
Write the equation of a line with slope -3 and y-intercept 5.

5. (Easy)
Write the equation of a line with slope 3 and y-intercept -5.

6. (Easy)
Write the equation of a line with slope -3 and y-intercept -5.

7. (Medium)
Graph the lines in questions 3 - 6.

8. (Medium)
Equation of a line is given as y = x - 3. What is the x-coordinate of a point on the line if its y-coordinate is -2? Draw the line on the xy-plane and display the point.

9. (Medium)
Graph line 2x + y = 1.

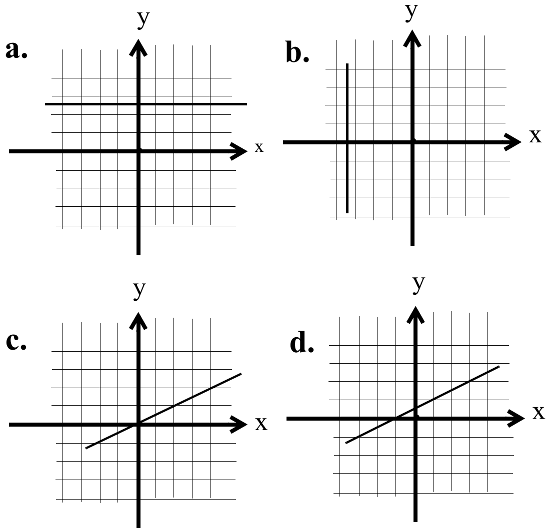
10. (Medium)
Draw 3 lines with slope = -1/2.

11. (Medium)
How many lines there are with slope 3?

12. (Medium)
What is the equation and x-intercept of a line with slope -2 and y-intercept -3? Draw the line on the xy-plane.

13. (Medium)
What is the equation and y-intercept of a line with slope 2 and x-intercept 3?
Draw the line on the xy-plane.

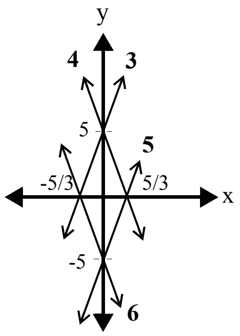
14. (Medium)
Give the equations of the lines shown below.



Each grid is one unit.

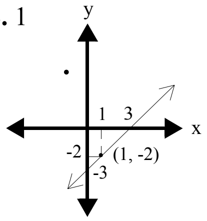
Answers: 1. slope = 1, x-intercept = 4, y-intercept = -4; **2.** -2; **3.** $y = 3x + 5$; **4.** $y = -3x + 5$; **5.** $y = 3x - 5$; **6.** $y = -3x - 5$;

7.

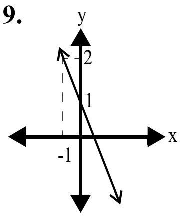


Question numbers between 3 and 6 are written for each line graph in bold.

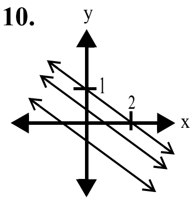
8. 1



9.



10.



11. infinite; **12.** Equation: $y = -2x - 3$, x-intercept = -3/2 (No graph is provided);
13. y-intercept = -6 and Equation: $y = 2x - 6$ (No graph is provided.);
14. a. $y = 2.5$, **b.** $x = -3.5$, **c.** $y = 0.5x$; **d.** $y = 0.5x + 0.5$

Equation of a Line With Slope and a Point

In the above section you learned how to write the equation of a line once you knew its slope and y-intercept. Note that y-intercept is only one special point on the line.

You can write the equation of a line if you know its slope and any point on the line. Here is how:

The equation of a line with slope m and passing through point (x_1, y_1) is:

$y - y_1 = m(x - x_1)$

This equation is easy to remember.
You can get the y-intercept easily as follows:

$y - y_1 = m(x - x_1) \Rightarrow$
$y = mx - mx_1 + y_1 = mx + (y_1 - mx_1) \Rightarrow$

y-intercept = $y_1 - mx_1$

Example: (Medium)

What is the equation of a line with slope 1/2 and passing through point (-1, 2)?

Solution:

Slope, $m = 1/2$, $x_1 = -1$ and $y_1 = 2$. Substituting these values into general equation:
$y - y_1 = m(x - x_1) \Rightarrow$
$$y - 2 = \frac{1}{2}(x - (-1)) = \frac{1}{2}(x + 1) \Rightarrow y = \frac{x}{2} + 2.5$$

Practice Exercises:

1. (Easy)
Graph 3 lines that go through the point (-1, 3).

2. (Easy)
How many lines go through point (2, -8)?

3. (Medium)
Write the equation of a line with slope 1/2, passing through point (2, 5).

4. (Medium)
Write the equation of a line with slope 1/2, passing through point (-2, -5).

5. (Medium)
Write the equation of a line with slope -1/2, passing through point (2, -5).

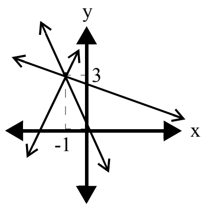
6. (Medium)
Write the equation of a line with slope -1/2, passing through point (-2, 5).

7. (Medium)
Write the equation of a line with slope -1/2, passing through point (-2, -5).

8. Graph the lines in questions 3- 7 above.

Answers:

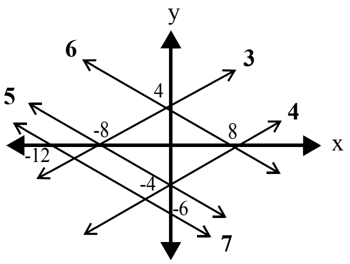
1.



2. Infinite; **3.** $y = \frac{1}{2}x + 4$; **4.** $y = \frac{1}{2}x - 4$;

5. $y = -\frac{1}{2}x - 4$; **6.** $y = -\frac{1}{2}x + 4$; 7. $y = -\frac{1}{2}x - 6$;

8.



Equation of a Line with 2 Points

You can also write the equation of a line if you know the coordinates of two points on the line. Here is how:

The equation of a line passing through points (x_1, y_1) and (x_2, y_2) is:

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

This formula is easy to remember. You can get the slope and the y intercept of the line by solving the above equation for y as follows:

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \Rightarrow (x - x_1)(y_1 - y_2) = (x_1 - x_2)(y - y_1)$$

$$\Rightarrow y = \frac{(x - x_1)(y_1 - y_2)}{x_1 - x_2} + y_1 =$$

$$\frac{(y_1 - y_2)}{x_1 - x_2}x - \frac{(y_1 - y_2)}{x_1 - x_2}x_1 + y_1 =$$

$$\frac{(y_1 - y_2)}{x_1 - x_2}x + \frac{x_1y_2 - y_1x_2}{x_1 - x_2} \Rightarrow$$

slope = $\frac{(y_1 - y_2)}{x_1 - x_2}$ and y-intercept = $\frac{x_1y_2 - y_1x_2}{x_1 - x_2}$

Example: (Medium)
What is the equation of a line passing through (0, 2) and (-1, -5)?

Solution:
The equation of a line passing through points (x_1, y_1) and (x_2, y_2) is:

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Substituting $x_1 = 0$, $y_1 = 2$, $x_2 = -1$ and $y_2 = -5$ into the above equation yields

$$\frac{x - 0}{0 - (-1)} = \frac{y - 2}{2 - (-5)} \Rightarrow x = \frac{y - 2}{7} \Rightarrow$$

$$y = 7x + 2$$

Checking the answer:

When x = 0, y should be 2:
Substituting x = 0 into y = 7x + 2, yields y = 2 as it should be.

When x = -1, y should be -5:
Substituting x = -1 into y = 7x + 2, yields y = -7 + 2 = -5 as it should be.

Practice Exercises:

1. (Medium)
Write the equation of a line passing through points (2, 5) and (3, 1).

2. (Medium)
Write the equation of a line passing through points (-2, 5) and (-3, 1).

3. (Medium)
Write the equation of a line passing through points (-2, -5) and (3, 1).

4. (Medium)
Write the equation of a line passing through points (2, -5) and (3, -1).

5. (Medium)
Write the equation of a line passing through points (-2, -5) and (-3, -1).

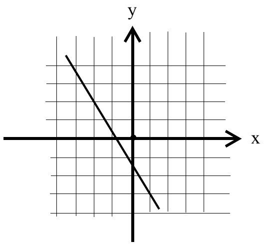
6. (Medium)
Write the equation of a line that aligns with x-axis.

7. (Medium)
Write the equation of a line that aligns with y-axis.

8. (Medium)
What are the slopes, x- and y- intercepts of the lines in questions 1 - 7?

9. (Medium)
Graph all the lines in questions 1 - 7.

10. (Medium)
What is the equation of the line shown in the figure?



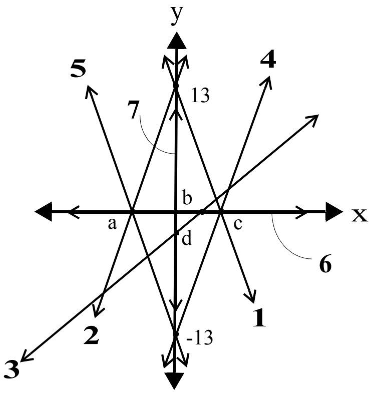
Hint: The line passes through points (-2, 2) and (1, -3).

Answers:
1. $y = -4x + 13$; **2.** $y = 4x + 13$;
3. $y = \frac{6}{5}x - \frac{13}{5}$; **4.** $y = 4x - 13$;
5. $y = -4x - 13$; **6.** $y = 0$; **7.** $x = 0$;

8.

Question	Slope	x-intercept	y-intercept
1	-4	13/4	13
2	4	-13/4	13
3	6/5	13/6	-13/5
4	4	13/4	-13
5	-4	-13/4	-13
6	0	x-axis	0
7	∞	0	y-axis

9.
a = -13/4
b = 13/6
c = 13/4
d = -13/5



Question numbers are shown near to each line in bold numbers

10. $y = -(5/3)x - 4/3$

Equations of Parallel Lines

Parallel lines have the same slope.

Examples:

1. (Easy)
$y = -3x + 7$, $y = -3x + 1$, $y = -3x - 7$ and $y = -3x - 2$ are all parallel because they all have slope -3.

2. (Easy)
$y = -3x + 7$ and $y = 3x + 7$ are not parallel, because they have different slopes: -3 and 3.

Practice Exercise:

1. (Easy)
Write an equation for a line parallel to the line $y = -5x + 0.5$

2. (Medium)
Write the equations for 3 lines, parallel to $y = -8x - 6$ and with y-intercepts -1, 0 and 1 respectively.

3. (Medium)
Write the equation for the line parallel to $y = 5x - 6$ and passing through point (0, 0).

4. (Medium)
What is the equation of the line parallel to $y = -x - 7$ and passing through point (3, -1)?

5. (Medium)
Which of the following lines are parallel?
Line m: $y = -5x - 3$,
Line n: $x = -5y$
Line o: $y - 5x = 7$
Line p: $y + 5x = -10$

(A) Line m and Line n.
(B) Line m and Line o.
(C) Line m and Line p.
(D) Line m and Line o and Line p.
(E) Line o and Line p.

Answers:

1. $y = -5x + c$, where c is a constant.;
2. $y = -8x - 1$, $y = -8x$, $y = -8x + 1$; **3.** $y = 5x$;
4. $y = -x + 2$; **5.** (C)

Equations of Perpendicular Lines

The multiplication of the slopes of two perpendicular lines is -1.

Examples:

1. $y = -2x + 7$, $y = 0.5x + 1$ are perpendicular lines because their slopes are -2 and 0.5 and $-2 \times 0.5 = -1$

2. $y = -2x + 7$ and $y = -0.5x + 7$ are not perpendicular, because their slopes are -2 and -0.5 and $-2 \times (-0.5) = 1$, not -1.

Practice Exercise:

1. (Medium)
Write an equation for a line perpendicular to the line $y = -5x + 0.5$

2. (Medium)
Write the equations for 3 lines, perpendicular to $y = -8x - 6$ and with y-intercepts -1, 0 and 1 respectively.

3. (Medium)
Write the equation for the line perpendicular to $y = 5x - 6$ and passing through point (0, 0).

4. (Medium)
What is the equation of the line perpendicular to $y = -x -7$ and passing through point (3, -1)?

5. (Medium)
Which of the following lines are perpendicular?
Line m: $y = -5x - 3$,
Line n: $x = 5y$
Line o: $- 5x = 7 - y$
Line p: $y - 5x = -10$

(A) Line m and Line n.
(B) Line m and Line o.
(C) Line m and Line p.
(D) Line m and Line o and Line p.
(E) Line o and Line p.

Answers: 1. $y = x/5 + c$, where c is a constant.;
2. $y = x/8 - 1$, $y = x/8$, $y = x/8 + 1$; **3.** $y = -x/5$;
4. $y = x/2 - 5/2$; **5.** (A)

Intersection of Two Lines

Two unparallel lines cross each other at one and only one point. The crossing point of two lines is a point that rests on both of the lines. The coordinates, (a, b), of the cross point of two lines, $y = m_1x + c_1$ and $y = m_2x + c_2$ is:

$$\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$$

where m_1, c_1 and m_2, c_2 are the slopes and y-intercepts of the first and second lines, respectively.

You don't need to memorize this formula. If you need it, you can easily drive it as follows:

Since point (a, b) is on first line, then $b = m_1a + c_1$
Since point (a, b) is on first line, then $b = m_2a + c_2$

Subtracting second equation from the first:

$$0 = (m_1 - m_2)a + (c_1 - c_2) \Rightarrow a = \frac{c_2 - c_1}{m_1 - m_2}$$

Substituting a into the first equation:

$$b = m_1 \frac{c_2 - c_1}{m_1 - m_2} + c_1 = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

Hence the point of intersection,

$$(a, b) = \left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2} \right)$$

Example: (Medium)
What is the intersection of $y = 4x + 1$ and $y = -2x - 3$?

Solution:
At the point of intersection of the two lines
$4x + 1 = -2x - 3 \Rightarrow 6x = -4 \Rightarrow x = -2/3$
Substituting $x = -2/3$ into the first equation yields:
$y = 4(-2/3) + 1 = -8/3 + 1 = -5/3$

Hence the intersection of $y = 4x + 1$ and $y = -2x - 3$ is
$(-2/3, -5/3)$

You could also arrive at the same conclusion by using the formula given above:

$$\left(\frac{-3-1}{4+2}, \frac{-4 \times 3 + 2 \times 1}{4+2} \right) = \left(\frac{-4}{6}, \frac{-10}{6} \right) = \left(\frac{-2}{3}, \frac{-5}{3} \right)$$

Practice Exercises:

1. (Medium)
What is the intersection of lines
$y = x + 1$ and $y = -x + 1$?

2. (Medium)
What is the intersection of lines
$y = x + 1$ and $y = x - 1$?

3. (Medium)
If the intersection of $y = 3x - 9$ and line p is $(2, -3)$ and the y-intercept of line p is -8, what is the equation for line p?

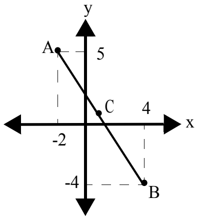
Answers: 1. (0, 1); **2.** None, they are parallel lines with slope 1.; **3.** $y = 2.5x - 8$

Mid-Point

If the coordinates of two points on a line are (x_1, y_1) and (x_2, y_2), then the coordinates of the mid-point is
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Examples:

1. (Medium)
In the figure, $\overline{AB}$ is a line segment. The coordinates of A and B are (-2,5) and (4,-4) respectively. What are the coordinates of the mid point, C, of $\overline{AB}$?



Solution:
The x-coordinate of C is $\frac{-2+4}{2} = 1$
The y-coordinate C is $\frac{5-4}{2} = \frac{1}{2}$
Hence the coordinates of mid-point C is (1,1/2)

2. (Hard)
Consider two points, A(u, v) and B(w, z). If the midpoint between A and B is (0, 2), and $(z - v)/(w - u) = 7$, what is the equation of a line passing through A and B?

Solution:
$(z - v)/(w - u) = 7$ is the slope of this line. Since the line that passes through A and B must pass through the mid-point, (0, 2), 2 is the y-intercept of the line. Hence the equation of the line is $y = 7x + 2$.

Practice Exercises:

1. (Medium)
What are the coordinates of mid-point between points (-0.3, -1.3) and (4, 5.1)?

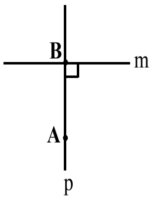
2. (Hard)
Line segment $\overline{AB}$ is divided into 4 congruent line segments, $\overline{AX}$, $\overline{XY}$, $\overline{YZ}$ and $\overline{ZB}$. If the coordinates of A and B are (-2, -2.9) and (4, 5.1), what is the coordinates of point X?

Hint: First find the mid-point of $\overline{AB}$ and then find the mid-point of $\overline{AY}$.

3. (Hard)
Consider two points, A(u, v) and B(w, z). If the midpoint between A and B is (2, 0), and $(w - u)/(z - v) = 7$, what is the equation of the line passing through A and B?

Answers: 1. (1.85, 1.9); **2.** (-0.5, -0.9); **3.** $y = x/7 - 2/7$

Distance Between a Line and a Point



The distance between a line m, and a point A, is the length of the line segment $\overline{AB}$, where B is the point on line m and $\overline{AB}$ and line m are perpendicular as shown in the figure.

To be able to find the distance between a line m and a point A, follow the steps below:

a. Find the equation of the perpendicular line, p, to line m.

b. Find the coordinates of the crossing point, B, between the line m and line p.

c. Find the distance between point A and point B.

Example: (Medium)
What is the distance between line y = 2x - 1 and the point (3, 4)?

Solution:

a. The slope of the line that is perpendicular to y = 2x -1 is -1/2 = -0.5. ➔
The equation of a line with slope -0.5 and passing through point (3, 4) is
y = -0.5(x - 3) + 4 = -0.5x + 5.5

b. The intersection of lines
y = 2x -1 and y = -0.5x + 5.5 is

$$\left(\frac{5.5+1}{2+0.5}, \frac{2\times 5.5-0.5}{2+0.5}\right) = (2.6, 4.2)$$

c. The distance between points (2.6, 4.2) and (3, 4) is:

$$\sqrt{(3-2.6)^2+(4-4.2)^2} = \sqrt{0.16+0.04} = \sqrt{0.2}$$

Practice Exercises:

1. (Hard)
What is the distance between line y = 5x + 1.5 and a point (-0.3, -1.3)?

2. (Hard)
The distance between line y = x and point A(0, m) is $\sqrt{5}$. What is the value of m?

Answers: **1.** 0.255; **2.** m = $\sqrt{10}$ or m = $-\sqrt{10}$

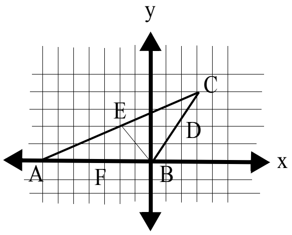
Triangles on x-y Plane

You can calculate several properties of a triangle if you can define a triangle on the xy-plane. Here are several examples.

Examples:

1. (Medium)
If the corners of a triangle are A = (-7, 0), B = (0, 0) and C = (3, 4), what is the length of its median $\overline{BE}$?

Solution:



Each grid is 1 unit.

Let's draw the figure first.

E is the mid-point between A and C.

x-coordinate of E is (-7 + 3)/2 = -2

y-coordinate of E is (0 + 4)/2 = 2

The length of the median, EB, is the distance between E and B.

$$EB = \sqrt{(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

2. Three sides of a triangle ABC is defined by lines, y = x + 1, y = -x + 2 and y = -0.5x - 2.
Prove the following.

a. (Medium)
ΔABC is a right triangle.

b. (Medium)
The vertices of the triangle are (1/2, 3/2), (-2, -1) and (8, -6)

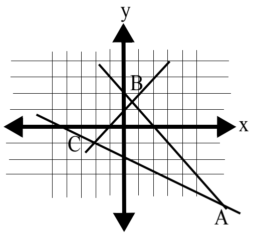
c. (Medium)
The side lengths are: $5/(\sqrt{2})$, $15/(\sqrt{2})$ and $5\sqrt{5}$

d. (Hard)
The heights are $\frac{5}{\sqrt{2}}$, $\frac{15}{\sqrt{2}}$ and $\frac{3\sqrt{5}}{2}$

e. (Medium)
The area of the triangle is 75/4

f. (Medium)
ΔABC is not an isosceles triangle.

Solution:



Each grid is 1 unit

You don't need a figure to solve this question. However, it helps if you have one.

a. The slope of y = x + 1 is 1 and the slope of y = -x + 2 is -1

Since the multiplication of the two slopes is

$1 \times (-1) = -1$, these two lines are perpendicular. Therefore the triangle is a right triangle.

b. The three intersections of three lines correspond to three vertices of the triangle. We will calculate two of them. You calculate the third one as a **practice exercise**.

At the intersection of the line y = -x + 2 and the line y = -0.5x - 2, both the x and y coordinates are equal on each line. Hence
-x + 2 = -0.5x - 2 ➔ x = 8
Substituting x = 8 into either of the equations yields y = -x + 2 = -8 + 2 = -6
So one of the vertices of this triangle is point (8, -6).

At the intersection of the line y = x + 1 and the line y = -x + 2, both the x and y coordinates are equal on each line. Hence
x + 1 = -x + 2 ➔ x = 1/2
Substituting x = 1/2 into either of the equations yields y = x + 1 = 1/2 + 1= 3/2
So the second vertex of this triangle is point (1/2, 3/2).

c. The side length is the distance between two vertices. We will calculate one of them. You calculate the other 2 as **practice exercises**.

Let's take (1/2, 3/2) and (8, - 6) as two vertices. The distance between these two points is

$$\sqrt{\left(8-\frac{1}{2}\right)^2+\left(-6-\frac{3}{2}\right)^2} = \sqrt{\frac{15^2}{4}+\frac{15^2}{4}} = \frac{15}{\sqrt{2}}$$

d. Since we have already proven that this is a right triangle, two of the side lengths, $5/(\sqrt{2})$ and $15/(\sqrt{2})$, are also equal to its two heights.

The third height is the distance between one of the vertices, (1/2, 3/2) and the line y = -0.5x - 2. To find this distance you first need to find the equation of a line which is perpendicular to y = -0.5x - 2 and passing through point (1/2, 3/2). It is
y = 2(x -1/2) + 3/2 = 2x + 1/2

You now need to calculate the intersection of these two perpendicular lines. At the intersection, -0.5x - 2 = 2x + 1/2 ➔ x = -1
Substituting x = -1 into one of the equations yields y = 2(-1) + 1/2 = -3/2. So the intersection of these two lines is (-1, -3/2)

As the last step to find the height, you need to find the distance between (1/2, 3/2) and (-1, -3/2). This distance is

$$\sqrt{\left(-1-\frac{1}{2}\right)^2+\left(-\frac{3}{2}-\frac{3}{2}\right)^2} = \sqrt{\frac{9}{4}+9} = \frac{3\sqrt{5}}{2}$$

e. You know all the side lengths and the heights. You can use any of the 3 pairs of sides and heights. We use the two perpendicular sides to calculate the area. The area is:

$$\frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)\left(\frac{15}{\sqrt{2}}\right) = \frac{75}{4}$$

f. Since none of the side lengths are equal, the triangle is not an isosceles triangle.

This example demonstrates that you can calculate all the properties of a triangle if you can define it on the xy-plane.

Practice Exercises:

1. (Medium)
What is the area of the triangle in the figure?

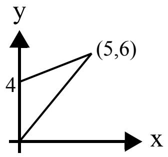


Figure is not drawn to scale.

2. (Medium)
If the area of the triangle in the figure is 12, what is the value of r?

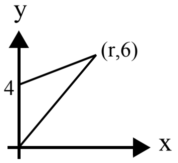


Figure is not drawn to scale.

3. (Medium)
Finish the exercises assigned in parts b and c of Example 2 above.

Answers: 1. 10; **2.** 6; **3.** No answer is provided.

Other Polygons on x-y Plane

Similar to the triangles, you can use coordinate geometry to calculate the properties of other polygons. Here are a few examples:

Examples:

1. (Medium)
Two neighboring corners of a square are (4, 1) and (-3, -2). What is the area of the square?

Solution:

The side length of the square is the distance between the two neighboring corners, (4, 1) and (-3, -2).

It is $\sqrt{(4+3)^2+(1+2)^2} = \sqrt{49+9} = \sqrt{58}$ ➔

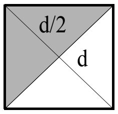
The area = $\sqrt{58} \times \sqrt{58} = 58$

2. (Hard)
Two opposite corners of a square are (4, 1) and (-3, -2). What is the area of the square?

Solution:

The diagonal length, d, of the square is the distance between the two opposite corners, (4, 1) and (-3, -2).

It is $\sqrt{(4+3)^2+(1+2)^2} = \sqrt{49+9} = \sqrt{58}$

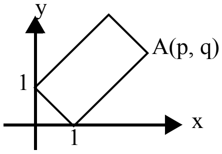


The area of a square is twice the area of the shaded triangle shown in the figure.

Therefore the area of the square =

$$d \cdot \frac{d}{2} = \frac{d^2}{2} = \frac{58}{2} = 29$$

3. (Very Hard)
In the figure, the area of the rectangle is 12. What are p and q?



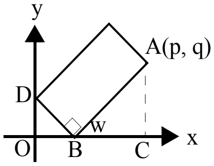
Solution:

The width of the rectangle is:

$\sqrt{1^2+1^2} = \sqrt{2}$

The length of the rectangle is the distance between A(p, q) and B(1, 0).

It is $\sqrt{(p-1)^2+q^2}$



The area of the rectangle =

$\sqrt{(p-1)^2+q^2} \cdot \sqrt{2} = 12$

On the other hand, ΔOBD is an isosceles right triangle, $\angle DBO = 45°$ ➔ $w = 45^o$ ➔
$\tan(w) = q/(p - 1) = 1$ ➔ $q = p - 1$

Substituting q in the above equation for the area of the rectangle yields:

The area of the rectangle =

$\sqrt{(p-1)^2+q^2} \cdot \sqrt{2} =$

$\sqrt{q^2+q^2} \cdot \sqrt{2} = 2q = 12$ ➔
$q = 6$ and $q = p - 1 = 6$ ➔ $p = 7$

Practice Exercises:

1. (Medium)
Two neighboring corners of a rectangle are (1,3) and (8,3). If the area of the rectangle is 21, what are the two possible coordinates of the other 2 corners?

2. (Medium)
A circle is inscribed inside a square. If the two neighboring corners of the square are (2, 3) and (5, 7), what is the area of the circle?

Answers: 1. {(1, 6) and (8, 6)} or {(1, 0) and (8, 0)};
2. $\pi \cdot \frac{25}{4} \cong 19.63$

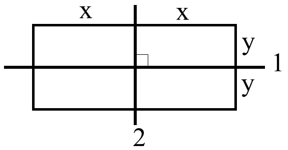
Symmetry

Mirror Symmetry

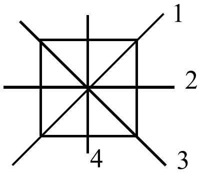
- **On the xy-plane, an object is symmetrical around a line if all the points of the object on one side of the line is the mirror image of the points of the object on the other side of the line.**

Example:

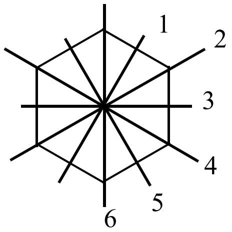
1. (Easy)
The following objects are symmetrical around the line(s) shown.



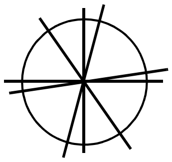
A rectangle is symmetric around 2 lines as shown in the figure.



A square is symmetric around 4 lines as shown in the figure.

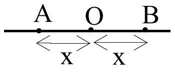


A regular hexagon is symmetric around 6 lines as shown in the figure.



A circle is symmetric around any line that goes through its center as shown in the figure.

- **Two points, A and B are symmetrical around point O, if O bisects $\overline{AB}$, as shown in the figure.**

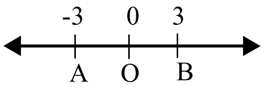


Examples:

1. (Easy)
On a number line, the coordinate of point A is -3. What is the coordinate of the point B that is symmetric to A around 0 (zero)?

Solution:

Let O be the point with coordinate 0 on the number line. If A and B are symmetric round point O, then O must bisect $\overline{AB}$. ➔

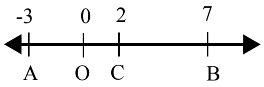


BO = AO = |-3 - 0| = 3 ➔
The coordinate of B is 0 + 3 = 3 as shown in the figure.

2. (Medium)
On a number line, the coordinate of point A is -3. What is the coordinate of the point B that is symmetric to A around 2?

Solution:

Let C be the point with coordinate 2 on the number line.

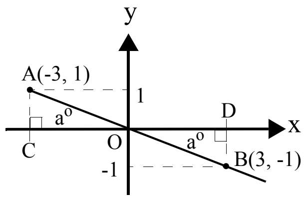


If A and B are symmetric around point C, then C must bisect $\overline{AB}$. ➔
BC = AC = |-3 - 2| = |-5| = 5 ➔
The coordinate of B is 2 + 5 = 7

3. (Medium)
On the xy-plane, the coordinates of point A is (-3, 1). What are the coordinates of the point B that is symmetric to A around point O (the origin)?

Solution:

In the figure, if A and B are symmetric around point O, then O must bisect $\overline{AB}$. ➔
BO = AO ➔
$\Delta OCA \cong \Delta ODB$ ➔



OD = OC = |-3| = 3 and BD = CA = |1| = 1 ➔
The coordinates of B is (3, -1) as shown in the figure.

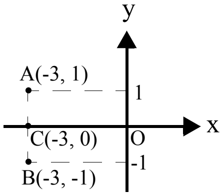
- **On the xy-plane, two objects are said to be symmetrical if they are the mirror image of each other around a line.**

Examples:

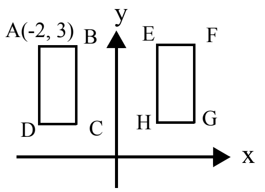
1. (Medium)
On the xy-plane, the coordinates of point A is (-3, 1). What is the coordinate of the point B that is symmetric to A around x-axis?

Solution:

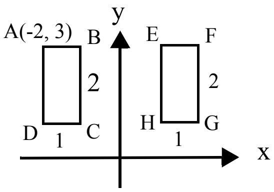
In the figure, if A and B are symmetric around the x-axis, then C must bisect $\overline{AB}$. ➔
BC = AC = 1 ➔
The coordinates of B is (-3, -1), as shown in the figure.



2. (Medium)
EFGH is a rectangle with width 1 and length 2. It is symmetric to ABCD around y-axis as shown in the figure. What are the coordinates of point G?



Solution:



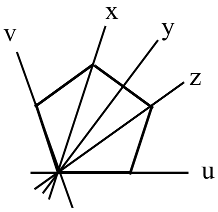
EFGH is a rectangle with width 1 and length 2. ➔ ABCD is also a rectangle with width 1 and length 2. ➔ The coordinates of point D is (-2, 3 - 2) = (-2, 1) ➔

The coordinates of point G is (2, 1).

Practice Exercises:

1. (Easy)

Around which of the 5 lines in the figure, the regular pentagon (5 sides with equal length) is symmetric?



2. (Easy)

How many lines of symmetry an equilateral triangle has?

3. (Medium)

A and B are two symmetrical points on a number line around point C with coordinate -8. If the coordinate of B is -3, what is the coordinate of the point A?

4. (Medium)

On the xy-plane, the coordinates of point A is (-3, 1). What is the coordinate of the point B that is symmetric to A around the y-axis?

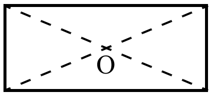
Answers: 1. y; **2.** 3; **3.** -13; **4.** (3, 1)

Rotational Symmetry

An object has rotational symmetry if it remains the same when you rotate it around a specific axis by a specific degree.

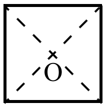
Examples:

1. (Easy)



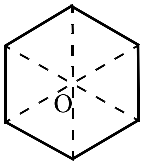
A rectangle will remain the same if you rotate it by 180^o and 360^o around the axis perpendicular to the plane of the rectangle and passing through the point O.

2. (Easy)



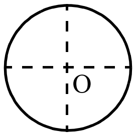
A square will remain the same if you rotate it by 90^o, 180^o and 360^o around the axis perpendicular to the plane of the square and passing through the point O.

3. (Medium)



A regular hexagon will remain the same if you rotate it by 60^o, 120^o, 180^o, 240^o, 300^o and 360^o around the axis perpendicular to the plane of the hexagon and passing through the point O.

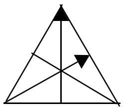
4. (Medium)



A circle will remain the same if you rotate it by any degree around the axis perpendicular to the plane of the circle and passing through the point O.

5. (Medium)

The triangle in the figure is equilateral. If the object is rotated x^o around an axis perpendicular to the plane of the triangle and passing through the point of intersection of the 3 line segments in the figure, it remains the same.



What is the value of x?

(A) 30

(B) 60

(C) 120

(D) 180

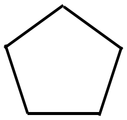
(E) None of the above.

Solution:

The equilateral triangle will remain the same if you rotate it 120^o. However, because of the arrow heads or lack of them on either end of the heights, makes it impossible to obtain the same figure when you rotate them by angles given in the answer choices. So the answer is (E).

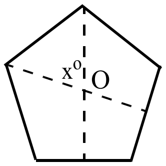
6. (Medium)

The regular pentagon shown in the figure will remain the same if you rotate it around an axis perpendicular to the plane of the pentagon and passing through the cross section of the angular bisectors of the pentagon by x^o. What is the value of x?



Solution:

Let's draw two of the angular bisectors as shown in the figure. If you rotate the pentagon around an axis perpendicular to the plane of the pentagon and passing through the point O by x^o, it will remain the same.



$x^o = 360/5 = 72^o$

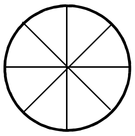
Practice Exercises:

1. (Medium)

A regular polygon with 9 sides will remain the same if you rotate it around an axis perpendicular to the plane of the polygon and passing through the crossing point of the angular bisectors of the polygon by x^o. What is the value of x?

2. (Medium)

The line segments in the figure divides the circle into equal pieces.

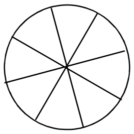


a. What figure you will get if you rotate the figure by 60^o, counterclockwise, around an axis perpendicular to the plane of the figure and passing through the center of the circle?

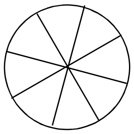
b. What figure you will get if you rotate the above figure by 60^o, clockwise, around an axis perpendicular to the plane of the figure and passing through the center of the circle?

c. By what angle you need to rotate the above figure around an axis perpendicular to the plane of the figure and passing through the center of the circle to get the same figure?

Answers: 1. 40; **2. a.**



b.

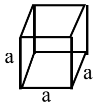
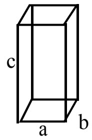


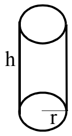
c. 45^o

3 - Dimensional Objects

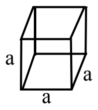
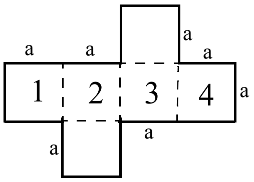
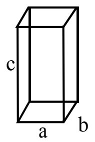
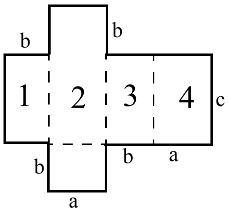
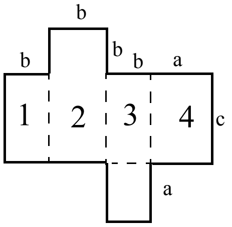
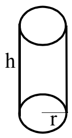
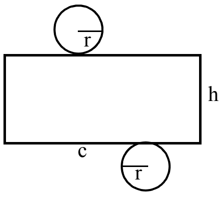
Most of the geometry questions in SAT are in plane geometry. However there are a few that have cubes, rectangular prisms, cylinders and other solid objects. Most of these questions have a Medium level of difficulty. You can find occasional Easy and Hard questions as well.

The table below shows the 3-Dimensional objects you need to know. It provides the number of corners, the number of edges, the number of sides, the surface area and the volume for each object.

Object	**Edges**	**Corners**	**Sides**	**Total Surface Area**	**Volume**
Cube 	12 edges length a	8	6 sides, with areas a^2 each	$6a^2$	a^3
Rectangular Prism 	12 edges, 4, length a 4, length b 4, length c	8	3 pairs of rectangular sides with areas ab, ac and bc each	2(ab+ac+bc)	abc

Cylinder  r - radius h - height	2 circular rings with length $2\pi r$ each	0	2 circles with areas, πr^2 each One rectangle with area $2\pi rh$	$2\pi r(r + h)$	$h\pi r^2$

You can create the above 3D figures from paper. The table below shows the shape you need to cut to make each of the 3D objects without overlap. Cut the shape outlined in column 2 and fold it through the dotted lines to make the 3D object in column 1.

Cube 		The squares at each end of the central rectangle can be at any of the 4 places available.
Rectangular Prism 	 	The rectangles at each end of the central rectangle can be at any of the 4 places available. If they are at 2nd or 4th positions they are $a \times b$ as shown in the first figure. If they are in 1 or 3 positions, they are $b \times a$ as shown in the second figure.
Cylinder 	 $c = 2\pi r$	The 2 circles, can be located anywhere as long as they are tangent to the rectangle and they are on opposite sides of the rectangle.

Examples:

1. (Easy)
What is the volume of a cube with the side length 2?

Solution:
$2^3 = 8$

2. (Easy)
What is the volume of a cylinder with diameter = height = 2?

Solution:
Diameter = 2 ➔ Radius = 1 ➔
Volume = $\pi r^2 \cdot h = 2\pi$

3. (Easy)
What is the surface area of a rectangular prism with dimensions: 2, 3 and 4?

Solution:
$2(2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4) = 2(6 + 8 + 12) = 2 \cdot 26 = 52$

4. (Medium)
Which of the following volumes is larger?

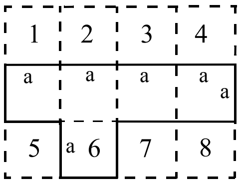
a. Volume of a cube with side length x.

b. Volume of a cylinder with diameter = height = x.

Solution:
You don't need to calculate the volumes individually. Since you can always fit a cylinder with diameter = height = x into a cube with side length x, the answer is:
The cube is larger than the cylinder.

5. (Medium)
If you would like to make a cube-shaped box without a lid from a sheet of paper, in what shape you need to cut the paper so that there won't be any overlap?

Solution:



Actually, this is only one of the possible solutions to the open lid cube. The square at position "6" can be at any position, indicated by numbers from 1 to 8. There are more possible solutions to this problem. See Exercise 4 below.

Practice Exercises:

1. (Easy)
What is the surface area of a cube with side length 2?

2. (Easy)
What is the surface area of a cylinder with diameter = height = 2?

3. (Medium)
From 8 cm^3 of clay, if you make

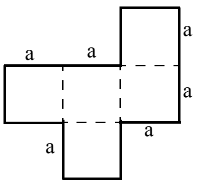
a. a cube, or

b. a cylinder with height = diameter,

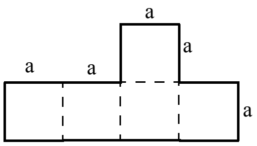
which of these shapes will have the smallest surface area?

4. (Hard)
A piece of paper is cut to make a cube with no lid. Which of the following figures will be most suitable to make such a cube?

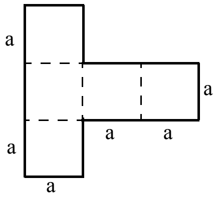
I.



II.



III.



(A) I only

(B) II only

(C) III only

(D) None

(E) I and II and III

Answers: 1. 24; **2.** 6π ; **3.** Cylinder; **4.** (E)

Exercises

Points, Lines and Angles

1. (Easy)
In the figure, O, A and B are on the same line.
$z^o = ?$

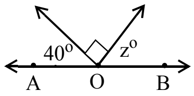


Figure is not drawn to scale.

2. (Easy)
In the figure, $y = 3x$. $x^o = ?$

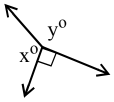


Figure is not drawn to scale.

3. (Easy)
In the figure, $AC \parallel BD$.
$y^o = ?$

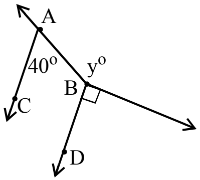


Figure is not drawn to scale.

4. (Easy)
In the figure, if $AB \parallel CD$,
$x - y = ?$

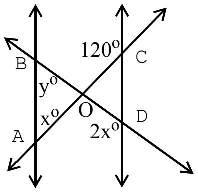


Figure is not drawn to scale.

5. (Medium)
Two crossing lines define a point. What is the minimum number of points that 4 lines define?

6. (Medium)
Two crossing lines define 4 angles. What is the minimum number of angles that 4 lines define?

7. (Medium)
$\overline{AB}$ and $\overline{CD}$ are two line segments with no common point. Which of the following is <u>always wrong</u>.

(A) $\overline{AD}$ can be parallel to both $\overline{AB}$ and $\overline{CD}$.

(B) $\overline{AB}$ and $\overline{CD}$ can be on the same line.

(C) $\overline{AB}$ and $\overline{CD}$ can be parallel.

(D) $\overline{AB}$ and $\overline{CD}$ can be perpendicular.

(E) None of the above.

8. (Medium)
m and n are two distinct lines. A and B are two points on m and n, respectively. Which of the following <u>must be wrong</u>.

(A) The distance between A and B can be infinite.

(B) The distance between A and B can be zero.

(C) $\overline{AB}$ can be perpendicular to line n.

(D) $\overline{AB}$ can be perpendicular to line m.

(E) None of the above.

9. (Medium)
In the figure, if $a = \frac{2}{5}b$ and $b = \frac{2}{5}c$, then $c^o = ?$
Approximate your result to the nearest integer.

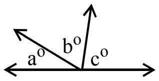


Figure is not drawn to scale.

10. (Medium)
In the figure, A, O and B are on the same line.
$\angle AOC = 110°$.
$z^o = ?$

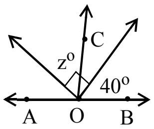


Figure is not drawn to scale.

11. (Medium)
In the figure, points A, O and F are on the same line.

$\angle BOE = ?$

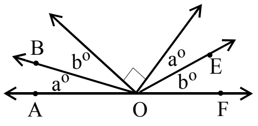


Figure is not drawn to scale.

12. (Medium)
In the figure, $\vec{m} \parallel \vec{n}$ and $y = 3x$. $x^o = ?$

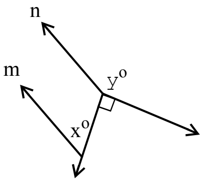


Figure is not drawn to scale.

13. (Medium)
In the figure, if $\vec{m} \parallel \vec{n}$ and $y = 3x$, then $x^o = ?$

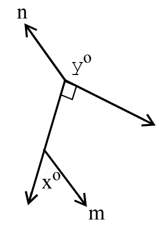


Figure is not drawn to scale.

14. (Medium)
In the figure, $\vec{m} \parallel \vec{n}$.
$y^o = ?$

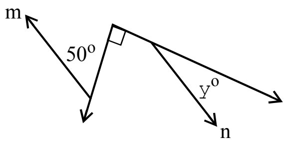


Figure is not drawn to scale.

15. (Hard)
Which of the following <u>can not</u> be the number of angles defined by 4 lines if no more than 2 lines cross each other at the same point?

(A) 20

(B) 18

(C) 16

(D) 12

(E) None of the above.

16. (Hard)
In the figure, the points A, B and E are aligned; and the points B, C and F are also aligned.

$\angle DCF = \frac{1}{3}\angle CBE$ and

$\angle CBE = \frac{1}{3}\angle BAO$

If $\overline{CO} \perp \overline{AO}$,
what is $\angle DCO$?

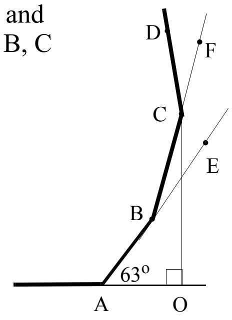


Figure is not drawn to scale.

17. (Hard)
In the figure, the points A, B and E are aligned; and the points B, C and F are also aligned.

$\angle DCF = \frac{1}{3}\angle CBE$ and

$\angle CBE = \frac{1}{3}\angle BAG$

If $\overline{CO} \parallel \overline{AG}$,
what is $\angle DCO$?

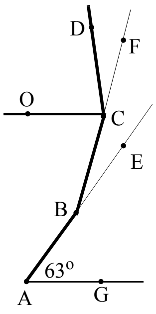


Figure is not drawn to scale.

18. (Hard)
In the figure, if $\overleftrightarrow{AD} \parallel \overleftrightarrow{EG}$,
what is $\angle BMF$?

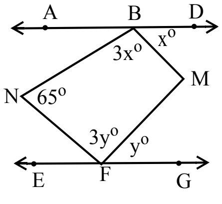


Figure is not drawn to scale.

Polygons

General

1. (Medium)
Which of the below <u>can not</u> be the inner angle of a regular polygon?

(A) 60^o
(B) 120^o
(C) 130^o
(D) 135^o
(E) 140^o

2. (Medium)
In the figure, if $\angle D = \angle C/2$, what is $\angle D$?

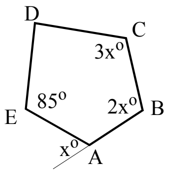


Figure is not drawn to scale.

3. (Hard)
In the figure, a part of a regular polygon is displayed. A, B, C and D are the four vertices of this polygon.

$\overline{AO}$ and $\overline{DO}$ are the medians of $\angle A$ and $\angle D$, respectively.

If $\angle AOD = 120^o$, how many vertices this polygon has?

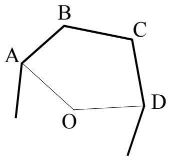


Figure is not drawn to scale.

4. (Hard)
In the figure, A, B, C and D are the four vertices of a regular polygon.

A, B and O are on the same line. D, C and O are on the same line as well.

If O = 120^o, how many vertices this polygon has?

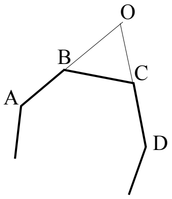


Figure is not drawn to scale.

Triangles: 3 - Sided Polygons

1. (Easy)
In the figure, x^o = ?

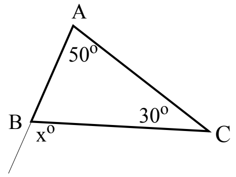


Figure is not drawn to scale.

2. (Easy)
In the figure, x^o = ?

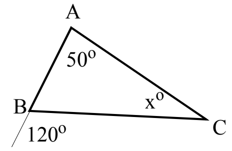


Figure is not drawn to scale.

3. (Easy)
The hypotenuse of a right triangle is 8 and the distance to the hypotenuse from the opposite corner is 3. What is the area of the triangle?

4. (Medium)
In the figure, what is the measure of $\angle C$?

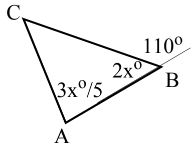


Figure is not drawn to scale.

5. (Medium)
If one side length of an isosceles right triangle is $2\sqrt{2}$, which of the following can be one of the other sides?

I. 2
II. $2\sqrt{2}$
III. 4

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and II and III

6. (Medium)
If one side length of a 30^{o} - 60^{o} - 90^{o} triangle is $4\sqrt{3}$, which of the following can be one of the other sides?

- **(A)** 4
- **(B)** 6
- **(C)** $8\sqrt{3}$
- **(D)** 12
- **(E)** All of the above.

7. (Medium)
If the periphery of an isosceles right triangle is 12, then what is the hypotenuse?

8. (Medium)
In the figure, $\overline{AB} \parallel \overline{DC}$, $AC = AB$, $\angle DCA = 50°$

What is $\angle ABC$?

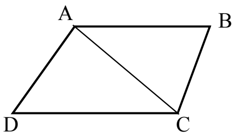


Figure is not drawn to scale.

9. (Medium)
$\angle ABC = 30°$,
$\angle BCD = 20°$,
$\angle DAC = 60°$

What is $\angle DCA$?

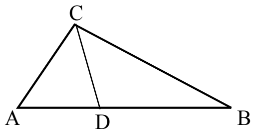


Figure is not drawn to scale.

10. (Medium)
For ΔABC and ΔDEF triangles,
$\overline{AB} \parallel \overline{DE}$, $\overline{AC} \parallel \overline{EF}$, $\overline{CB} \parallel \overline{DF}$,
$CB = 4$, $CA = 3$ and $FE = 2$

What is FD?

11. (Hard)
In the figure, z = ?

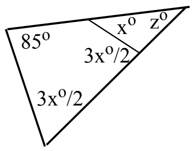


Figure is not drawn to scale.

12. (Hard)
In the figure, $\overline{AC} \parallel \overline{BD}$. If $y = x/2$, then z = ?

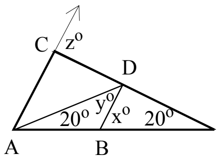


Figure is not drawn to scale.

13. (Hard)
In the figure, $\vec{m} \parallel \vec{n}$. If the distance between them is 2, then BC = ?

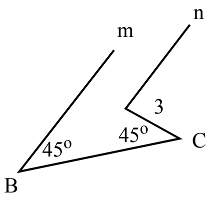


Figure is not drawn to scale.

14. (Hard)
One of the sides of a triangle is 4 and the area of the triangle is 10. Which of the following values can be the periphery of the triangle?

I. 11
II. 12
III. 13

- **(A)** II only
- **(B)** III only
- **(C)** I and II and III
- **(D)** None
- **(E)** The information given is not enough to find the periphery of the triangle.

15. (Hard)
One of the sides of a triangle is 3 and the area of the triangle is 9. What is the minimum value that the longest side of the triangle can take?

:

16. (Hard)
In the figure, $AB \parallel DE$,
$\overline{AC} \parallel \overline{EF}$, $EF = AC$,
$\angle C = 40°$,
$\angle B = 30°$
$\angle D = 20°$

$\angle F = ?$

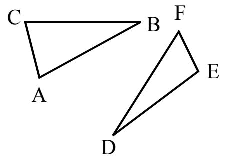


Figure is not drawn to scale.

17. (Hard)
In the figure,
B bisects $\overline{AD}$.
Area of $\Delta BDE = 6$
and DE = 3.
What is the periphery of ΔABC?

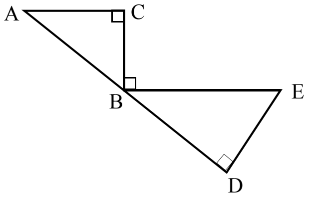
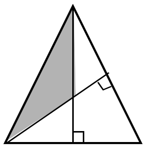


Figure is not drawn to scale.

18. (Hard)
Triangle in the figure is an equilateral triangle with side lengths $\sqrt{3}$.

What is the area of the shaded region?



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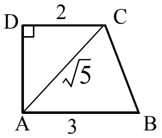
Quadrangles: 4 - Sided Polygons

Trapezoid

1. (Easy)
What is the area of the trapezoid ABCD?



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2. (Medium)
What is the area of the trapezoid ABCD?

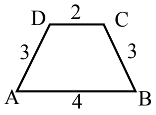


Figure is not drawn to scale.

3. (Hard)
$\overline{DE} \parallel \overline{CB}$ and $DC \parallel AB$

Point O is the intersection of $\overline{AC}$ and $\overline{DE}$
Area of $\Delta AEO = 4$

What is the area of the shaded region?

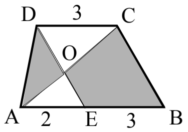
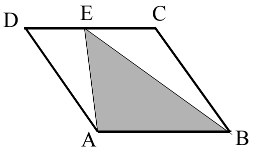


Figure is not drawn to scale.

Parallelogram

1. (Medium)
In the figure,
ABCD is a parallelogram. What is the ratio of the area of the shaded region to the area of ABCD?



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2. (Medium)
What are the lengths of the diagonals of the parallelogram ABCD?

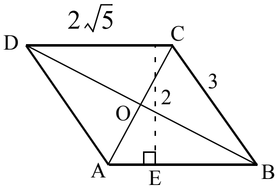
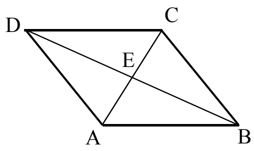


Figure is not drawn to scale.

3. (Medium)
ABCD is a parallelogram.
The area of $\Delta AEB = 12$

What is the area of ΔCEB?



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4. (Hard)
In the figure, ABCD is a parallelogram.
EB = DC and CE = AD.

$x^o = ?$

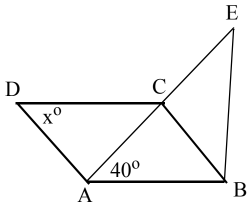


Figure is not drawn to scale.

5. (Hard)
ABCD is a parallelogram.
AC bisects $\angle ECB$ and E bisects $\overline{AD}$.

What is the periphery of ABCD?

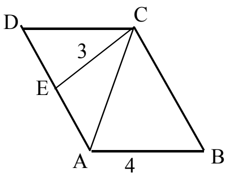


Figure is not drawn to scale.

Rectangles and Squares

1. (Easy)
ABCD and DFCE are two rectangles. What is the ratio of the area of ABCD to the area of DFCE?

(A) 0.8

(B) 1

(C) 1.25

(D) 1.5

(E) 2

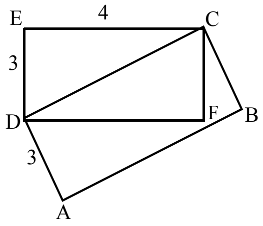


Figure is not drawn to scale.

2. (Medium)
In the figure, ABCD and EFGH are two squares. The 4 corners of EFGH bisects the 4 sides of ABCD.

If the area of ABCD is a, what is the area of EFGH?

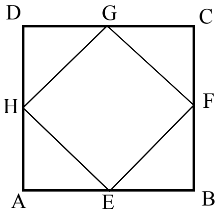
(A) a/2

(B) $a/(\sqrt{2})$

(C) $(2a)/\sqrt{2}$

(D) $\sqrt{3}a/2$

(E) 3a/4



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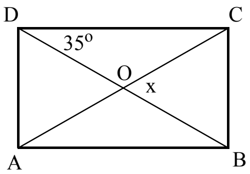
3. (Medium)
In the figure, ABCD is a rectangle.

x = ?



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4. (Medium)
In the figure, ABCD is a rectangle and AE = EO = FO = FB.

x = ?

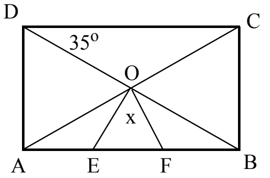


Figure is not drawn to scale.

5. (Medium)
ABFG and BCDE are 2 squares with areas a and 2a, respectively.

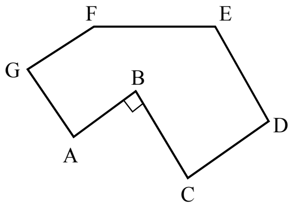


Figure is not drawn to scale.

EF = ?

(A) $\sqrt{3}a$

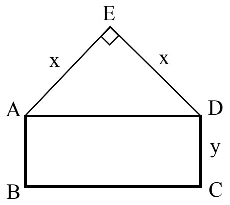
(B) $3\sqrt{a}$

(C) $\sqrt{3}a$

(D) 3a

(E) $\frac{a}{\sqrt{3}}$

6. (Hard)
The area of ΔADE is 2.25 and the area of the rectangle ABCD is 10.00. What is the value of y?



(A) 5

(B) 10/3

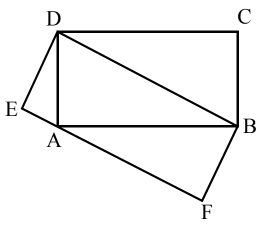
(C) 2.5

(D) 2

(E) 1.5

7. (Hard)
ABCD and EFBD are two rectangles. A is on $\overline{EF}$.

What is the ratio of the area of ABCD to the area of EFBD?



Circles

1. (Easy)
The area of a circle is π. What is its radius?

2. (Easy)
A man is jogging on a circular path with a radius of 30 feet. If he makes 15 lapses in one hour, what is his speed?

(A) π feet/min.

(B) 30 feet/min.

(C) 15π feet/min.

(D) 30π feet/min.

(E) 15 feet/min.

3. (Medium)
3 distinct points, A, B, C, are at equal distance from a fourth point D. On which of the following shapes can A, B, and C be?

I. Square
II. Rectangle
III. Circle

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) I, II, III

4. (Medium)
2 distinct points, A and B, are at equal distance from a third point C. On which of the following shapes can A, B, and C be?

I. Line
II. Rectangle
III. Circle

(A) I only

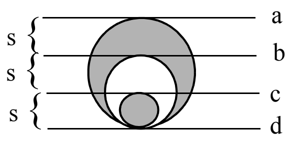
(B) II only

(C) III only

(D) I and III only

(E) I, II, III

5. (Hard)
a, b, c and d are 4 parallel tangent lines to 3 circles, as shown in the figure. They are separated from each other by an equal distance, s.



What is the ratio of the area of the smallest shaded circle to the area of the shaded crescent?

6. (Hard)
ΔABC is an equilateral triangle with side length $\sqrt{3}$. As shown in the figure, a circle is inscribed inside ΔABC.

What is the area of the circle?

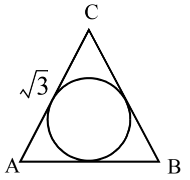
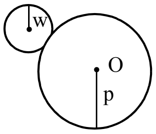


Figure not drawn to scale.

7. (Hard)
A small wheel of radius w is rolling on a circular path of radius p.

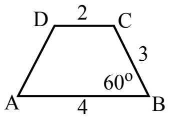
What percentage of the circumference
of the circle O is traveled by the wheel,

Provide your answer in terms of w and p.



Trigonometry

1. (Medium)
What is the area of the trapezoid ABCD?



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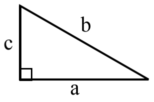
2. (Medium)
In the figure, if c/b = 1/2, what is a/c?



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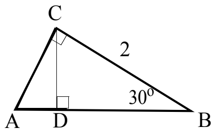
3. (Hard)
In the figure, AD = ?



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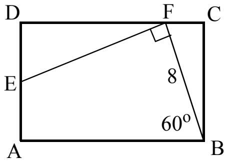
4. (Hard)
ABCD is a rectangle. If DE = EA, AB = ?



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5. (Hard)
In the figure, ABCD is a rectangle.

EF = ?

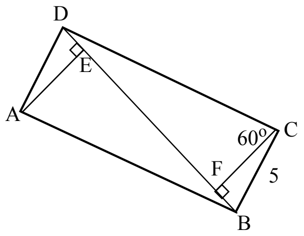
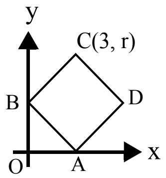


Figure is not drawn to scale

Coordinate Geometry

1. (Medium)
In the figure, ABCD is a square and OA = OB.

What is the value of r?



2. (Medium)
In the figure, BC= OB.

What is the value of r?

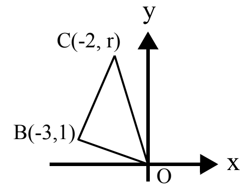
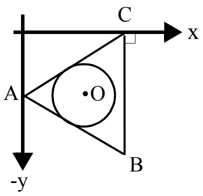


Figure is not drawn to scale.

3. (Hard)
ΔABC is an equilateral triangle. $\overline{AC}$ and $\overline{AB}$ are tangents to the circle O. The coordinates of the centre of the circle and the point A are: (3, -2) and (0, -2), respectively.



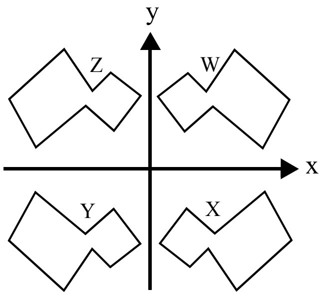
What is the area of the circle O?

Symmetry

1. (Easy)
How many lines of symmetry an isosceles triangle has?

2. (Easy)
A and B are two points on a number line with coordinates -5 and 27, respectively. What is the coordinate of the point around which A and B are symmetrical?

3. (Easy)
Which of the following pairs are not symmetric around any axis?

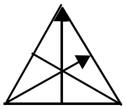


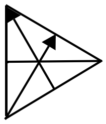
(A) W, Z
(B) Z, Y
(C) Y, X
(D) X, W
(E) W, Y

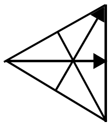
4. (Medium)
On the xy-plane, the coordinates of point A is (-3, 1). What is the addition of the x- and y-coordinates of the point B that is symmetric to A around point C(1, -1)?

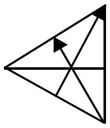
5. (Medium)
On the xy-plane, the coordinates of point A is (-3, 1). What is the multiplication of the x- and y-coordinates of the point B that is symmetric to A around the y = x line?

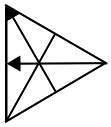
6. (Medium)
The triangle in the figure is equilateral. Which of the following will be obtained if the object is rotated 30^o counter clockwise, around an axis perpendicular to the plane of the triangle and passing through the point of intersection of the 3 line segments in the figure?

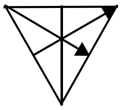


(A)


(B)


(C)


(D)


(E)


3 - Dimensional Objects

1. (Easy)
For a cube of edge length a, what is the total length of all the edges?

2. (Medium)
Consider the cube in the figure. What is the area of ABCD?

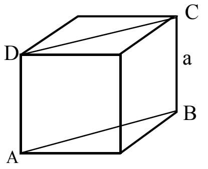
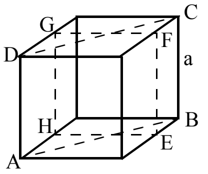


Figure is not drawn to scale.

3. (Medium)
The cube in the figure is cut in two different ways to create two objects in each case. Which cut creates the maximum number of corners?



- **a.** The cut through the ABCD plane.
- **b.** The cut through the EFGH plane.

4. (Medium)
A cylinder is cut though the dotted line shown in the figure, to obtain two cylinders.

What is the ratio of the total surface area of the final two cylinders to the surface area of the original cylinder?

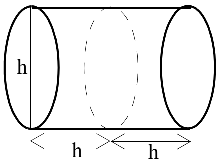


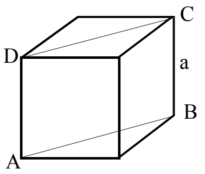
Figure is not drawn to scale.

5. (Medium)
A cylinder is inscribed inside a cube of edge length 5. What is the maximum value that the volume of the cylinder can take?

Hint: Base diameter and the height of the cylinder are both 5.

6. (Hard)
What is the length of the longest line segment you can fit inside of a cube with side length a?

7. (Hard)
If the cube in the figure is cut through the plane ABCD, what would be the difference between the original surface area of the cube and the total surface areas of the two resultant triangular prisms combined?



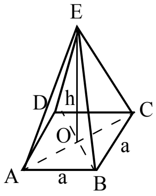
Hint: There is no need to calculate the surface areas of all 3 solids before and after the cut. Since all the surfaces of the original cube are also the surfaces of the two triangular prisms, the only extra surface to consider is the surface of the cut.

8. (Hard)
Create your own questions and answer them by replacing the cubes in questions 1, 2, 3, 5, 6 and 7, with a rectangular prism with side lengths a, b and c. Note that cube is a special rectangular prism with a = b = c.

9. (Hard)

The object shown in the figure is called a square right pyramid. It has a square base (ABCD) of edge length a and height OE = h. $\overline{OE}$ is perpendicular to the base of the pyramid.

CE = ?



10. (Hard)

What is its surface area of the pyramid in the previous question?

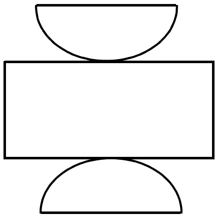
11. (Hard)

You are making a cradle-shape paper object with length l, width w and height h . If l > w and the height remains the same for the body and the ends of the cradle, which of the following will be the most suitable cut?

(A)

(B)

(C)



(D)

(E) None of the above.

Answers

Points, Lines and Angles

1. 50^{o}
2. 67.5^{o}
3. 130^{o}
4. 0
5. 0
6. 0
7. (E)
8. (E)
9. 115^{o}
10. 60^{o}
11. 135^{o}
12. 45^{o}
13. 45^{o}
14. 40^{o}
15. (B)
16. 179^{o}
17. 89^{o}
18. 73.75^{o}

Polygons

General

1. (C)
2. 75^{o}
3. 9
4. 12

Triangles: 3 - Sided Polygons

1. 80^{o}
2. 70^{o}
3. 12
4. 89^{o}
5. (E)
6. (E)
7. $\frac{12\sqrt{2}}{2+\sqrt{2}}$
8. 65^{o}
9. 70^{o}
10. 8/3
11. 23.75^{o}
12. 60^{o}
13. $5\sqrt{2}$
14. (D)
15. $\frac{3\sqrt{17}}{2}$
16. 90^{o}
17. 9.6
18. $\frac{\sqrt{3}}{4}$

Quadrangles: 4 - Sided Polygons

Trapezoid

1. 2.5
2. $6\sqrt{2}$
3. 27

Parallelogram

1. 1/2
2. AC = 3 and BD = 7
3. 12
4. 60^{o}
5. 20

Rectangles and Squares

1. 1.25
2. (A)
3. 70^{o}
4. 40^{o}
5. (A)
6. (B)
7. 1

Circles

1. 1
2. (C)
3. (E)
4. (E)
5. 1/5
π

Trigonometry

1. $(9\sqrt{3})/2$
2. $\sqrt{3}$
3. $1/(\sqrt{3})$
4. 10
5. 5

Coordinate Geometry

1. 6
2. 4
3. $9\pi/4$ or 7.069

Symmetry

1. 1
2. 11
3. (E)
4. 2
5. -3
6. (A)

3-Dimensional Objects

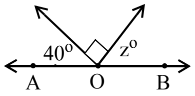
1. 12a
2. $\sqrt{2}a^2$
3. b
4. 6/5
5. $(125/4)\pi$
6. $\sqrt{3}a$
7. $2\sqrt{2}a^2$
8. No answer is provided.
9. $\sqrt{\frac{a^2}{2}+h^2}$
10. $2a\sqrt{\frac{a^2}{4}+h^2}+a^2$
11. (D)

Solutions

Points, Lines and Angles

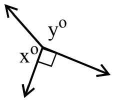
1. Answer: 50^o

The angle in the middle is 90^o. ➔
$40 + 90 + z = 180$ ➔
$z^o = 180^o - 90^o - 40^o = 50^o$



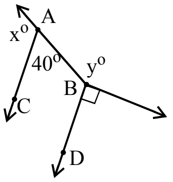
2. Answer: 67.5^o

$x + y + 90 = 360$ ➔
$y = 3x$ ➔ $x + 3x + 90 = 360$ ➔
$4x = 360 - 90 = 270$ ➔ $x^o = 67.5^o$



3. Answer: 130^o

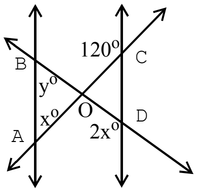
$x = 180 - 40 = 140$
$\overline{AC} \parallel \overline{BD}$ ➔ $\angle ABD = x^o$ ➔
$x + y + 90 = 360$ ➔
$140 + y + 90 = 360$ ➔
$y^o = 360^o - 140^o - 90^o = 130^o$



4. Answer: 0

$\overline{AB} \parallel \overline{CD}$ ➔
$\angle DCO = x° = 180° - 120° = 60°$

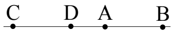
$\overline{AB} \parallel \overline{CD}$ ➔
$y = \angle CDO = 180° - 2x°$
$180° - 120° = 60°$
Hence $x^o - y^o = 60^o - 60^o = 0^o$



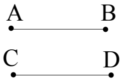
5. Answer: 0
All 4 lines can be parallel and never cross. So the answer is 0.

6. Answer: 0
All 4 lines can be parallel and never cross. So the answer is 0.

7. Answer: (E)
As you can see in the examples below, it is possible to have all the situations from (A) to (D) and still $\overline{AB}$ and $\overline{AB}$ not having any common points. So none of these cases are always wrong. So the answer is (E).

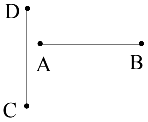


A, B, C and D are aligned in the way shown in the figure. This situation makes (A), (B) and (C) correct.



$\overline{AB} \parallel \overline{CD}$

This situation makes (C) Correct.



$\overline{AB} \perp \overline{CD}$

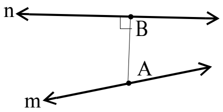
This situation makes (D) correct.

8. Answer: (E)

(A) may be correct, because you can place points A and B as far apart as you wish. Both lines extend indefinitely on both sides.

(B) may be correct, because the two lines may cross each other, and A and B may be the same crossing point. In this case their distance from one another is zero.

(C) may be correct. The figure illustrates how it can happen.

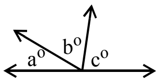


(D) may be correct for the same reason (C) is.

The answer is (E).

9. Answer: 115^o

$a + b + c = 180$
Substituting $a = \frac{2}{5}b$ and $b = \frac{2}{5}c$ into the above equation:

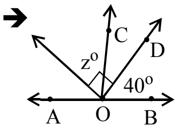


$\frac{2}{5}b + \frac{2}{5}c + c = \frac{2}{5} \cdot \frac{2}{5}c + \frac{2}{5}c + c =$

$\frac{39}{25}c = 180$ ➔ $c° = \frac{25}{39}180° \cong 115°$

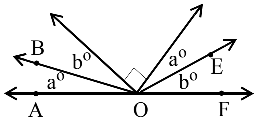
10. Answer: 60^o

$\angle AOC + \angle COD + 40° = 180°$ ➔
$110° + \angle COD + 40° = 180°$ ➔
$\angle COD = 180° - 110° - 40° = 30°$
$z° = 90° - \angle COD = 90° - 30° = 60°$



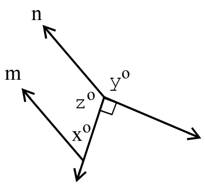
11. Answer: 135^o

$2a + 2b + 90 = 180$ ➔
$2a + 2b = 90$ ➔
$a + b = 45$
$\angle BOE = b° + 90° + a° = 90° + a° + b° = 90° + 45° = 135°$



12. Answer: 45^o

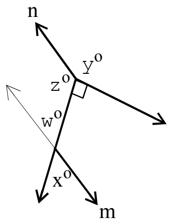
Let's label one more angle, z^o, in the figure.
$\vec{m} \parallel \vec{n}$ ➔ $z = 180 - x$
$z + y + 90 = 360$ ➔
Substituting z and y = 3x into the equation:
$180 - x + 3x + 90 = 360$ ➔
$2x = 90$ ➔ $x^o = 45^o$



13. Answer: 45^o

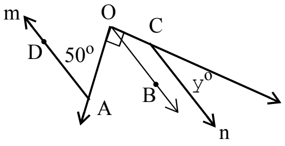
Let's extend $\vec{m}$ and label two more angles, z^o and w^o, in the figure.

$w = x$ and
$\vec{m} \parallel \vec{n}$ ➔ $z + w = 180$ ➔
$z = 180 - x$
$z + y + 90 = 360$ ➔
Substituting, z and $y = 3x$ into the equation:
$180 - x + 3x + 90 = 360$ ➔
$2x = 90$ ➔ $x^o = 45^o$



14. Answer: 40^o

Let's draw a parallel ray, $\overrightarrow{OB}$ to both $\vec{m}$ and $\vec{n}$ as shown in the figure.



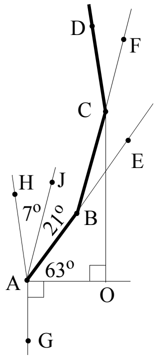
Because these 3 lines are parallel,
$\angle AOB = \angle DAO = 50°$ and $y° = \angle COB$
$\angle AOB + \angle COB = 90°$ ➔ $50 + y = 90$ ➔
$y^o = 90^o - 50^o = 40^o$

15. Answer: (B)

The answer has to be a multiple of 4, since at each intersection 4 angles are created. Among the answer choices, only 18 (B) is not divisible by 4. Hence the answer is (B).

16. Answer: 179^o

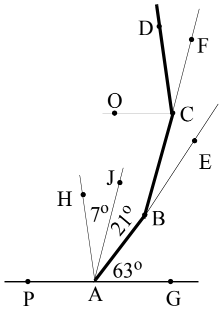
Let's draw line segments, $\overline{AG}$, $\overline{AH}$ and $\overline{AJ}$, such that,
$\overline{AG} \parallel \overline{CO}$, $\overline{AH} \parallel \overline{CD}$ and $\overline{AJ} \parallel \overline{BF}$
$\angle CBE = \frac{1}{3}\angle BAO = \frac{63°}{3} = 21°$
$\overline{AJ} \parallel \overline{BF}$ ➔
$\angle JAB = \angle CBE = 21°$
$\angle DCF = \frac{1}{3}\angle CBE = \frac{21°}{3} = 7°$
$\overline{AH} \parallel \overline{CD}$ and $\overline{AJ} \parallel \overline{BF}$ ➔
$\angle HAJ = \angle DCF = 7°$



$\angle HAG + 7° + 21° + 63° + 90° = 360°$ ➔
$\angle HAG = 179°$
$\overline{AH} \parallel \overline{CD}$ and $\overline{AG} \parallel \overline{CO}$ ➔
$\angle DCO = \angle HAG = 179°$

17. Answer: 89^o

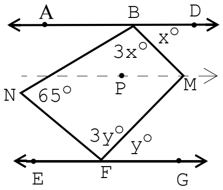
Let's extend AG and draw line segments,
$\overline{AH}$ and $\overline{AJ}$, such that,
$\overline{AH} \parallel \overline{CD}$ and $\overline{AJ} \parallel \overline{BF}$
$\angle CBE = \frac{1}{3}\angle BAG =$
$\frac{63°}{3} = 21°$ and $\overline{AJ} \parallel \overline{BF}$ ➔
$\angle JAB = \angle CBE = 21°$



$\angle DCF = \frac{1}{3}\angle CBE = \frac{21°}{3} = 7°$
$\overline{AH} \parallel \overline{CD}$ and $\overline{AJ} \parallel \overline{BF}$ ➔
$\angle HAJ = \angle DCF = 7°$
$\overline{AH} \parallel \overline{CD}$ and $\overline{AG} \parallel \overline{CO}$ ➔ $\angle HAP = \angle DCO$
$\angle HAP + 7° + 21° + 63° = 180°$ ➔
$\angle HAP = 180° - 91° = 89°$ ➔
$\angle DCO = \angle HAP = 89°$

18. Answer: 73.75^o

Let's draw $\overrightarrow{PM} \parallel \overleftrightarrow{AD}$ as shown in the figure.
$\overrightarrow{PM} \parallel \overleftrightarrow{AD}$ ➔ $\angle PMB = x°$
$\overrightarrow{PM} \parallel \overleftrightarrow{EG}$ ➔ $\angle PMF = y°$
$\angle BMF = x° + y°$



In the NFMB quadrangle, the total sum of all the inner angles is 360^o. ➔
$65 + 3x + 3y + \angle BMF = 360$ ➔
$65 + 3x + 3y + x + y = 360$ ➔
$65 + 4(x + y) = 360$ ➔
$\angle BMF = x° + y° = \frac{360 - 65}{4} = 73.5°$

Polygons

General

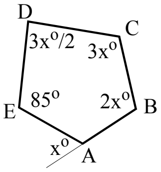
1. Answer: (C)

Let the measure of the inner angle be x.
The inner angles of a polygon add up to
$(n - 2)180 = nx$ ➔ $n = \frac{360}{180 - x}$
n is an integer and greater than two. If you substitute the values of x for cases (A), (B), (D) and (E), the above formula for n yields integer values of 3, 6, 8 and 9, respectively. However, for $x = 130^o$,
$n = \frac{360}{180 - 130} = \frac{360}{50} = 7.2$, which is not an integer. The answer is (C).

2. Answer: 75^o

$\angle A = 180° - x°$

Since the polygon is a pentagon,
$A + B + C + D + E =$
$(5 - 2)180 = 540^o$



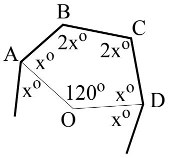
Substituting A, B, C, D and E into the above equation:

$180 - x + 2x + 3x + 3x/2 + 85 = 540^o$ ➔
$11x/2 = 275$ ➔ $x = 50^o$ ➔

$$D = \frac{3x}{2} = \frac{3 \times 50}{2} = 75°$$

3. Answer: 9

Let x be the half of an inner angle, as shown in the figure.



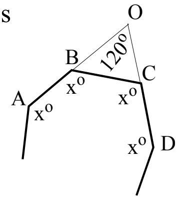
Since all the inner angles of the polygon are congruent, $B = C = 2x$.

OABCD is another polygon with 5 corners ➔
The addition of all of its inner angles is
$(5 - 2)180 = x + 2x + 2x + x + 120 =$
$6x + 120 = 540^o$ ➔ $x = 70^o$ ➔
One of the inner angles of the original polygon is
$2x = 2 \cdot 70 = 140°$
Let the number of vertices of the original polygon be n. The total sum of all the inner angles of this polygon is (n - 2)180. Only one angle's measure is

$$\frac{(n-2)180}{n} = 140 \text{ ➔ } (n-2)180 = 140n \text{ ➔}$$
$180n - 360 = 140n$ ➔ $40n = 360$ ➔
$n = 360/40 = 9$

4. Answer: 12

Let the measure of the inner angles of the polygon be x^o, as shown in the figure.



$\angle ABC = \angle BCD = x$ ➔

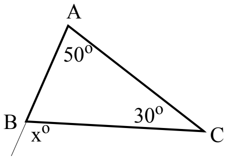
$\angle OBC = \angle BCO = 180 - x$ ➔
ΔOBC is an isosceles triangle.

Since $\angle O = 120°$, then
$\angle OBC = \angle BCO = (180 - 120)/2 = 30^o$ ➔
$x = 180 - 30 = 150$
Let the number of vertices of the original, regular polygon be n. The addition of the inner angles of this polygon is:
$(n - 2)180 = nx = 150n$ ➔ $n = 360/30 = 12$

Triangles: 3 - Sided Polygons

1. Answer: 80^o

The measure of an outer angle of a triangle is the addition of the two non-adjacent inner angles. Hence $x = 50 + 30 = 80$

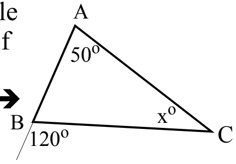


Alternate Solution:

$\angle ABC = 180 - 50 - 30 = 100^o$ ➔
$x° = 180° - \angle ABC = 180° - 100 = 80°°$

2. Answer: 70^o

The measure of an outer angle of a triangle is the addition of the two non-adjacent inner angles. Hence $120 = 50 + x$ ➔
$x = 120 - 50 = 70$

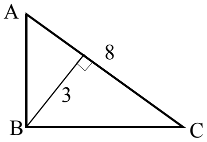


Alternate Solution:

$\angle ABC = 180° - 120° = 60°$
$x° = 180° - \angle ABC - 50° =$
$180° - 60° - 50° = 70°$

3. Answer: 12

Let's draw the figure first.

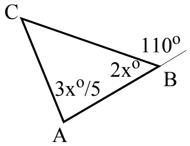


Hypotenuse, AC = 8, and the height, BD = 3

So the area $= \frac{8 \times 3}{2} = 12$

4. Answer: 89^o

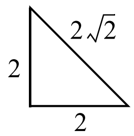
$2x = 180 - 110 = 70$ ➔ $x = 35$
$3x/5 + C = 110$ ➔
$\frac{3 \times 35}{5} + \angle C = 110$ ➔
$21 + \angle C = 110$ ➔ $\angle C = 89^o$



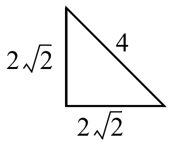
5. Answer: (E)

$2\sqrt{2}$ can be the length of the hypotenuse or the length legs.

If $2\sqrt{2}$ is the length of the hypotenuse, as shown in the figure, then the length of the legs are 2.
So, case I can be true.



If $2\sqrt{2}$ is the length of the one of the legs, then the other leg is also $2\sqrt{2}$. So case II can be true as well.
In this case the length of the hypotenuse =
$\sqrt{2\sqrt{2} + 2\sqrt{2}} = 4$ as shown in the figure.
So case III can also be true.



Since all 3 answer choices can be true, the answer is (E).

6. Answer: (E)

$4\sqrt{3}$ can be the length of any one of the 3 sides.

If $4\sqrt{3}$ is the length of the shortest leg, then the other leg = $4\sqrt{3} \cdot \sqrt{3} = 12$ and hypotenuse = $8\sqrt{3}$
So, cases (C) and (D) can be true. At this point you don't have to go any further. Since more than one of the answer choices may be true, the answer is (E), "All of the above."

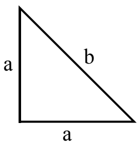
For the sake of completeness, let's prove that Case (A) and Case (B) can also be true.

If $4\sqrt{3}$ is the length of the hypotenuse then the shortest leg = $4\sqrt{3}/2 = 2\sqrt{3}$ and the other leg is $2\sqrt{3} \cdot \sqrt{3} = 6$
So, case (B) can be true.

If $4\sqrt{3}$ is the length of the longest leg, then the other leg = $4\sqrt{3}/\sqrt{3} = 4$ and hypotenuse = $8\sqrt{3}$
So, cases (A) and (C) can also be true.

7. Answer: $\dfrac{12\sqrt{2}}{2+\sqrt{2}}$

The figure displays an isosceles right triangle.
Using the Pythagorean Theorem,
$b = \sqrt{a^2 + a^2} = \sqrt{2}a$



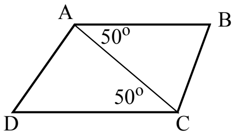
Periphery = $2a + \sqrt{2}a = a(2 + \sqrt{2}) = 12$ ➔

$a = \dfrac{12}{2+\sqrt{2}}$ ➔ hypotenuse = $\sqrt{2}a = \dfrac{12\sqrt{2}}{2+\sqrt{2}}$

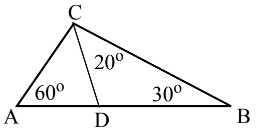
8. Answer: 65^o

$\overline{AB} \parallel \overline{DC}$ ➔
$\angle CAB = \angle DCA = 50°$.

AC = AB ➔
ΔCAB is isosceles ➔
$\angle ACB = \angle ABC = \dfrac{180° - 50°}{2} = 65°$



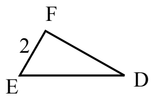
9. Answer: 70^o
The figure illustrates the information provided in the question.

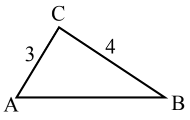


From the figure, you can easily see that:

$\angle ADC = 20° - 30° = 50°$
$\angle DCA = 180° - 50° - 60° = 70°$

10. Answer: 8/3





Since all three sides are parallel,
$\Delta ABC \sim \Delta EDF$ as shown in the figure. Notice which corners correspond to each other on each triangle.

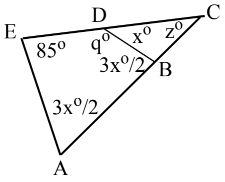
AC/BC = EF/DF ➔ $3/4 = 2/(DF)$ ➔ DF = 8/3

11. Answer: 23.75^o

In the figure, q = 180^o - x

Using the formula for the inner angles of a polygon for ABDE quadrangle:

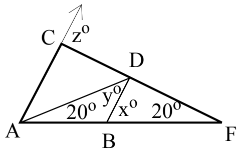
85 + q + 3x/2 + 3x/2 = 360 ➔



85 + 180 - x + 3x = 360 ➔ 2x = 95 ➔ x = 47.5

3x/2, is one of the outer angles of ΔBCD. Hence the addition of two inner angles, x and z is 3x/2:
x + z = 3x/2 ➔ z = 3x/2 - x = x/2 = 47.5/2 = 23.75

12. Answer: 60^o
In the figure,
$\angle ABD = 180 - x$ ➔
In ΔABD,
$20 + y + \angle ABD = 180$ ➔
20 + x/2 + 180 - x = 180 ➔
x = 40 ➔ y = 20



$\overline{AC} \parallel \overline{BD}$ ➔ $\angle CAB = x = 40$

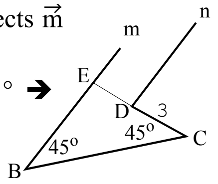
z is an outer angle of ΔAFC. ➔
$z = \angle CAF + \angle F$ ➔ z = 40 + 20 = 60

13. Answer: $5\sqrt{2}$

Let's extend $\overline{CD}$ until it intersects $\vec{m}$ at point E.

$\angle E = 180° - 45° - 45° = 90°$ ➔

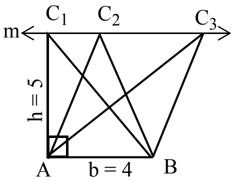
ΔEBC is an isosceles right triangle.



$\vec{m} \parallel \vec{n}$ and $\angle E = 90°$ ➔

DE is the distance between $\vec{m}$ and $\vec{n}$ ➔
EB = EC = 2 + 3 = 5
Using the Pythagorean Theorem:
$BC = \sqrt{25 + 25} = 5\sqrt{2}$

14. Answer: (D)
As you can see in the figure, if you fix the 2 corners of the triangle, A and B, such that AB = 4, the third corner, C, can be anywhere on the line m, which is parallel to $\overline{AB}$.



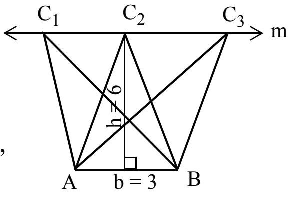
As long as the point C is on line m, the height, h, of the triangle is the same. For example the heights of the triangles ABC_1, ABC_2 and ABC_3 are all h.

The area of ΔABC for any of the triangles is $\frac{h \times b}{2} = \frac{h \times 4}{2} = 2h = 10$ ➔ h = 5

You can also see that the smallest AC can get is when AC = h = 5. AC = 5 occurs when $\overline{AB} \perp \overline{AC}$ and the third corner, C, is at C_1 position in the above figure. So the sum total of the three sides, AB + AC + BC, can not be less than 4 + 5 + 5 = 14. Hence the periphery of ΔABC has to be more than 14.

Therefore none of the cases can be the periphery of the triangle, because all three of them are less than 14. Hence the answer is (D).

15. Answer: $\frac{3\sqrt{17}}{2}$



As you can see in the figure, if you fix the 2 corners, A and B, of the triangle such that AB = 3, the third corner, C, can be anywhere on line m, which is parallel to $\overline{AB}$.

As long as the point C is on line m, the height, h, of the triangle is the same. For example the heights of the triangles ABC_1, ABC_2 and ABC_3 are all h.

The area of the triangle = $\frac{h \times b}{2} = \frac{h \times 3}{2} = 9$ ➔ h = 6

You can also see that the longest sides of triangles ABC_1, ABC_2, ABC_3 are BC_1, $BC_2 = AC_2$ and AC_3 respectively.

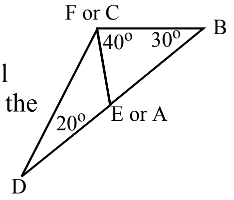
As point C moves from left to right, the longest side decreases in length first until AC = BC, when C is at point C_2. It then starts increasing again as you move point C to the right of C_2.

In other words, the minimum value of the longest side occurs when the third corner C is at the position C_2 in the above figure.

Now if you consider the triangle ABC_2, you can easily calculate the minimum length of the longest side as: $BC_2 = AC_2 =$

$$\sqrt{h^2 + \left(\frac{b}{2}\right)^2} = \sqrt{36 + \frac{9}{4}} = \frac{\sqrt{153}}{2} = \frac{3\sqrt{17}}{2}$$

16. Answer: 90^o



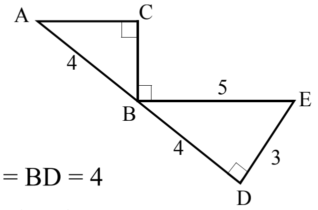
It is easy to answer the question if you slide the two triangles such that the parallel sides are aligned as shown in the figure.

For visual aid, also put the data in the figure.

$\angle CAD = 30 + 40 = 70^o$

$\angle F = \angle DFE = 180 - 70 - 20 = 90^o$

17. Answer: 9.6



Since BDE is a right triangle, the area of $\Delta BDE = (3 \cdot BD)/2 = 6$ ➔ BD = 4

B bisects AD ➔ AB = BD = 4

Since BDE is a right triangle,

$$BE = \sqrt{BD^2 + DE^2} = \sqrt{9 + 16} = 5$$

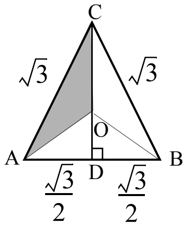
$$\left.\begin{aligned} &\angle D = \angle C = 90° \\ &\overline{AC} \parallel \overline{BE} ➔ \\ &\angle A \cong \angle DBE \end{aligned}\right\} ➔ \Delta ACB \sim \Delta BDE ➔$$

AC/BD = AB/BE ➔ AC/4 = 4/5 ➔
AC = 16/5 = 3.2 and

CB/DE = AB/BE ➔ CB/3 = 4/5 ➔
CB = 12/5 = 2.4

Periphery of ABC = 2.4 + 3.2 + 4 = 9.6

18. Answer: $\frac{\sqrt{3}}{4}$



Let the height of the triangle, CD = h.

Since ΔABC is equilateral, D bisects $\overline{AB}$ as shown in the figure.

Using Pythagorean Theorem for ΔCDB:

$$h = \sqrt{(\sqrt{3})^2 - \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3 - \frac{3}{4}} = \sqrt{\frac{9}{4}} = 3/2$$

The area of the ΔABC =

$$h \times AD = \frac{1}{2}\left(\frac{3}{2} \times \sqrt{3}\right) = \frac{3\sqrt{3}}{4}$$

The shaded area is 1/3 of the area of the triangle ABC because the areas of AOC, AOB and BOC are equal in an equilateral triangle.

So the answer is $\frac{\sqrt{3}}{4}$.

Quadrangles: 4 - Sided Polygons

Trapezoid

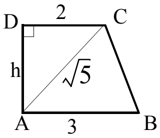
1. Answer: 2.5

ΔACD is a right triangle ➔

$AD^2 = (\sqrt{5})^2 - 2^2 =$

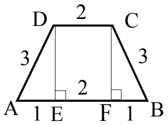
$5 - 4 = 1$ ➔ $h = AD = 1$

Area = $(3 + 2)h/2 = 5/2 = 2.5$



2. Answer: $6\sqrt{2}$

As shown in the figure, if you draw perpendicular line segments, $\overline{DE}$ and $\overline{CF}$ from D and C to $\overline{AB}$, then EFCD is a rectangle with DE = CF and EF = DC = 2



On the other hand, AD = BC = 3 ➔

$\Delta AED \cong \Delta BFC$ ➔

AE = FB = (AB - EF)/2 = (4 - 2)/2 = 1

Using the Pythagorean Theorem for ΔAED:

$DE^2 = 3^2 - 1^2 = 9 - 1 = 8$ ➔ $DE = \sqrt{8} = 2\sqrt{2}$

The area of ABCD = $\left(\frac{1}{2}\right)(4 + 2)2\sqrt{2} = 6\sqrt{2}$

3. Answer: 27

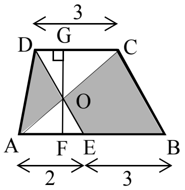
Area of $\Delta AEO = 4$ ➔

$(2 \times OF)/2 = 4$ ➔ OF = 4

$DC \parallel AB$ ➔

$\angle OEA \cong \angle ODC$ and

$\angle OAE \cong \angle OCD$



$\overline{DE}$ crosses $\overline{AC}$ at point O ➔

So $\Delta AOE \sim \Delta COD$ ➔ OA/OC = AE/CD = 2/3

Similarly, $\Delta OFA \sim \Delta OGC$ ➔

OF/OG = 4/OG = OA/OC = 2/3 ➔ OG = 6 ➔

Area of $\Delta ODC = \frac{3 \cdot 6}{2} = 9$ and

Area of ABCD =

$\frac{(AB + DC)(OF + OG)}{2} =$

$\frac{(2 + 3 + 3) \cdot (4 + 6)}{2} = \frac{8 \cdot 10}{2} = 40$

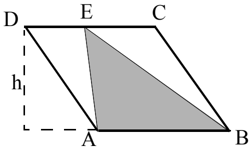
The area of the shaded region =
The area of ABCD -
The area of ΔAEO - The area of ΔDOC =
40 - 4 - 9 = 27

Parallelogram

1. Answer: 1/2

The area of the shaded region = $h \cdot AB/2$

The area of the parallelogram = $h \cdot AB$



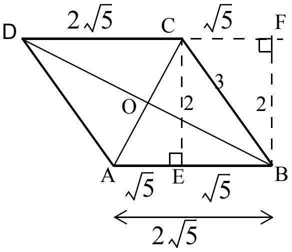
Hence the area of the shaded region to the area of the parallelogram ABCD =

$\frac{h \cdot AB/2}{h \cdot AB} = \frac{1}{2}$

2. Answer: AC = 3 and BD = 7

Let's redraw the figure with some additions.

In the figure,
BF = EC = 2
AB = DC = $2\sqrt{5}$
D, C and F are on the same line.



Finding AC:

Using the Pythagorean Theorem for the right triangle ΔCEB:

$EB = \sqrt{3^2 - 2^2} = \sqrt{5}$ ➔

$AE = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$ ➔

In ΔABC, the height, $\overline{CE}$, bisects base $\overline{AB}$ ➔

ΔABC is isosceles ➔ AC = 3

Finding BD:

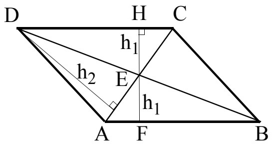
In the right triangle ΔDBF,

$DB = \sqrt{DF^2 + FB^2} = \sqrt{(2\sqrt{5} + \sqrt{5})^2 + 2^2} =$

$\sqrt{(3\sqrt{5})^2 + 2^2} = \sqrt{49} = 7$

3. Answer: 12

The heights, h_1, and the bases, DC and AB, of ΔCED and ΔBAE are the same. Hence the areas of these two triangles are the same.



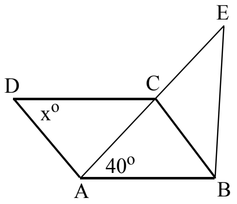
The heights, h_1, and the bases, DC and AB, of ΔCED and ΔBAE are the same. Hence the areas of these two triangles are the same as well.

Therefore the diagonals of a parallelogram divides the parallelogram into 4 triangles of equal areas.

➔ Area of ΔCEB = Area of $\Delta BEA = 12$

4. Answer: 60^o

$EB = DC = AB$ ➔
ΔABE is isosceles ➔
$\angle E = \angle CAB = 40°$



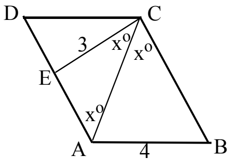
$CE = AD = CB$ ➔
ΔECB is isosceles ➔
$\angle CBE = \angle E = 40°$

$\angle BCE = 180 - 40 - 40 = 100°$
$\angle ACB = \angle DAC =$
$180 - \angle BCE = 180 - 100 = 80°$

$\angle DAB = 40 + 80 = 120°$ ➔
$x° = 180 - \angle DAB = 180 - 120 = 60°$

5. Answer: 20

Let $x = \angle ECA$ as shown in the figure.



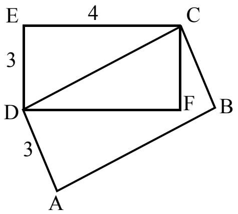
$\overline{DA} \parallel \overline{CB}$ ➔ $\angle ACB =$
$\angle EAC = x$ ➔

ΔAEC is isosceles with
$AE = EC = 3$
E bisects $\overline{DA}$ ➔ $ED = AE = 3$ ➔
$BC = AD = 2 \cdot 3 = 6$ ➔
The periphery of ABCD = 6 + 6 + 4 + 4 = 20

Rectangles and Squares

1. Answer: 1.25

Area of DFCE =
$3 \times 4 = 12$

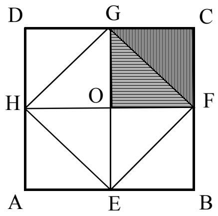


Using Pythagorean Theorem for ΔDCE
$DC = \sqrt{4^2 + 3^2} = 5$
Area of ABCD =
$DC \times AD = 5 \times 3 = 15$
(Area of ABCD)/(Area of DFCE) = 15/12 = 1.25

The answer is (C).

2. Answer: (A)

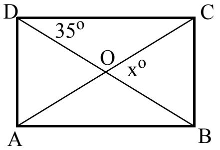
To get the answer, you can use the Pythagorean Theorem to calculate the side length of EFGH in terms of the area of ABCD, a, and then calculate the area. However it is faster to see the correct answer visually. Here is how:



In the figure, the two shaded triangles OFG and CFG are congruent. So the areas of these two triangles are equal. EFGH is composed of 4 of these triangles and ABCD is composed of 8 of these triangles. Hence the area of ABCD is twice as large of the area of EFGH.
The area of EFGH = a/2.

3. Answer: 70^o

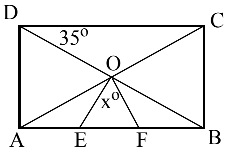
In a rectangle, the diagonals bisect each other. So OCD is an isosceles triangle with
$\angle OCD = \angle ODC = 35°$



x is an outer angle of ΔOCD. Hence it is equal to the addition of the two non-adjacent inner angles:
$\angle OCD + \angle ODC = 35° + 35° = 70°$

4. Answer: 40^o

In a rectangle, the diagonals bisect each other. So OCD is an isosceles triangle with
$\angle OCD = \angle ODC = 35°$



Sum total of the inner angles of $\Delta OCD = 180°$ ➔

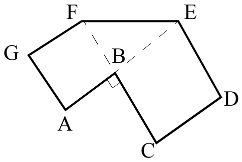
$\angle COD = 180 - 35 - 35 = 110°$

$\overline{DC} \parallel \overline{AB}$ ➔ $\angle OBF = \angle ODC = 35°$
ΔABO is isosceles ➔ $\angle OAE = \angle OBF = 35°$
$OF = FB$ ➔ ΔOBF is an isosceles triangle ➔
$\angle FOB = \angle FBO = 35°$.
$AE = EO$ ➔ ΔOAE is an isosceles triangle ➔
$\angle AOE = \angle OAE = 35°$

$\angle AOB = \angle COD = 110°$
$x = \angle AOB - \angle AOE - \angle FOB =$
$110 - 35 - 35 = 40^o$

5. Answer: (A)

$\overline{AB} \perp \overline{BC}$ ➔ ΔBFE is a right triangle.
Using the Pythagorean Theorem:
$EF^2 = BF^2 + EB^2 =$

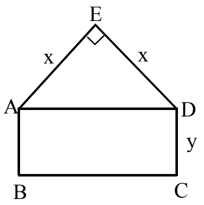


Area of ABFG + Area of BCDE =
$a + 2a = 3a$ ➔ $EF = \sqrt{3}a$

The answer is (A).

6. Answer: (B)

The area of the triangle =
$x^2/2 = 2.25$ ➔ $x = 3/(\sqrt{2})$



ΔADE is a right triangle. ➔

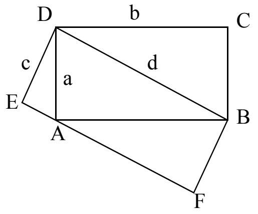
$AD = \sqrt{x^2 + x^2} = \sqrt{2}x = 3$

The area of the rectangle =
$3y = 10$ ➔ $y = 10/3$

The answer is (B).

7. Answer: 1

Let the lengths of the two sides of ABCD be a and b and the lengths of the sides of EFBD be c and d, as shown in the figure.



E = C = 90^o and $\angle EDA = \angle CDB$ (Their arms are perpendicular to each other.) ➔

$\Delta ADE \sim \Delta BDC$ ➔

$\frac{DE}{AE} = \frac{DC}{BC}$ ➔ $\frac{c}{AE} = \frac{b}{a}$ ➔ $c = AE\frac{b}{a}$ and

$\frac{BD}{BC} = \frac{AD}{AE}$ ➔ $\frac{d}{a} = \frac{a}{AE}$ ➔ $d = \frac{a^2}{AE}$

The area of EFBD = $cd = AE\frac{b}{a} \times \frac{a^2}{AE} = ab$ = The area of ABCD ➔
(The area of EFBD) / (The area of ABCD) = 1

Circles

1. Answer: 1

Let r be the radius of the circle.

Area of the circle = $\pi = \pi r^2$ ➔ r = 1

2. Answer: (C)

Let c be the circumference of the circular path.

$c = 2\pi 30 = 60\pi$

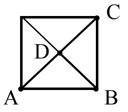
The distance that he travels in one hour:

$15c = 2\pi 30 \times 15 = 900\pi$ ➔

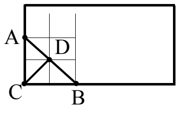
The speed in feet/min. = $\frac{900\pi}{60} = 15\pi$

3. Answer: (E)

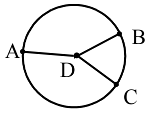
A, B and C can be on any of the shapes given in the answer choices. The below figure illustrates how AD = BD = CD when A, B and C are on a square, a rectangle and a circle.



Square



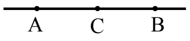
Rectangle

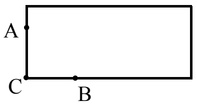


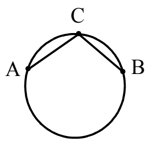
Circle

4. Answer: (E)

A, B and C can be on any of the shapes given in the answer choices. The below figure illustrates how AC = BC when A, B and C are on a line, a rectangle and a circle.

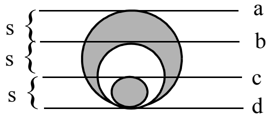






5. Answer: 1/5

In the figure, the radii of the three circles are s/2, s and 3s/2, from the smallest to the largest.



The area of the small shaded circle: $\pi\frac{s^2}{4}$

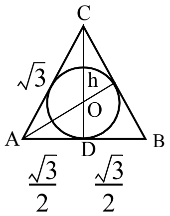
The area of the shaded crescent =
Area of the largest circle -
Area of the middle circle $= \pi\frac{9s^2}{4} - \pi s^2 = \pi\frac{5s^2}{4}$

(The area of small shaded circle) /
(The area of the shaded croissant) =

$\left(\pi\frac{s^2}{4}\right) / \left(\pi\frac{5s^2}{4}\right) = \frac{1}{5}$

6. Answer: $\frac{\pi}{4}$

Since ΔABC is equilateral, $\angle A = \angle B = \angle C = 60°$ and the center of the circle, O, is the intersection of the two heights of ΔABC.



Also, the heights, the angular bisectors and the medians are one and the same. ➔

AD = DB = $\frac{\sqrt{3}}{2}$ as shown in the figure.

ΔADC is a 30^o-60^o-90^o right triangle. ➔

$h^2 = CD^2 = AC^2 - AD^2 = 3 - 3/4 = 9/4$ ➔ h = 3/2

Consider the two triangles, ΔADO and ΔCDA. They are both 30^o-60^o-90^o triangles. ➔

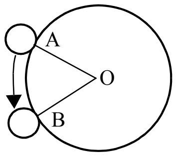
ADO ~ CDA ➔ OD/AD = AD/CD ➔

The radius of the circle = OD = $AD^2/CD = (3/4)/(3/2) = 1/2$ ➔

The area of the circle is $\pi r^2 = \frac{\pi}{4}$

7. Answer: (100w/p)%

The arc $\overset{\frown}{AB}$ traveled by the wheel after a complete turn is the circumference of the wheel, which is $2\pi w$.



The circumference of the circular path is $2\pi p$.

The percentage of the circumference of the circle O traveled =

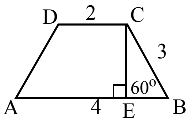
$\frac{2\pi w}{2\pi p} \cdot 100 = \frac{100w}{p}$ %

Trigonometry

1. Answer: $(9\sqrt{3})/2$

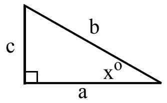
Height, CE = 3sin(60) = $\frac{3\sqrt{3}}{2}$

Area = $\frac{2+4}{2} \cdot \frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$



2. Answer: $(\sqrt{3})$
c/b = 1/2 ➔ sin(x) = 1/2 ➔
x = 30^o

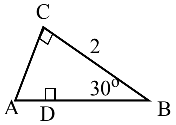
a/c = cot(x) = cot(30) = $\sqrt{3}$



3. Answer: $1/(\sqrt{3})$
sin(30) = 1/2 =
CD/CB = CD/2 ➔ CD = 1

$\angle A = 180 - 90 - 30 = 60°$ ➔

tan(60) =
CD/AD = 1/AD = $\sqrt{3}$ ➔
AD = $1/(\sqrt{3})$



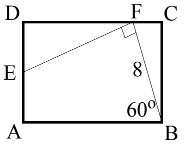
4. Answer: 10
$\overline{AB} \parallel \overline{DC}$ ➔ $\angle BFC = 60°$ ➔
FC = 8cos(60) = 8/2 = 4

Let DE = EA = a ➔
BC = AD = 2a = 8sin(60) =

$\frac{8\sqrt{3}}{2} = 4\sqrt{3}$ ➔ $a = 2\sqrt{3}$

$\angle EFD = 180 - 90 - 60 = 30°$
$DF = DE \times \cot(30) = a \times \cot(30) =$

$2\sqrt{3} \cdot \sqrt{3} = 6$
AB = DC = DF + FC = 6 + 4 = 10



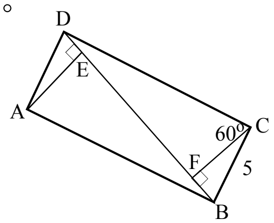
5. Answer: 5
$\angle FCB = 90 - 60 = 30°$

$\overline{AE} \parallel \overline{CF}$ ➔ $\angle EAB =$
$\angle FCD = 60°$ ➔
$\angle DAE =$
90 - 60 = 30^o ➔
DE = BF = 5sin(30) =
5/2 and

CF = 5cos(30) = $5\frac{\sqrt{3}}{2}$

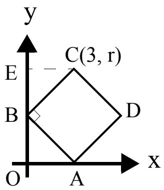
$\angle FDC = 90 - 60 = 30°$ ➔

tan(30) = CF/FD = $\left(5\frac{\sqrt{3}}{2}\right)/(FD) = \frac{1}{\sqrt{3}}$ ➔
FD = 15/2 ➔
EF = DF - DE = 15/2 - 5/2 = 10/2 = 5



Coordinate Geometry

1. Answer: 6
OA = OB ➔ ΔOAB is an isosceles, right triangle. ➔
$\angle OBA = \frac{180-90}{2} = 45°$ ➔
$\angle EBC = \angle ECB = 45°$ ➔
ΔCEB is an isosceles, right triangle. ➔ EB = EC = 3
ABCD is a square ➔ BA = BC ➔
$\Delta AOB \cong \Delta CEB$ ➔ OB = EB = 3 ➔
r = EB + BO = 3 + 3 = 6

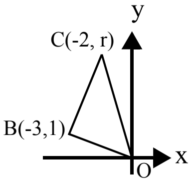


2. Answer: 4
BC= OB ➔
$\sqrt{(-2+3)^2 + (r-1)^2} =$

$\sqrt{(-3)^2 + (1)^2}$ ➔

$\sqrt{1 + (r-1)^2} = \sqrt{10}$

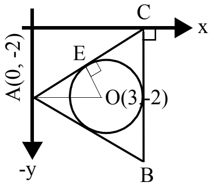
$1 + (r - 1)^2 = 10$ ➔ r - 1 = 3 ➔ r = 4



3. Answer: $\frac{9\pi}{4}$ or 7.069

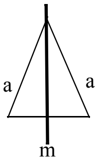
ΔABC is an equilateral triangle. ➔ $\angle A = 60°$

$\overline{AC}$ and $\overline{AB}$ are tangents to the circle O ➔
$\overline{AO}$ bisects $\angle A$ ➔
$\angle OAE = 60/2 = 30°$
sin(30) = EO/AO = EO/3 = 1/2 ➔
The radius of the circle, r = EO = 3/2 ➔
The area of the circle =
$\pi r^2 = \pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4} = 7.069$

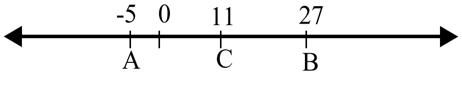


Symmetry

1. Answer: 1
There is only one line of symmetry and it is shown in the figure. Either side of the line m is the mirror image of the other side.



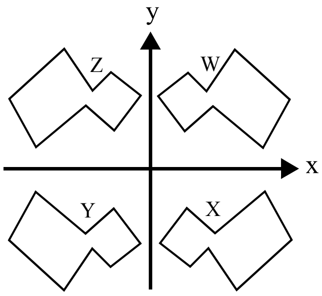
2. Answer: 11



Let the point in question be C. If the coordinates of A and B are -5 and 27 respectively, then
AB = 27 - (-5) = 27 + 5 = 32
A and B are symmetrical around C. ➔
C bisects $\overline{AB}$. ➔ AC = CB = AB/2 = 32/2 = 16 ➔
C's coordinate is -5 + 16 = 11
The coordinates of A, B and C and the origin are shown in the above figure.

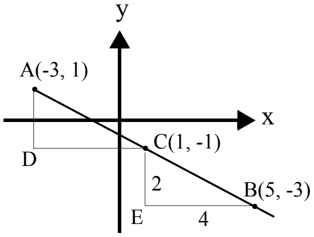
3. Answer: (E)

W and Z are symmetric around the y-axis.
Z and Y are symmetric around the x-axis.
Y and X are symmetric around the y-axis.
X and W are symmetric around the x-axis.
W and Y are not symmetric around any axis.
So the answer is (E).



4. Answer: 2

A and B are symmetric around C ➔ A, B and C are on the same line and AC = BC, as shown in the figure.



Let (A_x, A_y), (B_x, B_y), (C_x, C_y) be the x- and y-coordinates of A, B and C respectively.

AC = BC ➔

$B_x - C_x = C_x - A_x$ ➔

$B_x - 1 = 1 - (-3) = 4$ ➔ $B_x = 4 + 1 = 5$

and

$B_y - C_y = C_y - A_y$ ➔

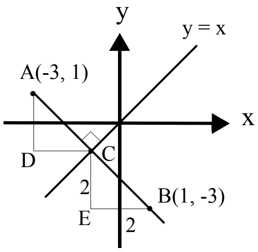
$B_y - (-1) = -1 - 1 = -2$ ➔ $B_y = -2 - 1 = -3$➔

The coordinates of point B is (5, -3) ➔

The addition of the x- and y-coordinates of point B is $5 - 3 = 2$

5. Answer: -3

Let $\overleftrightarrow{m}$ be the line represented by y = x.
A and B are symmetrical around line $\overleftrightarrow{m}$. ➔
$\overline{AB} \perp \overleftrightarrow{m}$ and AC = BC, where C is the intersection of line m and $\overline{AB}$, as shown in the figure.



$\overline{AB} \perp \overleftrightarrow{m}$ ➔ The equation for the $\overleftrightarrow{AB}$ is $y = -x + c$, where c is a constant.

Since $\overleftrightarrow{AB}$ passes through point A(-3, 1), then
$1 = -(-3) + c$ ➔ $c = 1 - 3 = -2$ ➔

C is at the intersection of y = x and y = -x - 2 ➔

At point C, x = -x - 2 ➔ x = - 1 and y = -1 ➔

The coordinates of C is (-1, -1)

Let (A_x, A_y), (B_x, B_y), (C_x, C_y) be the x- and y-coordinates of A, B and C respectively.

AC = BC ➔

$B_x - C_x = C_x - A_x$ ➔

$B_x - (-1) = -1 - (-3) = 2$ ➔ $B_x = 2 - 1 = 1$

and

$B_y - C_y = C_y - A_y$ ➔

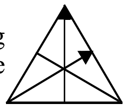
$B_y - (-1) = -1 - 1 = -2$ ➔ $B_y = -2 - 1 = -3$➔

The coordinates of point B is (1, -3) ➔

The multiplication of the x- and y-coordinates of B is $1 \cdot (-3) = -3$

6. Answer: (A)

When you rotate the triangle 30^o, counter clockwise, the arrow pointing up will point toward upper left. So the answer is either (A) or (D).



At the same time, the other arrow will still point upper right but more steeply. In fact, it will align with the left side of the original triangle. The only answer choice that satisfies these two requirements is (A).

3 - Dimensional Objects

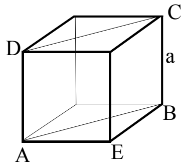
1. Answer: 12a

A cube has 12 equal length edges. Hence the total length of its edges is 12a.

2. Answer: $\sqrt{2}a^2$

AE = EB = a and
$\angle AEB = 90°$ ➔
Using Pythagorean Theorem,
$AB = \sqrt{a^2 + a^2} = \sqrt{2}a$

Area of ABCD =
$AB \times CB = \sqrt{2}a \times a = \sqrt{2}a^2$

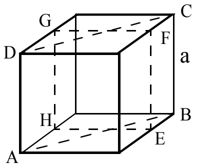


3. Answer: b

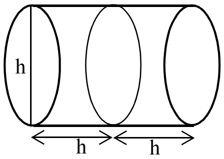
Case a creates two triangular prisms with 6 corners each.

Case b creates two rectangular prisms with 8 corners each.

Hence the answer is b.



4. Answer: 6/5



Let O, B and S are the total area, base area and the side area of the original cylinder, respectively. Original area = O = S + 2B =

$$2\pi\left(\frac{h}{2}\right)^2 + \pi h \times 2h = \frac{5\pi h^2}{2}$$

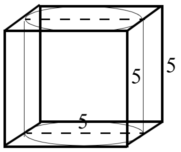
Extra area created by the cut = 2B
Total area after cut = 2B + O

$$2\pi\left(\frac{h}{2}\right)^2 + \frac{5\pi h^2}{2} = 3\pi h^2$$

Total area after cut / Original area =

$$\frac{3\pi h^2}{\left(\frac{5\pi h^2}{2}\right)} = \frac{3}{\left(\frac{5}{2}\right)} = \frac{6}{5}$$

5. Answer: $(125/4)\pi$

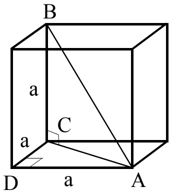


As shown in the figure, the height and the diameter of the base area of the cylinder are 5.

The volume of the cylinder = (Base area) x (Height) =

$$\pi\left(\frac{5}{2}\right)^2 \cdot 5 = \frac{125}{4}\pi$$

6. Answer: $\sqrt{3}a$



The longest length inside a cube is AB, as shown in the figure.

Using the Pythagorean Theorem for the right triangle ΔDAC :

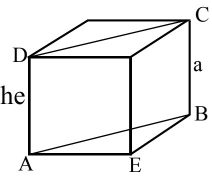
$AC^2 = 2a^2$

Using the Pythagorean Theorem for the right triangle ΔACB :

$AB^2 = AC^2 + CB^2 = 2a^2 + a^2 = 3a^2$ ➔

$AB = \sqrt{3}a$

7. Answer: $2\sqrt{2}a^2$



By cutting the cube through ABCD, the only extra surface area created is the area of ABCD for each prism. Hence the answer is twice the area of ABCD.

Using the Pythagorean Theorem for the right triangle ΔABE :

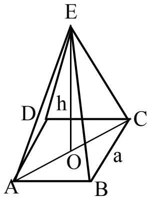
$AB = \sqrt{a^2 + a^2} = \sqrt{2}a$

The area of ABCD = $AB \cdot BC = \sqrt{2}a \cdot a = \sqrt{2}a^2$ ➔

The total extra area = $2\sqrt{2}a^2$

8. No answer is provided.

9. Answer: $\sqrt{\frac{a^2}{2} + h^2}$



Using the Pythagorean Theorem for the right triangle ΔABC :

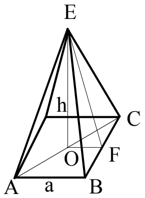
$AC = \sqrt{2}a$

Using the Pythagorean Theorem for the right triangle ΔCOE :

$$CE = \sqrt{OE^2 + OC^2} =$$

$$\sqrt{h^2 + \left(\frac{\sqrt{2}a}{2}\right)^2} = \sqrt{\frac{a^2}{2} + h^2}$$

10. Answer: $2a\sqrt{\frac{a^2}{4} + h^2} + a^2$



Using the Pythagorean Theorem on right triangle ΔFOE :

$$FE = \sqrt{OE^2 + OF^2} = \sqrt{\frac{a^2}{4} + h^2}$$

Base area = a^2
Side area = 4 x (Area of ΔBCE)

Area of $\Delta BCE = \frac{a \cdot EF}{2}$

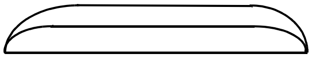
The total surface area = $2a\sqrt{\frac{a^2}{4} + h^2} + a^2$

11. Answer: (D)

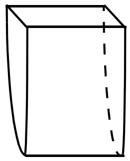
A cradle has a half-cylinder and two semi circles on each end. You can eliminate case (A) because it has two full circles. The 3-D object that you can make by using the shapes that are given in the rest of the cases are displayed below.
Both case (C) and (D) displays a cradle shape, however, in the case of (C), w > l. Hence the answer is (D).

Note that you can make the half-cylinder body of the cradle by using a rectangle.

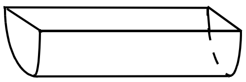
(B)



(C)



(D)



7

ALGEBRA

In this chapter we assume that you have studied Chapter 5, Arithmetic, and are familiar with the topics explained there. If you need to, go back to Chapter 5 and study the appropriate section at your level.

This chapter has two parts. Simple Algebra and Advanced Algebra.

Many of the Easy questions in the SAT are about **Simple Algebra**. This part is especially important for the students with math scores less than 500. The methods suggested here will also help the advanced students and enable them to increase their speed.

The second half, **Advanced Algebra**, is very important to the advanced students. There are many Medium and Hard level questions in SAT about Advanced Algebra.

The format is the same as in the other math subjects sections. New concepts are explained by examples. Most sections have Practice Exercises. At the end of the chapter you can find exercises and their solutions.

Simple Algebra

One Variable Simple Equations

Many of the Easy Questions in SAT are about one variable equations. Sometimes the equation is given and you need to solve the equation to find the unknown. Other times the question is a word question and you need to construct the equation before solving it.

Since most of the questions in SAT about this subject are Easy, in this section we provide several Easy examples.

Definition

One variable equation is an equation with only one unknown.

Examples:

Equation	Variable	Coefficient
$4x = 0$	x	4 (of x)
$-3b + 1 = 3$	b	-3 (of b)
$2c^2 - 3 = 15$	c	2 (of c^2)
$y^2 - 2y + 2 = -2(y - 3)$	y	1 (of y^2)
$\frac{2x-4}{7x-3} = 2$	x	12 (of x)

The equations in the first column are all one variable equations.

The variables of these equations are displayed in the second column.

The constant that is multiplied by the variable (appears in front of the variable) is called the **coefficient** of the variable. The last column in the above table displays the coefficients of the variables.

The expression in the last example, $(2x - 4)/(7x - 3)$, is called a **rational expression** because it is the division of two expressions, 2x - 1 and 7x - 3. We can easily simplify it as follows:

$\frac{2x-4}{7x-3} = 2$ ➔ $2x - 4 = 2(7x - 3)$ ➔ $2x - 4 = 14x - 6$ ➔

$12x - 2 = 0$. So the coefficient of x is 12.

Practice Exercises:

1. (Easy)
Write 3 examples of rational expression.

2. (Easy)
What is the variable and the coefficient in the following expressions?

a. $-4d - 2 = 3$

b. $2a^3 - 8 = 1$

c. $\frac{3x-4}{7x-8-3x} = -4$

d. $\frac{20x-4}{7x-3} = 2$

Answers:
1. $(s^4 - 3s^3 + 1)/(s^2 + 2s)$;
2. a. d and -4; **b.** a and 2; **c.** x and 19; **d.** x and 6

How to Solve a One-Variable Simple Equation

Follow the steps below to solve a one-variable equation:

1. Perform all the operations to remove all the parentheses and rational expressions, if any. Order of the operations are explained for different expressions in chapter 5. If necessary, study those sections.
2. Carry the terms with the unknown to the one side of the equal sign. Remember that the sign of the term changes, from + to - and from - to +, as you move it from one side of equal sign to the other.
3. Carry the other terms to the other side of the equal sign. Remember that the sign of the term changes, from + to - and from - to +, as you move it from one side of equal sign to the other.
4. Add or subtract all the like terms on both sides of the equal sign.
5. Solve for the unknown by dividing both sides of the equal sign by the coefficient of the unknown.

Examples:

1. (Easy)
$4x = 0$, then $x = 0$

2. (Easy)
If $-3b + 1 = 3$, then b = ?

Solution:
$-3b + 1 = 3$ ➔ $-3b = 3 - 1$ ➔ $-3b = 2$ ➔ $b = -2/3$

3. (Easy)
If $-3a - 4 = 7a + 1$, then a = ?

Solution:
$-3a - 4 = 7a + 1$ ➔ $-3a - 7a = 1 + 4$ ➔ $-10a = 5$ ➔ $a = -5/10 = -1/2$

4. (Easy)
If $-4(5 - u) = 3(2u + 3)$, then $u = ?$

Solution:
$-4(5 - u) = 3(2u + 3)$ ➔ $-20 + 4u = 6u + 9$ ➔
$4u - 6u = 9 + 20$ ➔ $-2u = 29$ ➔ $u = -29/2$

5. (Easy)
If $\frac{2x + 3}{1 - x} = 5$, then $x = ?$

Solution:
$\frac{2x + 3}{1 - x} = 5$ ➔ $2x + 3 = 5 - 5x$ ➔ $7x = 2$ ➔ $x = 2/7$

6. (Medium)
If $2c^2 - 3 = 15$, then $x = ?$

Solution:
$2c^2 - 3 = 15$ ➔ $2c^2 = 18$ ➔ $c^2 = 18/2 = 9$ ➔
$c = 3$ or $c = -3$.
Note that square of both 3 and -3 is 9.

Sometimes, the equation looks complicated with higher powers and/or more than one variables. You can simplify these equations by working through them. Therefore if a question is in the beginning of a section and looks complicated, don't be intimidated. Instead, try to simplify the question.

Examples:

1. (Medium)
If $2(x^2 - 3x + 3) + x(3 - 2x) = x + 1$,
then $x = ?$

Solution:
$2(x^2 - 3x + 3) + x(3 - 2x) = x + 1$ ➔
$2x^2 - 6x + 6 + 3x - 2x^2 = x + 1$ ➔
$-3x + 6 = x + 1$ ➔ $6 - 1 = 3x + x$ ➔
$4x = 5$ ➔ $x = 5/4$

2. (Medium)
If $2a - b - c^2 - 12 = -(c^2 - b + 1) + 2(a - 4)$,
then $a = ?$, $b = ?$ and $c = ?$

Solution:
$2a - b - c^2 - 12 = -(c^2 - b + 1) + 2(a - 4)$ ➔
$2a - b - c^2 - 12 = -c^2 + b - 1 + 2a - 8$ ➔
$-b - b = 12 - 1 - 8$ ➔ $-2b = 3$ ➔ $b = -3/2$
a and c can be any real number.

3. (Medium)
If $y^2 - 2y + 2 = -2(y - 3)$,
then $x = ?$

Solution:
$y^2 - 2y + 2 = -2(y - 3)$ ➔
$y^2 - 2y + 2 = -2y + 6$ ➔
$y^2 = 4$ ➔ $y = 2$ or $y = -2$

Note that the square of both 2 and -2 is 4.

4. (Medium)
If $\frac{2x - 4}{1 - x} = \frac{4}{x}$, then $x^{-2} = ?$

Solution:
$\frac{2x - 4}{1 - x} = \frac{4}{x}$ ➔ $4(1 - x) = x(2x - 4)$ ➔
$4 - 4x = 2x^2 - 4x$ ➔ $2x^2 = 4$ ➔
$x^2 = 2$ ➔ $x^{-2} = 1/2$

Practice Exercises:

1. (Easy)
$3x - 5 = 8$, then $x = ?$

2. (Easy)
$-3x - 5 = -8$, then $x = ?$

3. (Easy)
$3(x - 1) + 2x = 4x + 7$, then $x - 1 = ?$

4. (Easy)
$\frac{2x + 3}{10 - x} = \frac{1}{5}$, then $x = ?$

5. (Medium)
$2 - 7(c - 3 - 2c) + 5c = -6(-2c - 5) - 8 - c$, then $c = ?$

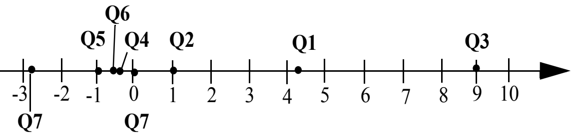
6. (Medium)
$5n^2 + 9n = 3(n^2 - 5) + 2n^2$, then $1/n = ?$

7. (Medium)
$\frac{x - 4}{-1 + 2x} = \frac{-2}{x}$, then $x - \sqrt{2} = ?$

8. (Medium)
Mark your solutions on a number line for the questions above.

Answers:
1. 13/3; **2.** 1; **3.** 9; **4.** -5/11; **5.** -1; **6.** -3/5; **7.** 0 or $-2\sqrt{2}$

8.



Question numbers are written in bold.

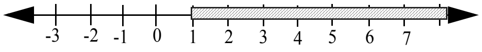
Inequalities

One variable inequalities are similar in solution to one variable equalities, except the following:

- The equality sign is replaced by the inequality sign.
- When finding the unknown, you find a range of values instead of just one value.

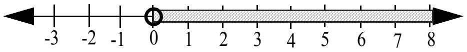
Examples:

1. (Easy)
If $x - 1 > 0$, then $x > 1$. The solution to the inequality is a range, rather then a specific value. You can show this range on a number line as shown below.



Note that 1 is excluded.

2. (Easy)
$4x > 0$, then $x > 0$



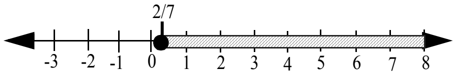
Note that zero is excluded.

3. (Easy)
If $\frac{2x+3}{1-x} \ge 5$, then what is the range of x?

Solution:

$\frac{2x+3}{1-x} \ge 5 \Rightarrow 2x + 3 \ge 5 - 5x \Rightarrow$

$7x \ge 2 \Rightarrow x \ge \frac{2}{7}$



Note that 2/7 is included.

- **Important: when you <u>multiply</u> or <u>divide</u> both sides of an inequality with a negative number, the inequality sign changes its direction.**

Examples:

1. (Easy)
If $2x > 1$, then $-2x < -1$. Both sides of the inequality is multiplied by -1, hence the inequality sign is changed from ">" to "<".

2. (Easy)
If $x^2 \ge 9$, then $-x^2 \le -9$. Both sides of the inequality is multiplied by -1, hence the inequality sign is changed from "$\ge$" to "$\le$."

3. (Easy)
$-5a < 3$, then $a > -3/5$. In this example, to find the unknown, you must divide both sides of the inequality by -5, hence you must change the sign from "<" to ">".

4. (Easy)
If $-y^2 + 6 \le 0$, then $y^2 - 6 \ge 0$. Both sides of the inequality is multiplied by -1, hence the inequality sign is changed from "$\le$" to "$\ge$."

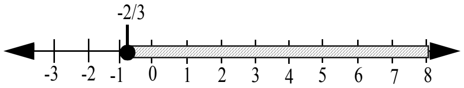
5. (Medium)
If $-3b + 1 \le 3$, what is the range of b?

Solution:

$-3b + 1 \le 3 \Rightarrow -3b \le 3 - 1 \Rightarrow$

$-3b \le 2 \Rightarrow b \ge -\frac{2}{3}$.

Note that the direction of the sign has changed in the last step.



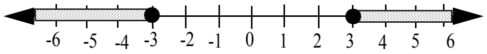
Note that -2/3 is included.

6. (Medium)
If $2c^2 - 3 \ge 15$, what is the range of c?

Solution:

$2c^2 - 3 \ge 15 \Rightarrow 2c^2 \ge 18 \Rightarrow c^2 \ge 9 \Rightarrow$
$c \ge 3$ or $c \le -3$.

Note that the square of the numbers greater than 3 and smaller than -3 are both greater than 9.



Note that -3 and 3 are included.

7. (Medium)
If $\frac{2x-4}{1-x} < \frac{4}{x}$, what is the range of x^2?

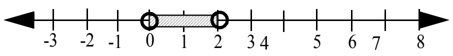
Solution:

$\frac{2x-4}{1-x} < \frac{4}{x} \Rightarrow 2x^2 - 4x < 4 - 4x \Rightarrow$

$2x^2 < 4 \Rightarrow x^2 < 2$

Since the square of any real number is greater than or equal to 0 (never negative),

$0 \le x^2 < 2$



Note that 0 is included and 2 is excluded.

Practice Exercises:

1. (Easy)
If $3x - 5 \ge 8$, what is the range of x? Mark your solution on a number line.

2. (Medium)
If $-3x - 5 \geq -8$, what is the range of x? Mark your solution on a number line.

3. (Medium)
If $3(x - 1) + 2x < 4x + 7$, what is the range of x? Mark your solution on a number line.

4. (Medium)
If $2 - 7(c - 3 - 2c) + 5c > -6(-2c - 5) - 8 - c$, what is the range of c? Mark your solution on a number line.

5. (Medium)
If $5n^2 + 9n < 3(n^2 - 5) + 2n^2$, what is the range of n? Mark your solution on a number line.

6. (Medium)
If $-3a - 4 > 7a + 1$, what is the range of a? Mark your solution on a number line.

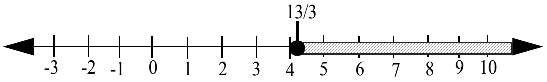
7. (Medium)
If $-4(5 - u) < 3(2u + 3)$, what is the range of u? Mark your solution on a number line.

8. (Medium)
If $\frac{2x-4}{1-x} < \frac{4}{x}$, what is the range of x?
Mark your solution on a number line.

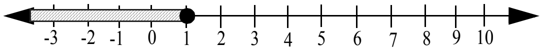
9. (Medium)
If $\frac{2x-4}{1-x} < \frac{4}{x}$, what is the range of x^{-2}?
Mark your solution on a number line.

Answers:

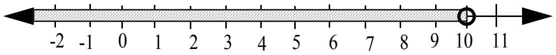
1. $x \geq \frac{13}{3}$; **2.** $x \leq 1$; **3.** $x < 10$; **4.** $c > -1$; **5.** $n < -5/3$; **6.** $a < -1/2$; **7.** $u > -29/2$; **8.** $-\sqrt{2} < x < \sqrt{2}$; **9.** $x^{-2} > \frac{1}{2}$



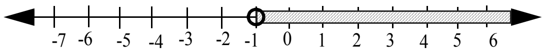
Question 1. Note that 13/3 is included.



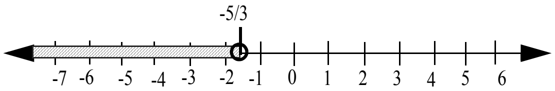
Question 2. Note that 1 is included.



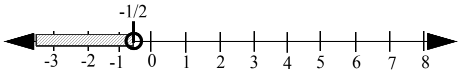
Question 3. Note that 10 is excluded.



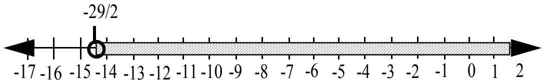
Question 4. Note that -1 is excluded.



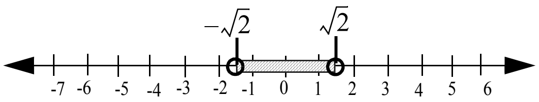
Question 5. Note that -5/3 is excluded.



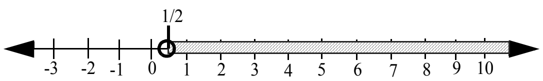
Question 6. Note that -1/2 is excluded.



Question 7. Note that -29/2 is excluded.



Question 8. Note that $\sqrt{2}$ and $-\sqrt{2}$ are excluded.



Question 9. Note that 1/2 is excluded.

Equations With Multiple Unknowns

In SAT, there are questions involving equations with multiple unknowns.

Sometimes, you are asked to solve one unknown in terms of the others. In this case, you can use all the rules that you have learned in the previous section. The best way of understanding these questions is to practice as much as possible. Go through all the examples carefully first and solve all the exercises.

Examples:

1. (Easy)
Express each variable in terms of the other in the expression: $2c + y = 8$

Solution:
Express y in terms of c:
$2c + y = 8 \Rightarrow y = 8 - 2c$

Express c in terms of y:
$2c + y = 8 \Rightarrow 2c = 8 - y \Rightarrow c = (8 - y)/2$

2. (Easy)
Express each variable in terms of the other two in the expression: $2d - h + 3e = e$

Solution:
Express d in terms of h and e:
$2d - h + 3e = e \Rightarrow 2d = e + h - 3e \Rightarrow d = (h - 2e)/2$

Express h in terms of d and e:
$2d - h + 3e = e \Rightarrow h = 2d + 3e - e = 2(d + e)$

Express e in terms of d and h:
$2d - h + 3e = e \Rightarrow 3e - e = h - 2d \Rightarrow e = h/2 - d$

Practice Exercises:

1. (Easy)
If $x - 7y = 3y$, what is x in terms of y and what is y in terms of x?

2. (Easy)
If $-u/2 - 7v = 3v - 2u$, what is u in terms of v and what is v in terms of u?

Answers:
1. $x = 10y$ and $y = x/10$; **2.** $u = 20v/3$ and $v = 3u/20$

Sometimes, the equation looks complicated with higher powers and/or many variables. You can simplify these equations by working through them. Therefore if a question is in the beginning of a section and looks complicated, don't be intimidated, instead, try to simplify the question.

Examples:

1. (Easy)
$2(x^2 -3x + 3) + x(3 - 2x + y) = x + 1 + xy - y \Rightarrow$
$2x^2 - 6x + 6 + 3x - 2x^2 + xy = x + 1 + xy - y \Rightarrow$
$-3x + 6 = x + 1 - y \Rightarrow 4x = 5 + y \Rightarrow$
$x = (y + 5)/4$ or $y = 4x - 5$

2. (Easy)
$2a - b - c^2 - 12 = -(c^2 - b + 1) + 2(a - 4) + d \Rightarrow$
$2a - b - c^2 - 12 = -c^2 + b - 1 + 2a - 8 + d \Rightarrow$
$2b = 3 + d \Rightarrow b = (d + 3)/2$ or $d = 3 - 2b$
a and c can take any real value.

In both of the above examples, the question looks difficult, but is simple.

Practice Exercises:

1. (Easy)
If $3(x - 7y) - 3 = 3(-y - 8 + 2x)$, what is x in terms of y and what is y in terms of x?

2. (Easy)
If $-u^2 - 7v + 4 - uv = 3v - u(u + v - 8)$, what is u in terms of v and what is v in terms of u?

3. (Medium)
If $(x + y)^2 - xy + y + 1 = (x - y)(x + y) + 2y^2 + xy + x$, what is x in terms of y and what is y in terms of x?

Answers:
1. $x = 7 - 6y$ and $y = (7 - x)/6$;
2. $u = (2 - 5v)/4$ and $v = (2 - 4u)/5$;
3. $x = y + 1$ and $y = x - 1$

In other times more than one equation is given and you are asked to solve for all the unknowns.

In general, if you are given n independent equations with n unknowns, you can solve for all the unknowns.

If you are given 2 independent equations with 2 unknowns, you can solve for both of the unknowns. In this situation do the following:

Step 1: Take one of the equations (let's call this Equation 1) and express the first variable in terms of the second one.

Step 2: Substitute the expression you find in Step 1 in the other equation, Equation 2, and solve for second variable.

Step 3: Substitute the value of the second variable you found in Step 2 in the expression you found in Step 1and solve for the first variable.

Examples:

1. (Easy)
If 3n - 1 = s and n = 2/3, then s = ?

Solution:

In this example you can skip the first two steps, because the value of one of the variables, n, is already given in the question.

In the third step, substitute n = 2/3 in the first equation,

$$s = 3 \cdot \frac{2}{3} - 1 = 1$$

Check:

To make sure that your answer is correct, you can substitute the values of the unknowns into the original equations and see if they are satisfied. Here is how:
Substitute n = 2/3 and s = 1 in 2n - 1 = s:
3n - 1 = 2 - 1 = 1
Since 1 = 1 is a correct statement, your answer is correct.

2. (Easy)
x + y = 3 - 5x
x + y = y
x = ?, y = ?

Solution:

Step 1: x + y = y ➔ x = 0
Step 2: Substitute x = 0 into x + y = 3 - 5x ➔ y = 3
Step 3: Both x and y are found, so skip Step 3.

3. (Medium)
2x + y = 3 and
x + y = 1
x = ?, y = ?

Solution:

Step 1: x + y = 1 ➔ x = 1 - y
Step 2: Substitute x = 1 - y into 2x + y = 3:
2(1 - y) + y = 3 ➔ 2 - 2y + y = 3 ➔ y = -1
Step 3: Substitute y = -1 into x = 1 - y ➔ x = 2

Check:

Substitute x = 2 and y = -1 into the equations yields
2x + y = 2 · 2 – 1 = 4 - 1 = 3 and
x + y = 2 - 1 = 1
Since 3 = 3 and 1 = 1 are correct statements, your answers are correct.

Practice Exercises:

1. (Easy)
If 2n - 8 = 3s + 2 and n = -2, then s = ?

2. (Easy)
If x + y = 3 + x and x - 3y = y, then x = ?, y = ?

3. (Medium)
If x + y = 3 + 7x and x - 3y = -y - 6, then x = ?, y = ?

Answers: 1. -14/3; **2.** x = 12, y = 3; **3.** x = 0, y = 3

Occasionally, there are questions involving three unknowns. If you are given three independent equations with three unknowns, you can solve for all the three unknowns.

In these cases, one of the equations involves only one of the variables. You can calculate the value of this variable easily and substitute it in the remaining two equations.

It is also possible that two of the three equations involve only the two unknowns. In this situation, use the three steps explained before to get the value of these two unknowns and substitute the results in the third equation and solve for the third unknown.

Examples:

1. (Medium)
2x - y - 7z = 9
y = 2z + 1
z = 1
x = ?, y = ?

Solution:

The value of z is already given in the third equation.
Substituting z = 1 in second equation yields
y = 2 · 1 + 1 = 2 + 1 = 3
Substituting z = 1 and y = 3 in the first equation yields 2x - 3 - 7 = 9 ➔ x = 19/2

Check:

x = 19/2, y = 3 and z = 1 ➔
2x - y - 7z = 19 - 3 - 7 = 9 = 9 and
y = 2z + 1 ➔ 3 = 2 + 1= 3
Since 9 = 9 and 3 = 3 are correct statements, your answers are correct.

2. (Medium)
2x - 2y - 7z = -33
y = 2z + x
z - 2 = 1
x = ?, y = ?, z = ?

Solution:

z - 2 = 1 ➔ z = 3
Substituting z = 3 in second equation, y = 2z + x, yields y = 6 + x
Substituting z = 3 and y = 6 + x in the first equation, 2x - 2y - 7z = -33, yields

2x - 2(6 + x) - 21 = -33 ➔
2x - 12 - 2x - 21= -33 ➔ Since the first equation is satisfied for all real values of x, x can be any real number.

3. (Medium)
4y = 2z
2x - 2y - 7z = 9
z - 2 = y
x = ?, y = ?, z = ?

Solution:

The first and the last equations have y and z only as unknowns. So let's use these equations to solve for y and z. 4y = 2z ➔ z = 2y
Substituting z = 2y in z - 2 = y yields
2y - 2 = y ➔ y = 2
Substituting y = 2 in z = 2y yields z = 4
Substituting y = 2 and z = 4 in 2x - 2y - 7z = 9 yields 2x - 4 - 28 = 9 ➔ x = 41/2

4. (Hard)
Joe works in a department store and makes $6.50/hour. He works 40 hour/week.

Jim mows the lawns for $20 per lawn. It takes him 1.5 hours to mow one lawn. He mows each lawn once in every other week.

If, on the average, Jim and Joe make the same money each week, who works longer?

Solution:

Let i, n and t be the Jim's weekly income, the number of lawns Jim mows and the total time Jim has to spend each week, respectively. We have 3 unknowns. We need to find 3 independent equations to solve these 3 unknowns.

Joe makes 6.5×40 = $260 in a week. Since Jim and Joe make the same amount, then
i = 260 (1st equation)

Jim makes $20 \times \frac{n}{2} = i$. (2nd equation).

Note that we divide n by 2 because each lawn is mowed once in every other week, not once every week.

Since each lawn takes 1.5 hours to mow,
$1.5 \times \frac{n}{2} = t$ (3rd equation)
Combining the first and the second equations,
$20 \times \frac{n}{2} = 260$ ➔ n = 26 homes
Substituting n in the third equation, t = 19.5 hours

Hence Joe works longer to make the same amount of money.

Practice Exercises:

Before solving the following questions, write down the equations one after another in different lines as they are given in the Examples. This helps you organize the question and solve it faster, without error.

1. (Easy)
If y + 3 = 2z + 1, 2x - 3y - 4z = 10 and z = -2, then x = ?, y = ?

2. (Medium)
If y = 2z + x, z/2 = 1- 3z/2 and 2x + y - z/2 = -3 then x = ?, y = ?, z = ?

3. (Medium)
If 4y = 2z - 1, 2x - y - 7z = 9 and z - 2 = 3y - 1, then x = ?, y = ?, z = ?

Answers: 1. x = -8, y = -6; **2.** x = -5/4, y = -1/4, z = 1/2; **3.** x = 5/2, y = -1/2, z = -1/2

Equations with Powers

You are already familiar with the powers of numbers. In this section we emphasize the powers of expressions (e.g.) $(x + 1)^3$ and expression powers of numbers (e.g.) 3^{x+1}.

All the rules you learned in the "Powers" section of Chapter 5, Arithmetic, also apply to the expressions. Review this section if you have any difficulty understanding the examples or answering the Practical Exercises below.

Examples:

1. (Easy)
$$x^{-3} = \frac{1}{x^3}$$

2. (Easy)
$$\frac{1}{x^{-3}} = x^3$$

3. (Medium)
If $x^{-3} = 27$, what is the value of x^{-2}?

Solution:

If $x^{-3} = 27$ ➔ $x^{-3} = 3^3$ ➔ $\frac{1}{x} = 3$ ➔
$$x^{-2} = \frac{1}{x^2} = 3^2 = 9$$

4. (Medium)
If $x^{-1/3} = \frac{1}{2}$, what is the value of $\left(\frac{1}{x}\right)^{-2} + x - 1$

Solution:

If $x^{-1/3} = \frac{1}{2}$ ➔ $\frac{1}{x^{1/3}} = \frac{1}{2}$ ➔ $x^{1/3} = 2$ ➔
$x = 2^3 = 8$ ➔
$\left(\frac{1}{x}\right)^{-2} + x - 1 = x^2 + x - 1 = 8^2 + 8 - 1 = 71$

Practice Exercises:

1. (Easy)
If $q^4 = 9$, then $q^2 = ?$

2. (Easy)
If $q^{-4} = 9$, then $q^2 = ?$

3. (Medium)
If $q^{1/4} = 9$, then $q = ?$

4. (Medium)
If $q^{-1/4} = 9$, then $q = ?$

5. (Medium)
If $2^{2b-5} = 4^{\frac{3}{4}}$, then $b = ?$

6. (Medium)
If $2^{2b-5} = \frac{4^{1/4}}{8^b}$, then $b = ?$

7. (Medium)
If $x^{3/2} \cdot x^{-2} = 4$, then $x = ?$

8. (Medium)
If $(x^{1/2})^{-3/2} = 8$, then $x = ?$

9. (Medium)
If $\frac{h \cdot h^{4-\frac{1}{2}}}{h^2} = 32$, then $h = ?$

Answers: 1. 3; **2.** 1/3; **3.** 6561; **4.** 1/6561; **5.** 13/4; **6.** 11/10; **7.** 1/16; **8.** 1/16; **9.** 4

Radical Equations

Definition

The term $\sqrt[n]{x}$, which indicates a root, is called a **radical**. $\sqrt[n]{x}$ is read "x radical n," or "the nth root of x."

The special case $\sqrt[2]{x}$ is also written as $\sqrt{x}$ and is called the **square root** of x.

$\sqrt[3]{x}$ is called the **cube root** of x.

An equation that contains a radical is called a radical equation.

Radicals can be expressed by rational powers.

- **For n is odd: $\sqrt[n]{x} = x^{1/n}$, where x is a real number.**
- **For n is even and <u>non-zero</u>: $\sqrt[n]{x} = |x^{1/n}|$, where x is a real <u>non-negative</u> number.**

Examples:

1. (Medium)
$\sqrt[3]{8} = 8^{(1/3)} = 2$

2. (Medium)
$\sqrt[3]{-8} = (-8)^{(1/3)} = -2$

3. (Medium)
$\sqrt[2]{x} = |x^{1/2}|$

4. (Medium)
$\sqrt[2]{25} = |25^{1/2}| = 5$
Note that $25^{1/2}$ has two values, 5 and -5.
However, $\sqrt[2]{25} = \sqrt{25}$ equals only positive (5) of these two solutions.

5. (Medium)
$\sqrt[8]{x} = |x^{1/8}|$

6. (Medium)
$\sqrt[6]{64} = |64^{1/6}| = 2$,
Note that both 2^6 and $(-2)^6$ equals 64. However only positive value, 2, equals $\sqrt[6]{64}$

7. (Medium)
$\sqrt[-8]{x} = |x^{-1/8}| = 1/|x^{1/8}|$

- **For odd values of y,** $\sqrt[\frac{y}{z}]{x} = \sqrt[y]{x^z} = x^{z/y}$**, where x is a real number and z is an integer.**
- **For non-zero, and even values of y,** $\sqrt[\frac{y}{z}]{x} = \sqrt[y]{x^z} = \left|x^{z/y}\right|$**, where x^z is a positive real number and z is an integer.**
- **For y = 1, the above formula becomes** $\sqrt[\frac{1}{z}]{x} = \sqrt[1]{x^z} = x^z$

Examples:

1. (Medium)

$\sqrt[\frac{1}{3}]{2} = 2^3 = 8$

2. (Medium)

$\sqrt[\frac{1}{3}]{-2} = (-2)^3 = -8$

3. (Medium)

$\sqrt[\frac{2}{3}]{4} = \sqrt[2]{4^3} = \sqrt{64} = 8$

4. (Medium)

$\sqrt[\frac{8}{3}]{256} = \left|256^{3/8}\right| = \left|256^{1/8}\right|^3 = 2^3 = 8$

Note that $256^{1/8}$ is 2 or -2, but in radical equations, we only use the positive value, 2.

5. (Medium)

$\sqrt[\frac{8}{3}]{-256} = \left|(-256)^{3/8}\right| = \left|(-256)^{1/8}\right|^3$ is not defined, because 1/8th power of a negative number is not defined.

6. (Medium)

$\sqrt[\frac{7}{3}]{-128} = (-128)^{3/7} = \left((-128)^{1/7}\right)^3 =$

$(-2)^3 = -8$

- **Distribution over Multiplication:** $\sqrt[n]{x \cdot y} = \sqrt[n]{x} \cdot \sqrt[n]{y}$

Examples:

1. (Medium)

$\sqrt[3]{8 \cdot 125} = \sqrt[3]{8} \cdot \sqrt[3]{125} = 2 \cdot 5 = 10$

2. (Medium)

$\sqrt[3]{8 \cdot (x+1)} = \sqrt[3]{8} \cdot \sqrt[3]{(x+1)}$

$2 \cdot \sqrt[3]{(x+1)}$

- **Distribution over Division:** $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Examples:

1. (Medium)

$\sqrt[3]{\frac{8}{125}} = \frac{\sqrt[3]{8}}{\sqrt[3]{125}} = \frac{2}{5}$

2. (Medium)

$\sqrt[3]{\frac{x^6}{125}} = \frac{\sqrt[3]{x^6}}{\sqrt[3]{125}} = \frac{x^{(6/3)}}{125^{(1/3)}} = \frac{x^2}{5}$

- **Negative/Positive Conversion:** $\sqrt[-n]{x} = \frac{1}{\sqrt[n]{x}}$

Examples:

1. (Medium)

$\sqrt[-3]{(x-1)^3} = \frac{1}{\sqrt[3]{(x-1)^3}} = \frac{1}{x-1}$

2. (Medium)

$\sqrt[-8]{81} = \left|81^{-1/8}\right| = 1/\left|81^{1/8}\right| = 1/(\sqrt{3})$

Practice Exercises:

1. (Medium)

$\sqrt[4]{64} = ?$

(A) $2 \cdot \sqrt[4]{2}$
(B) $2\sqrt{2}$
(C) 4096
(D) 16777216
(E) Undefined

2. (Medium)

$\sqrt[4]{-64} = ?$

(A) $2 \cdot \sqrt[4]{2}$
(B) $2\sqrt{2}$
(C) $-2 \cdot \sqrt[4]{2}$
(D) 16777216
(E) Undefined

3. (Medium)

$\sqrt[3]{64} = ?$

(A) 4
(B) 8
(C) 512
(D) 262144
(E) Undefined

4. (Medium)

$\sqrt[3]{-64}\ = ?$

- **(A)** -4
- **(B)** -8
- **(C)** -512
- **(D)** -262144
- **(E)** Undefined

5. (Medium)

$\sqrt[\frac{1}{4}]{64}\ = ?$

- **(A)** $2\sqrt{2}$
- **(B)** 1/16777216
- **(C)** 4096
- **(D)** 16777216
- **(E)** Undefined

6. (Medium)

$\sqrt[\frac{1}{4}]{-64}\ = ?$

- **(A)** $-2\cdot\sqrt[4]{2}$
- **(B)** 1/16777216
- **(C)** -16777216
- **(D)** 16777216
- **(E)** Undefined

7. (Medium)

$\sqrt[\frac{1}{3}]{64}\ = ?$

- **(A)** 4
- **(B)** 1/262144
- **(C)** 1/512
- **(D)** 262144
- **(E)** 1/4

8. (Medium)

$\sqrt[\frac{1}{3}]{-64}\ = ?$

- **(A)** -4
- **(B)** 1/262144
- **(C)** 262144
- **(D)** -262144
- **(E)** Undefined

9. (Medium)

$\sqrt[-\frac{1}{3}]{64}\ = ?$

- **(A)** -4
- **(B)** 1/262144
- **(C)** -1/262144
- **(D)** -262144
- **(E)** Undefined

10. (Medium)

$\sqrt[-\frac{1}{3}]{-64}\ = ?$

- **(A)** -4
- **(B)** 1/262144
- **(C)** -1/262144
- **(D)** -262144
- **(E)** Undefined

11. (Medium)

$\sqrt[-\frac{1}{4}]{64}\ = ?$

12. (Medium)

$\sqrt[-\frac{1}{4}]{-64}$

13. (Medium)

$\sqrt[\frac{2}{3}]{4}\ = ?$

14. (Medium)

$\sqrt[\frac{2}{3}]{-4}\ = ?$

15. (Medium)

$\sqrt[-\frac{2}{3}]{4}\ = ?$

16. (Medium)

$\sqrt[-\frac{2}{3}]{-4}\ = ?$

17. (Medium)

$\sqrt[\frac{3}{2}]{4}\ = ?$

18. (Medium)

$\sqrt[\frac{3}{2}]{-4}\ = ?$

19. (Medium)

$\sqrt[-\frac{3}{2}]{4}\ = ?$

20. (Medium)

$\sqrt[-\frac{3}{2}]{-4}\ = ?$

21. (Medium)

If $\sqrt[4]{16\cdot x}\ =\ 3$, then x = ?

22. (Medium)

If $\sqrt[4]{\dfrac{16}{x}}\ =\ 3$, then x = ?

23. (Medium)

If $\sqrt[-4]{16\cdot x}\ =\ 3$, then x = ?

24. (Medium)

If $\sqrt[-4]{\frac{16}{x}} = 3$, then x = ?

25. (Medium)

If $\overset{\frac{2}{3}}{\sqrt[3]{x}} = 8$, then x = ?

Answers:
1. (B); **2.** (E); **3.** (A); **4.** (A); **5.** (D); **6.** (D); **7.** (D); **8.** (D); **9.** (B); **10.** (C); **11.** 1/16777216; **12.** 1/16777216; **13.** 8; **14.** Undefined; **15.** 1/8; **16.** Undefined; **17.** $2 \cdot \sqrt[3]{2}$; **18.** $2 \cdot \sqrt[3]{2}$; **19.** $1/(2 \cdot \sqrt[3]{2})$; **20.** $1/(2 \cdot \sqrt[3]{2})$ **21.** 81/16; **22.** 16/81; **23.** 1/1296; **24.** 1296; **25.** 4

Absolute Value

Definition:

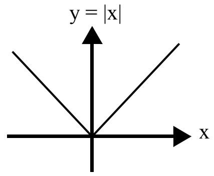
The absolute value of a number is its distance on the number line from the origin. The absolute value of the number w is denoted by |w|.

Absolute value of any number can never be negative. It is always zero or a positive number.

$|x| = x$ if $x \geq 0$

$|x| = -x$ if $x < 0$
$|0| = 0$

Figure shows the graph of |x|. As you can see, |x| is positive even for the negative values of x.

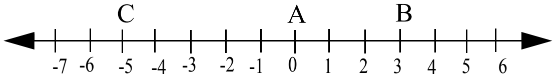


Examples:

1. (Easy) |2.3| = 2.3

2. (Easy) |-2.3| = -(-2.3) = 2.3

3. (Medium)



In the above figure, the distances of points A, B and C from the origin are as follows:
Point A: |0| = 0
Point B: |3| = 3
Point C: |-5| = 5

Here is what you need to know about the absolute value:

- $|q| = |-q|$
  Example: (Easy) $|8| = |-8| = 8$
- $|a \cdot b| = |a| \cdot |b|$
  Example: (Medium) $|2 \cdot (-3)| = |2| \cdot |-3| = 6$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $(b \neq 0)$
  Example: (Medium) $\left|\frac{-2}{-3}\right| = \frac{|-2|}{|-3|} = \frac{2}{3}$
- $|a^n| = |a|^n$
  Examples:
  1. (Medium) $|3^2| = |3|^2$
  2. (Medium) $|(-2)^3| = |-2|^3 = |2|^3$
- $|a + b| \leq |a| + |b|$
  Examples:
  1. (Medium) $|4 + 5| = |4| + |5| = 9$
  2. (Medium) $|4 + (-5)| = |4 - 5| = |-1| = 1 <$ $|4| + |-5| = 9$
- If |a| = n, then a = n or a = -n.

Examples:

1. (Medium)
|x| = 5 ➔ x = 5 or x = -5

2. (Easy)
If 5|x| = 5/6, then x = ?

Solution:
5|x| = 5/6 ➔ |x| = 1/6 ➔ x = 1/6 or x = -1/6

3. (Medium)
Prove that for all values of x, |x| - |-x| = 0

Solution:
Since |x| = |-x| for all values of x, |x| - |-x| = 0

4. (Medium)
If x < 0, |x| - |-x| - |-5x| = ?

Solution:
x < 0 ➔ |x| = -x
|x| - |-x| - |-5x| = - |-5x| = -|-5||x| = -5(-x) = 5x

5. (Medium)
If |x - 1| = 5, then x = ?

Solution:
|x - 1| = 5 ➔ x - 1 = 5 or x - 1 = -5
x - 1 = 5 ➔ x = 6
x - 1 = -5 ➔ x = -4
The answer is x = 6 or x = -4

6. (Hard)
If $|x - 1| + 2|x| = 5$, then $x = ?$

Solution:

The solution depends on the value of x. In these type of solutions, you need to consider the positive and negative values of each of the terms with absolute value.
For $x - 1 \geq 0$ ➔ $x \geq 1$ ➔
$|x - 1| = x - 1$ and $|x| = x$ ➔
$|x - 1| + 2|x| = x - 1 + 2x = 5$ ➔ $3x = 6$ ➔
$x = 2$

For $0 \leq x < 1$
$|x - 1| = -x + 1$ and $|x| = x$ ➔
$|x - 1| + 2|x| = -x + 1 + 2x = 5$ ➔ $x = 4$
But $x = 4$ is not a number between 0 and 1. Hence there is no solution to this equation for $0 \leq x < 1$.

For $x < 0$
$|x - 1| = -x + 1$ and $|x| = -x$ ➔
$|x - 1| + 2|x| = -x + 1 - 2x = 5$ ➔ $-3x = 4$ ➔
$x = -4/3$

So the answer is $x = 2$ or $x = -4/3$

Check:

Let's check the answers by substituting them in the original equation.

For $x = 2$,
$|x - 1| + 2|x| = |2 - 1| + 2|2| = 1 + 4 = 5$

For $x = -4/3$,
$|x - 1| + 2|x| = |-4/3 - 1| + 2|-4/3| =$
$|-7/3| + 8/3 = 7/3 + 8/3 = 15/3 = 5$

Practice Exercises:

1. (Easy)
$|3| = ?$

2. (Easy)
$|-3| = ?$

3. (Easy)
$|0| = ?$

4. (Easy)
$\left|3 - \sqrt{10}\right| = ?$

5. (Easy)
If $x = -2$, then $|x - 5| + |3 + 2x| = ?$

6. (Medium)
If $x < 0$, then $|x| + |-x| = ?$

7. (Medium)
If $x < 0$, then $|2x| + |-x/2| - |-4x| = ?$

8. (Medium)
If $x < y < z < 0$, then $\left|\frac{x^2y^3}{z^3}\right| = ?$

9. (Medium)
For what values of p and q, $|p + q| \neq |p| + |q|$?

(A) $p = q = 0$
(B) $p > 0$ and $q > 0$
(C) $p < 0$ and $q > 0$
(D) $p < 0$ and $q < 0$
(E) $p = 0$ and $q < 0$

10. (Medium)
For what values of p and q, $|p + q| < |p| + |q|$?

(A) $p = q = 0$
(B) $p > 0$ and $q > 0$
(C) $p < 0$ and $q > 0$
(D) $p < 0$ and $q < 0$
(E) $p = 0$ and $q < 0$

11. (Medium)
If $|x|/6 = 5/6$, then $x = ?$

12. (Medium)
If $|x + 1| = 5$, then x

13. (Medium)
If $|2x - 3| = x$, then $x = ?$

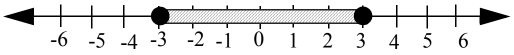
Answers:
1. 3; **2.** 3; **3.** 0; **4.** $\sqrt{10} - 3$; **5.** 8; **6.** -2x;**7.** 3x/2; **8.** $\frac{x^2y^3}{z^3}$;
9. (C); **10.** (C); **11.** 5 or -5; **12.** 4 or -6; **13.** 1 or 3

Inequalities with Absolute Values

Most of the questions about inequalities with absolute values fall in one of the following two cases:

- If $|x| \leq a$, then $-a \leq x \leq a$, where a is a positive constant.

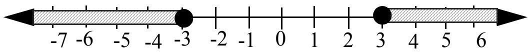
  For example: (Medium)
  If $|x| \leq 3$, then $-3 \leq x \leq 3$.
  You can display it on a number line as follows:



Note that -3 and 3 are included.

- If $|x| \geq a$, then $x \leq -a$ or $x \geq a$, where a is a positive constant.

  For example: (Medium)
  If $|x| \geq 3$, then $x \leq -3$ or $x \geq 3$
  You can display it on a number line as follows:

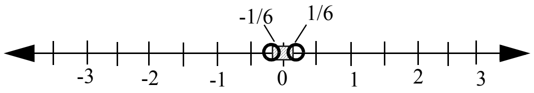


Note that -3 and 3 are included.

Examples:

1. (Medium)
If $5|x| < 5/6$, then what is the range of x?

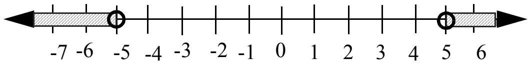
Solution:
$5|x| < 5/6$ ➔ $|x| < 1/6$ ➔ $1/6 < x < 1/6$



Note that -1/6 and 1/6 are excluded.

2. (Medium)
If $|x|/6 > 5/6$, then what is the range of x?

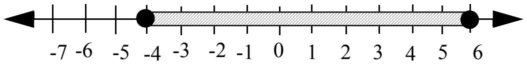
Solution:
$|x|/6 > 5/6$ ➔ $|x| > 5$ ➔ $x > 5$ or $x < -5$



Note that -5 and 5 are excluded.

3. (Medium)
If $|x - 1| \leq 5$, then what is the range of x?

Solution:
$|x - 1| \leq 5$ ➔ $-5 \leq x - 1 \leq 5$ ➔ $-4 \leq x \leq 6$



Note that -4 and 6 are included.

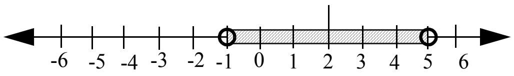
4. (Hard)
On a number line, which of the following describes all the points between -1 and 5?

I. $-1 < x < 5$
II. $|x - 2| < 3$
III. $-1 < |x| < 5$

(A) I only
(B) II only
(C) III only
(D) I and II
(E) I and III

Solution:
I is obviously correct.

II describes the collection of points centered at 2 and with a distance from this center less than 3, as shown in the figure.



Note that -1 and 5 are excluded.

Hence it gives all the points between $2 - 3 = -1$ and $2 + 3 = 5$

$|x|$ can never be negative. So the lower limit, -1, of the inequality may be changed to 0 as follows: $0 \leq |x| < 5$ or $|x| < 5$, which is a collection of the points between -5 and 5, not -1 and 5. Hence III is not correct.

The answer is (D), I and II.

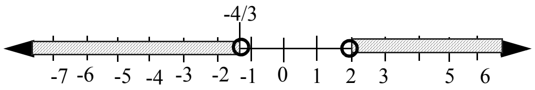
5. (Hard)
If $|x - 1| + 2|x| > 5$, then what is the range of x?

Solution:
The solution depends on the value of x. In these type of solutions, you need to consider the positive and negative values of each term with an absolute value.
For $x - 1 \geq 0$, $x \geq 1$
$|x - 1| = x - 1$ and $|x| = x$ ➔
$|x - 1| + 2|x| = x - 1 + 2x > 5$ ➔ $3x > 6$ ➔ $x > 2$

For $0 \leq x < 1$,
$|x - 1| = -x + 1$ and $|x| = x$ ➔
$|x - 1| + 2|x| = -x + 1 + 2x > 5$ ➔ $x > 4$
But $x > 4$ is not between 0 and 1.
Hence there is no solution to this equation for $0 \leq x < 1$

For $x < 0$,
$|x - 1| = -x + 1$ and $|x| = -x$ ➔
$|x - 1| + 2|x| = -x + 1 - 2x > 5$ ➔
$-3x > 4$ ➔ $x < -4/3$

The answer is $x > 2$ or $x < -4/3$



Note that -4/3 and 2 are excluded.

Practice Exercises:

1. (Medium)
If $|5x| < 5$, then x = ?
Show your answer on the number line.

2. (Medium)
If $|x/5| > 5$, then x = ?
Show your answer on the number line.

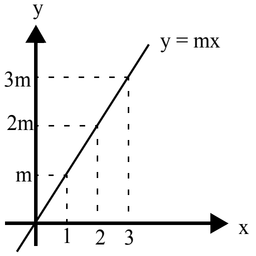
3. (Hard)
If $\left|\frac{x-1}{-5}\right| \le 2$, then x = ?
Show your answer on the number line.

Answers:
1. $-1 < x < 1$; **2.** $x > 25$ or $x < -25$; **3.** $-9 \le x \le 11$

Direct and Inverse Proportionality

Direct Proportionality

- If y is **directly proportional** to x, then y = mx, where m is called the proportionality constant.
- If y is directly proportional to x, with proportionality constant m (y = mx), then x is directly proportional to y with proportionality constant 1/m (x = y/m).
- **Proportionality symbol** is $\propto$. If a is proportional to b, then $a \propto b$
- You can graph the equation y = mx as follows:



In the figure, m, the proportionality constant, is the slope of the line.

Examples:

1. (Easy)
If a is proportional to b with proportionality constant 3, what is the value of a when b = 2?

Solution:
If a is proportional to b with proportionality constant 3, then a = 3b.
If b = 2, $a = 3 \times 2 = 6$

2. (Easy)
a is proportional to b and b = 8 when a = 1. What is the value of the proportionality constant?

Solution:
If $a \propto b$, then a = kb, where k is the proportionality constant.
b = 8 when a = 1 ➔ $1 = k \times 8$ ➔ k = 1/8

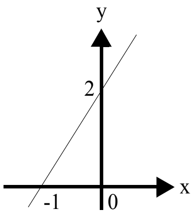
3. (Easy)
If a + 4 is proportional to b - 2 with proportionality constant 3, what is the value of a when b = 2?

Solution:
$(a + 4) \propto (b - 2)$ with proportionality constant 3
➔ a + 4 = 3(b - 2)
b = 2 ➔ a + 4 = 3(2 - 2) = 0 ➔ a = -4

4. (Medium)
In the figure, graph of a straight line is shown.

Which of the following is true?

(A) $y \propto x$
(B) $(y + 2) \propto x$
(C) $(y - 2) \propto x$
(D) $(y - 2) \propto (x + 1)$
(E) $(y + 2) \propto (x - 1)$



Solution:
The figure shows the line, y = 2x + 2 ➔
y - 2 = 2x ➔ $(y - 2) \propto x$ with proportionality constant 2.

The answer is (C).

Practice Exercises:

1. (Easy)
If a is proportional to b with proportionality constant 3/2, what is the value of a when b = 2?

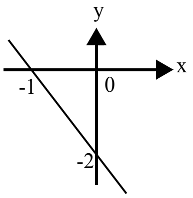
2. (Easy)
a is proportional to b and b = 8 when a = 1/2. What is the value of the proportionality constant?

3. (Easy)
If a is proportional to b with proportionality constant 2, graph a versus b.

4. (Easy)
a is proportional to b and a = 2 when b = 4. Graph b versus a. What is the proportionality constant?

5. (Easy)
If 2a - 4 is proportional to -b - 1 with proportionality constant 3, what is the value of a when b = -2?

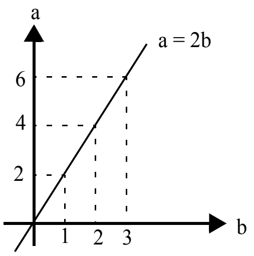
6. (Medium)
In the figure, graph of a straight line is shown.



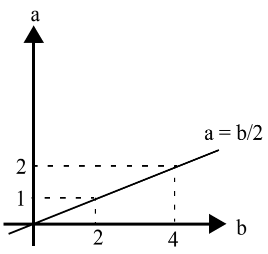
Which of the following is true?

(A) $y \propto -x$
(B) $(y + 2) \propto -x$
(C) $(y - 2) \propto -x$
(D) $(y - 2) \propto (-x + 1)$
(E) $(y + 2) \propto (-x - 1)$

Answers: **1.** 3; **2.** 1/16; **5.** 7/2; **6.** (B)



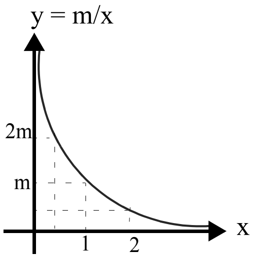
Q. 3



Q. 4

Inverse Proportionality

- If y is inversely proportional to x, then y = m/x or xy = m, where m is the proportionality constant.
- If y is inversely proportional to x, y = m/x, then x is inversely proportional to y, x = m/y, where m is the same proportionality constant for both.
- You can graph the equation y = m/x as follows:



Examples:

1. (Easy)
If a is inversely proportional to b with proportionality constant 3, what is the value of a when b = 2?

Solution:
If a is inversely proportional to b with proportionality constant 3, then a = 3/b.
If b = 2, a = 3/2 = 1.5

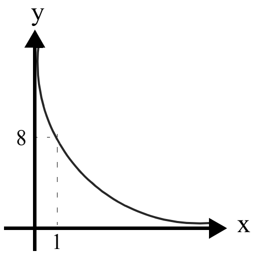
2. (Easy)
If a is inversely proportional to b and b = 8 when a = 1. What is the value of the proportionality constant?

Solution:
If a is inversely proportional to b, then a = k/b.
If b = 8 when a = 1, then 1 = k/8 ➔ k = 8.

3. (Easy)
If a + 4 is inversely proportional to b - 2 with proportionality constant 3, what is the value of a when b = 3?

Solution:
a + 4 is inversely proportional to b - 2 with proportionality constant 3 ➔ a + 4 = 3/(b - 2)
If b = 3, then a + 4 = 3/(3 - 2) = 3 ➔ a = -1

4. (Medium)
In the figure, y is inversely proportional to x with proportionality constant, m. What is the value of m?



Solution:
y is inversely proportional to x with proportionality constant, m. ➔ y = m/x
In the figure, when x = 1, y = 8 ➔ 8 = m/1 ➔ m = 8

Practice Exercises:

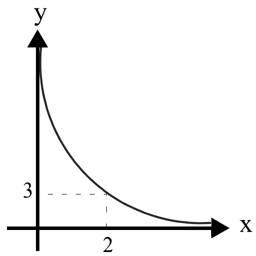
1. (Easy)
If a is inversely proportional to b with proportionality constant 3/5, what is the value of a when b = 2?

2. (Easy)
If a is inversely proportional to b with proportionality constant 3/5, what is the value of b when a = 2?

3. (Easy)
If a is inversely proportional to b and b = 3/8 when a = 1/2. What is the value of the proportionality constant?

4. (Easy)
If a - 4 is inversely proportional to b^2 with proportionality constant 2, what is the value of a when b = 3?

5. (Medium)
In the figure, y is inversely proportional to x with proportionality constant, m. What is the value of m?



Answers: 1. 3/10; **2.** 3/10; **3.** 3/16; **4.** 38/9; **5.** 6

Mixed Proportionality

Sometimes one quantity is directly or inversely proportional to more than one quantities. Below we will examine this situation in three different cases.

Case 1:
If z is directly proportional to y and x, then $z = mxy$, where m is the proportionality constant.

For example: (Easy)
a is directly proportional to b and c. If b = 2 and c = 3, then $a = m(2 \cdot 3) = 6m$, where m is a constant.

Case 2:
If z is directly proportional to y and inversely proportional to x, then $z = my/x$, where m is the proportionality constant.

For example: (Easy)
a is directly proportional to b and inversely proportional to c. If b = 2 and c = 3, then $a = m\frac{2}{3}$, where m is the proportionality constant.

Case 3:
If z is inversely proportional to y and inversely proportional to x, then $z = m/(xy)$, where m is a constant.

For example: (Easy)
a is inversely proportional to b and inversely proportional to c. If b = 2 and c = 3, then $a = \frac{m}{2 \cdot 3} = \frac{m}{6}$, where m is a constant.

Examples:

1. (Easy)
If a is inversely proportional to b and directly proportional to c. If the proportionality constant is 3, what is the value of c when b = 2 and a = 6?

Solution:
a is inversely proportional to b and directly proportional to c ➔ a = 3c/b ➔ c = ab/3 ➔
When a = 6 and b = 2, $c = \frac{6 \cdot 2}{3} = 4$

2. (Easy)
If a is inversely proportional to b and c, and a = 1 when b = 8 and c = 2. What is the value of the proportionality constant?

Solution:
If a is inversely proportional to b, then
a = k/(bc)
a = 1, b = 8, c = 2 ➔ 1 = k/16 ➔ k = 16

3. (Easy)
a + 4 is inversely proportional to b - 2 and directly proportional to c. If the proportionality constant is 3, what is the value of a when b = -3 and c = 5?

Solution:
a + 4 is inversely proportional to b - 2 and directly proportional to c with the proportionality constant 3 ➔ a + 4 = 3c/(b - 2)
If b = -3 and c = 5, then a + 4 = 15/(-3 - 2) = -3 ➔ a = -7

Practice Exercises:

1. (Easy)
a is directly proportional to b and c. If the proportionality constant is 3, what is the value of c when b = 2 and a = 6?

2. (Easy)
a is inversely proportional to b and directly proportional to c. a = 1 when b = 8, c = 2. What is the value of c when a = 4 and b = 6?

3. (Easy)
a + 4 is inversely proportional to b^2 and directly proportional to c. The proportionality constant is 3. What is the value of a when b = 3 and c = 6?

Answers: 1. 1; **2.** 6; **3.** -2

Advanced Algebra

Functions

Definition

When a variable, y, is expressed in terms of another variable x, y is said to be a function of x if:

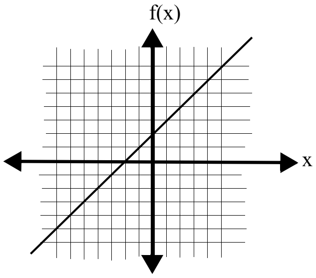
a. It is defined for all values of x within a specified domain.

b. There is only one value of y for each x.

The notation used to express a function is f(x). Here, x is the independent variable.

Examples:

1. (Medium)
$y = x + 2$ for all real values of x. Is y a function of x?



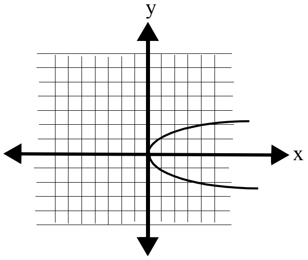
Each grid is 1 unit

Solution:

y is a function of x because y is defined for all real values of x and there is only one and only one value of y for each value of x as shown in the figure.

This function is written as $f(x) = x + 2$.

2. (Medium)
$y = x^{1/2}$ for all values of $x \geq 0$. Is y a function of x?



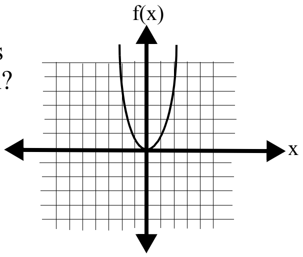
Each grid is 1 unit

Solution:

This is not a function because there are 2 values for each positive x as shown in the figure.

For example, when x = 4, y is 2 or -2.

3. (Medium)
$y = x^2$ for all real values of x. Is y a function of x?



Each grid is 1 unit

Solution:

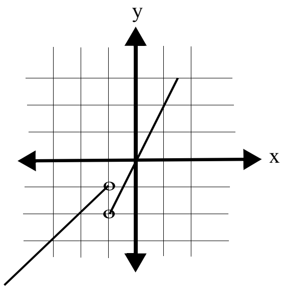
y is a function of x because y is defined for all real values of x and there is one and only one value of y for each value of x.

This function is written as $f(x) = x^2$.

As shown in the figure, for each value of y, there are more than one x. This is permissible.

4. (Medium)
The relationship between x and y is given as follows:
$y = x$ for $x < -1$
$y = 2x$ for $x > -1$
Is y a function of x for all real values of x?

Solution:



o- Points that are excluded
Each grid is 1 unit

Figure shows the relationship between x and y.

You can see that y is not a function of x, because it is defined only for $x < -1$ and $x > -1$, but not defined for $x = -1$

This example shows you the importance of noticing the differences between $<$, $\leq$ and $>$, $\geq$ signs.

Note that for $-2 < y < -1$, there are 2 values of x for each value of y. This is permissible.

Domain and Range of a Function

The values of x for which the function is defined is called the domain of the function.

The domains of functions in the Examples presented in the previous section are:

Example 1: (Medium)
All real numbers between $-\infty$ and ∞ .

Example 2: (Medium)
In this example, y is not a function of x. It is a function of x only for x = 0. At x = 0, there is only one value of y: $y = x^{1/2} = 0$

Example 3: (Medium)
All real numbers between $-\infty$ and ∞ .

Example 4: (Hard)
In this example, y is not a function of x. It is a function only in the domain $x < -1$ and $x > -1$, not for $x = -1$.

The values that y can take is called the range of a function.

The ranges of functions in the Examples presented in the previous section are:

Example 1: (Medium)
All real values between $-\infty$ and ∞ .

Example 2: (Medium)
For domain $x = 0$, the range is $y = x^{1/2} = 0$. For other values of x, y not a function of x.

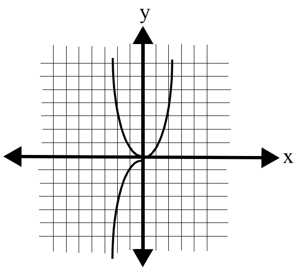
Example 3: (Medium)
All real values between 0 and ∞.

Example 4: (Medium)
For domain
$x < -1$ and $x > -1$, the range is all real values between $-\infty$ and ∞, as shown in the figure.

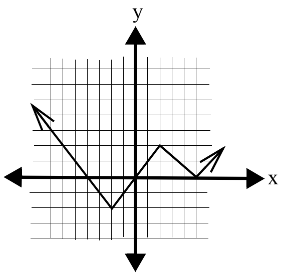
For $x = -1$, y is not a function of x.

Practice Exercises:

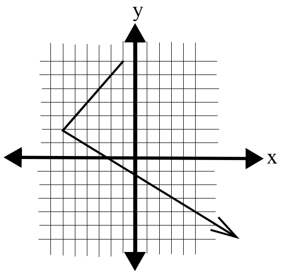
1. For the following graphs, determine if the graph represents a function. If so, determine its domain and its range. If not, find a domain in which it is a function.



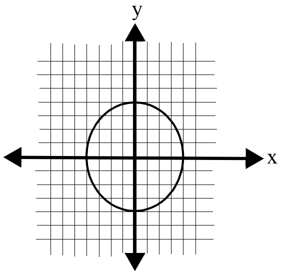
(a)



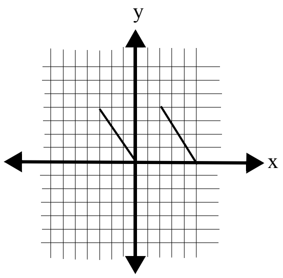
(b)



(c)



(d)



(e)

Each grid is 1 unit for all five cases.

2. Consider each of the following relationships between x and y. For each case, determine if y is a function of x. If so, give its domain and range. If not, find a domain in which it is a function.

   a. (Medium)
   $y = x$ for $-1 < x$
   $y = 2x$ for $0 < x$

   b. (Medium)
   $y = 2$ for $x < 0$
   $y = 3$ for $x > 0$

   c. (Medium)
   $y = 2$ for $x < 0$
   $y = 3$ for $x \geq 0$

Answers:
1. a. Not a function. It is a function only for $x \geq 0$;
1. b. A function, domain: all real numbers between $-\infty$ and ∞, Range: all real numbers between -2 and ∞;
1. c. Not a function. It is a function for $x > -1$;
1. d. Not a function. It is a function for $x = -4$ and $x = 4$;
1. e. Not a function. It is a function for $-3 \leq x \leq 0$ and $2 \leq x \leq 5$;
2. a. Not a function. It is a function for $-1 < x < 0$;
2. b. Not a function. It is a function for all real numbers except 0;
2. c. A function. Domain: all real numbers. Range: 2 and 3.

Determining the Values of Functions

To determine the value of a function for a given value of the variable, you need to substitute the value of the variable into the function.

For example: (Easy)

If $t(u) = u^2 + 1$, then $t(3) = 3^2 + 1 = 10$

To determine the value of the independent variable for a given value of the function, you need to substitute the value of the function and solve for the variable.

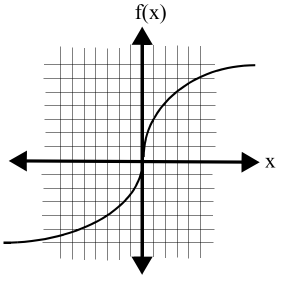
For example: (Easy)
$f(c) = 2c - 7$
If $f(c) = 5$, the value of c can be found as follows:
$5 = 2c - 7 \Rightarrow c = (5 + 7)/2 = 6$

Sometimes the graph of a function is given and you are asked to read the value from the graph.

Examples:

1. (Easy)
In the figure, read

f(1)
f(3)
f(-3)
f(-7)



Each grid is 1 unit.

Solution:
f(a) is the value of f(x) when x = a.

In the figure,
f(1) = 3
f(3) = 5
f(-3) = -4
f(-7) = -5.5

2. In the figure in Example 1, find the value of x when:

f(x) = 0
f(x) = 5.5
f(x) = -4.5

Solution:
In the figure,
f(x) = 0 when x = 0
f(x) = 5.5 when x = 4
f(x) = -4.5 when x = -4

Practice Exercises

1. (Easy)
Find the values of $f(x) = (x - 1)^2 - 1$ for each case:

a. f(0)

b. f(-1)

c. f(1)

d. f(2)

2. (Easy)
If g(z) = 1/z + 4, find the values of z for each case:

a. g(z) = 2

b. g(z) = -2

c. g(z) = 0

d. g(z) = 8

3. (Medium)
For the function in the figure, find the following values:

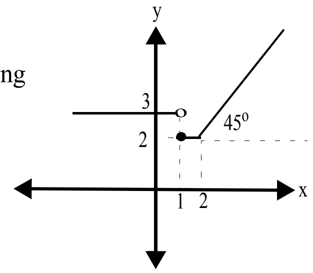
a. f(-1) =

b. f(0) =

c. f(1) =

d. f(2) =

e. f(10) =



● - Points that are included
o - Points that are excluded

Answers: 1. a. 0; **b.** 3; **c.** -1; **d.** 0; **2. a.** -0.5; **b.** -1/6; **c.** -0.25; **d.** 0.25; **3. a.** 3; **b.** 3; **c.** 2; **d.** 2; **e.** 10

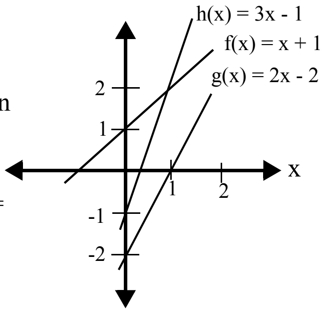
Addition and Subtraction of Functions

You can add or subtract two functions by adding or subtracting their similar terms.

Examples:

1. (Easy)
f(x) = x + 1
g(x) = 2x - 2
What is the addition of f(x) + g(x)?

Solution:
h(x) = f(x) + g(x) =
x + 1 + 2x -2 =
3x - 1



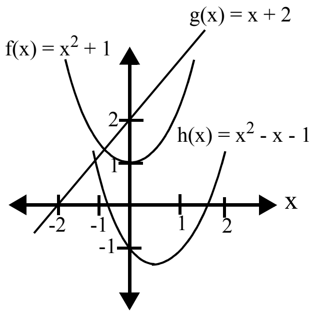
The figure shows the graphs of f(x), g(x) and h(x).

2. (Easy)
If f(2) = 1, g(2) = - 2, then what is f(x) + g(x) at x = 2?

Solution:
f(x) + g(x) at x = 2 is f(2) + g(2) = 1 - 2 = -1

3. (Easy)
$f(x) = x^2 + 1$
g(x) = x + 2
f(x) - g(x) = ?

Solution:
h(x) = f(x) - g(x) =
$x^2 + 1 - x - 2 =$
$x^2 - x - 1$



The figure shows the graphs of f(x), g(x) and h(x).

4. (Easy)
$f(x) = x^2 + 7x - 1$
$g(x) = 2x^3 - 3x^2 - 1/x + 2^x + x - 4$
$h(x) = f(x) + g(x)$

What is the value of h(-1)?

Solution:
$h(x) = f(x) + g(x) =$
$x^2 + 7x - 1 + 2x^3 - 3x^2 -1/x + 2^x + x - 4 =$
$2x^3 - 2x^2 + 8x - 1/x + 2^x - 5 \Rightarrow$
$h(-1) = -2 - 2 - 8 + 1 + 1/2 - 5 = -15.5$

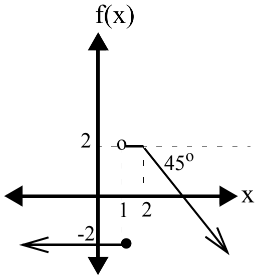
Better Solution:
$f(-1) = 1 - 7 - 1 = -7$
$g(-1) = -2 - 3 + 1 + 1/2 - 1 - 4 = -8.5$
$h(-1) = f(-1) + g(-1) = -7 - 8.5 = -15.5$

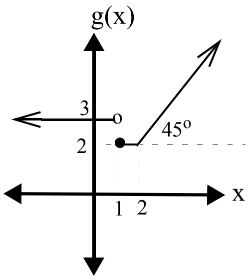
Practice Exercises:

1. (Easy)
$f(x) = 2x^2 - 1$
$g(x) = 2x^3 + (-1/x)^2 + 2^{-2x}$
$h(x) = f(x) + g(x)$

What is the value of h(-1)?

2. (Medium)
Two functions, f(x) and g(x) are displayed below. Draw a graph of f(x) + g(x).





● - Points that are included
o - Points that are excluded

3. (Medium)
f(x) and g(x) are two functions. $f(3) - g(3) = 0$ and $f(x) \neq g(x)$. Provide three possible forms of f(x) and g(x).

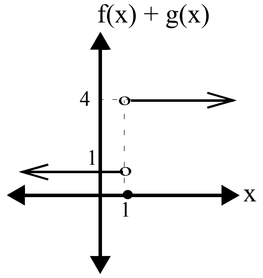
4. (Hard)
Two functions, f(u) and m(u) are defined as follows:

$f(u) = u + 4$ for $u < 3$
$f(u) = u - 4$ for $u > 3$
$f(u) = 10$ for $u = 3$

$m(u) = 8$ for $u > 4$
$m(u) = 2u$ for $u \leq 4$

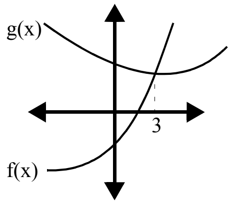
Evaluate $m(u) - f(u) = ?$

Answers:
1. 4;
2.



● - Points that are included
o - Points that are excluded

3.
$f(x) = x - 3$, $g(x) = 2^x - 8$
$f(x) = x^2 - 6$, $g(x) = |x|$



4. $m(u) - f(u) = u - 4$ for $u < 3$
$m(u) - f(u) = -4$ for $u = 3$
$m(u) - f(u) = u + 4$ for $3 < u \leq 4$
$m(u) - f(u) = -u + 12$ for $u > 4$

Multiplication of Functions

You can multiply 2 functions by multiplying each term of the first function by each term of the second function.

For example: (Medium)

$f(z) = 2z - 8$ and $t(z) = z^2 + 4z + 16 \Rightarrow$

$g(z) = f(z)t(z) = (2z - 8)(z^2 + 4z + 16) =$

$2z \cdot z^2 + 2z \cdot 4z + 2z \cdot 16 - 8 \cdot z^2 - 8 \cdot 4z - 8 \cdot 16 =$

$2z^3 + 8z^2 + 32z - 8z^2 - 32z - 128 = 2z^3 - 128$

Practice Exercises:

1. (Easy)

$f(x) = 2x^2 - 1$

$g(x) = 2x^3 + (-1/x)^2 + 2^{-2x}$

$h(x) = f(x)g(x)$

What is the value of h(-1)?

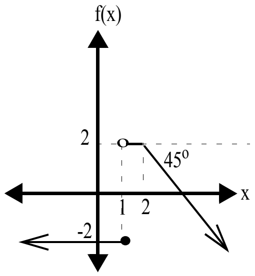
Hint: It is easier to calculate f(-1) and g(-1) first.

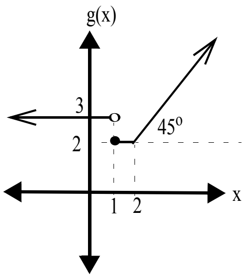
2. (Medium)

f(x) and g(x) are two functions. $f(3)g(3) = 1$ and $f(x) \neq (1/g(x))$. Provide three possible forms of f(x) and g(x).

3. (Medium)

Two functions, f(x) and g(x) are displayed below. Draw f(x)g(x).





● - Points that are included

o - Points that are excluded

4. (Hard)

$f(u) = u + 4$ for $u < 3$

$f(u) = u - 4$ for $u > 3$

$f(u) = 10$ for $u = 3$

$m(u) = 8$ for $u > 4$

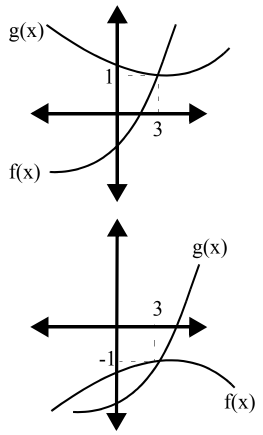
$m(u) = 2u$ for $u \leq 4$

$m(u)f(u) = ?$

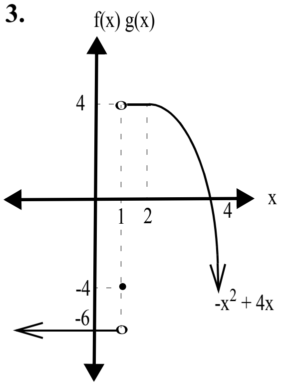
Answers: 1. 3;

2.

$f(x) = x - 2$, $g(x) = 2^x - 7$



3.



● Points that are included

o Points that are excluded

4. $m(u)f(u) = 2u^2 + 8u$ for $u < 3$

$m(u)f(u) = 60$ for $u = 3$

$m(u)f(u) = 2u^2 - 8u$ for $3 < u \leq 4$

$m(u)f(u) = 8u - 32$ for $u > 4$

Division of Functions

You can divide the functions as well. However, in SAT you will only need to deal with a few divisions through **factorization**.

There are in 4 types of factorization in SAT:

- Factoring the common multiplier:
  If all the terms are multiplied by the same number, you can factor this number out.
  For example: (Easy)
  $3 \cdot 2^x - 6x^3 + 12 = 3(2^x - 2x^3 + 4)$
- Factoring the common variable:
  If all the terms have the same variable, you can factor this variable out.
  For example: (Easy)
  $-6x^3 + 12x^2 = (-6x^2)(x - 2)$
- $a^2 - b^2 = (a + b) \cdot (a - b)$.
  For example: (Medium)
  $x^2 - 7^2 = (x + 7) \cdot (x - 7)$
- $a^2 + 2ab + b^2 = (a + b)^2$
  Examples:
  1. (Medium)
  $y^2 + 6y + 9 = (y + 3)^2$
  2. (Medium)
  $y^2 - 10y + 25 = (y - 5)^2$

Examples:

1. (Easy)

$$\frac{2c+4}{2} = \frac{2(c+2)}{2} = c+2$$

2. (Medium)

$$\frac{2x^2+x}{3x} = \frac{x(2x+1)}{3x} = \frac{2x+1}{3}$$

3. (Medium)

$$\frac{3^c}{6^c-3^{2c}} = \frac{3^c}{2^c \cdot 3^c - 3^c \cdot 3^c} = \frac{3^c}{3^c(2^c-3^c)} = \frac{1}{2^c-3^c}$$

If you have a difficulty in understanding this example, study the Powers section in Chapter 5, Arithmetic.

4. (Medium)

$$\frac{9x^2+12x+4}{3x+2} = \frac{(3x+2)^2}{3x+2} = 3x+2$$

5. (Medium)

$$\frac{3x^2-12x+12}{3x-6} = \frac{3(x^2-4x+4)}{3(x-2)} = \frac{3(x-2)^2}{3(x-2)} = x-2$$

Practice Exercises:

1. (Medium)

$$\frac{x^2+2x+1}{x+1} = ?$$

2. (Medium)

$$\frac{x^2-2x+1}{x-1} = ?$$

3. (Medium)

$$\frac{x^2+2xa+a^2}{x+a} = ?$$

4. (Medium)

$$\frac{x^2-2xa+a^2}{x^2-a^2} = ?$$

5. (Medium)

$$\frac{x^2-a^2}{x-a} = ?$$

6. (Medium)

$$\frac{x^2-a^2}{(x+a)^2} = ?$$

Answers:

1. $x + 1$; **2.** $x - 1$; **3.** $x + a$; **4.** $(x - a)/(x + a)$;
5. $x + a$; **6.** $(x - a)/(x + a)$

More about Functions

To score high in SAT you need to know some properties of the functions. These questions are of two types.

Type A:

You are provided the exact form of the function, f(x), and you are asked to find

- **a.** $f(x) + a$
- **b.** $f(x + a)$
- **c.** $af(x)$
- **d.** $f(ax)$

where a in a constant.

Examples:

1. (Easy)
If $f(x) = x^2 - 8$, $f(x) + 7.7 = ?$

Solution:
$f(x) + 7.7 = x^2 - 8 + 7.7 = x^2 - 0.3$

2. (Easy)
If $f(x) = x^2 - 8$, $3f(x) = ?$

Solution:
$3f(x) = 3(x^2 - 8) = 3x^2 - 24$

3. (Medium)
If $f(x) = 2x - 1$, $f(x + 3) = ?$

Solution:
Substitute x + 3 for x:
$f(x + 3) = 2(x + 3) - 1 = 2x + 6 - 1 = 2x + 5$

4. (Medium)
If $f(x) = 2x - 1 + 3^{x+5}$, $f(x - 4) = ?$

Solution:
Substitute x - 4 for x:
$f(x - 4) = 2(x - 4) - 1 + 3^{(x-4)+5} =$
$2x - 8 - 1 + 3^{x-4+5} = 2x - 9 + 3^{x+1}$

5. (Medium)
If $f(x) = x^2 - 8$, $f(3x) = ?$

Solution:
Substitute 3x for x:
$f(3x) = (3x)^2 - 8 = 9x^2 - 8$

6. (Medium)
If $f(x) = x^2 - 8 + 3^x$ and $g(x) = -x/2$, then $f(g(x)) = ?$

Solution:

Substitute g(x) for x in f(x):

: $f(g(x)) = f(-x/2) = (-x/2)^2 - 8 + 3^{-x/2} =$

$$\frac{x^2}{4} - 8 + \frac{1}{3^{x/2}}$$

7. (Medium)
If $f(x) = x^2 - 8 + 3^x$ and $g(x) = -x/2$, then $g(f(x)) = ?$

Solution:

Substitute f(x) for x in g(x):

: $g(f(x)) = g(x^2 - 8 + 3^x) = -(x^2 - 8 + 3^x)/2 =$

$$-\frac{x^2}{2} + 4 - \frac{3^x}{2}$$

Practice Exercises:

1. (Easy)
If $f(x) = -x^2 - 3$, then $f(x) - 7 = ?$

2. (Easy)
If $f(x) = -x^2 - 3$, then $-2f(x) - 7 = ?$

3. (Easy)
If $f(x) = -x^2 - 3$, then $-2(f(x) - 7) = ?$

4. (Medium)
If $f(x) = -x^2 - 3$, then $f(-x) - 7 = ?$

5. (Medium)
If $f(x) = -x^2 - 3$ and $g(x) = 2 - 5x$,
then $f(g(x)) - 7 = ?$

6. (Medium)
If $f(x) = (x - 3)/2$ and $g(x) = -x^2 - 3$,
then $-(g(f(x)))/3 - 7 = ?$

Answers: 1. $-x^2 - 10$; **2.** $2x^2 - 1$; **3.** $2x^2 + 20$;
4. $-x^2 - 10$; **5.** $-25x^2 + 20x -14$; **6.** $(-x^2 + 6x - 105)/12$

Type B:

Only the graph of a function, f(x), is provided. You are asked to identify the graphs of

- **a.** f(x) + a
- **b.** f(x + a)
- **c.** af(x)
- **d.** f(ax)

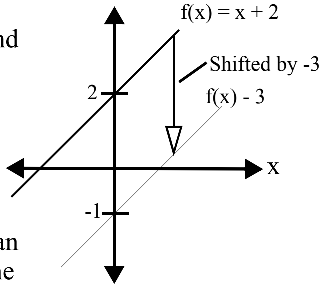
where a in a constant.

- **f(x) + a shifts the graph by "a" along the y axis.**

Examples:

1. (Medium)
$f(x) = x + 2$

Display both f(x) and f(x) - 3 on the same graph.

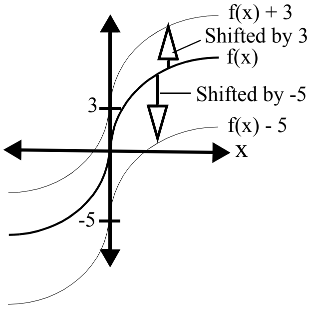


Solution:

$f(x) - 3 =$
$x + 2 - 3 = x - 1$

In the figure, you can see that f(x) - 3 is the shifted version of f(x).

2. (Medium)
If f(x) is a function displayed in the figure, display f(x) + 3 and f(x) - 5 on the same figure.



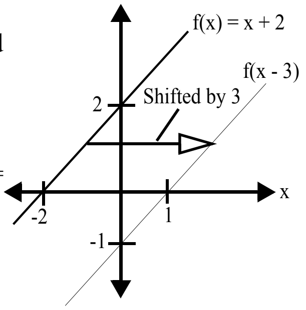
Solution:

See the figure. Note that you don't need to know the exact formule of f(x) to answer the question.

- **f(x + a) shifts the graph by "-a" along the x axis.**

Examples:

1. (Medium)
$f(x) = x + 2$
Display both f(x) and f(x - 3) on the same figure.



Solution:

$f(x - 3) = (x - 3) + 2 =$
$x - 1$

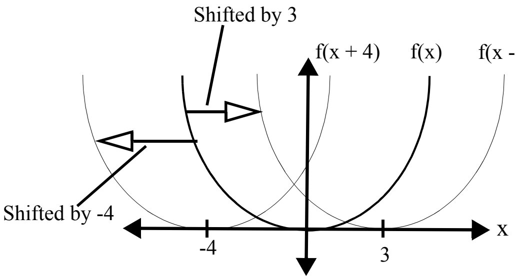
Both functions are displayed in the figure.

Note that you can get f(x - 3) by shifting f(x) by 3 to the right.

2. (Hard)
If f(x) is a function displayed in the below figure, display f(x - 3) and f(x + 4) on the same figure.

Solution:

See the figure below. Note that you don't need to know the exact formula of f(x) to answer the question. All you need to do is to shift $f(x)$ by 3 to the right to get $f(x - 3)$ and by -4 to the left to get $f(x + 4)$.



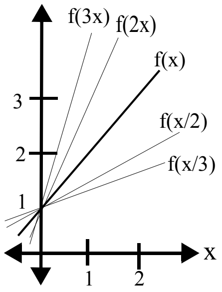
- **If $f(x) = a$ at $x = b$, then $f(cx) = a$ at $x = b/c$**

For $c > 1$, $f(cx)$ "squeezes" the graph by a factor of c around the y axis, without changing the y-intercept.

For $0 < c < 1$, $f(ax)$ "expends" the graph by a factor of c around the y axis, without changing the y-intercept.

Examples:

1. (Medium)
$f(x) = x + 1$
Display $f(x)$, $f(2x)$, $f(3x)$, $f(x/2)$ and $f(x/3)$ on the same figure.

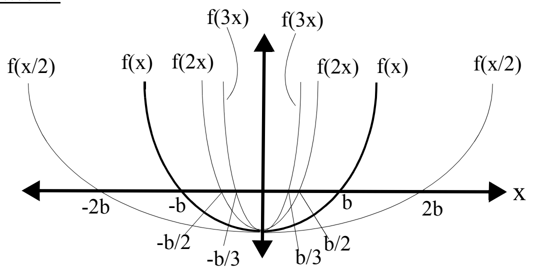


Solution:
$f(2x) = 2x + 1$ with slope 2.
$f(3x) = 3x + 1$ with slope 3.
$f(x/2) = x/2 + 1$ with slope 1/2
$f(x/3) = x/3 + 1$ with slope 1/3.

See the figure. Notice that y-intercept remains the same for all five functions.

2. (Hard)
If $f(x)$ is a function displayed in the figure below, display $f(2x)$, $f(3x)$ and $f(x/2)$ on the same figure.

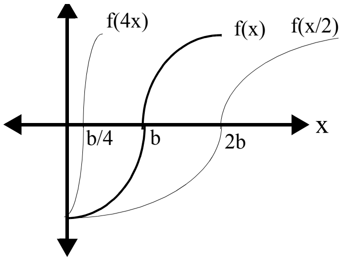
Solution:



Note that you don't need to know the exact formula of $f(x)$ to answer the question.

Notice that $f(2x)$ and $f(3x)$ are the "squeezed" versions of $f(x)$ and $f(x/2)$ is the "expanded" version of $f(x)$. Also notice that y-intercept remains the same for all four functions.

3. (Hard)
$f(x)$ is displayed in the figure below. Draw the graph of $f(4x)$ and $f(x/2)$.

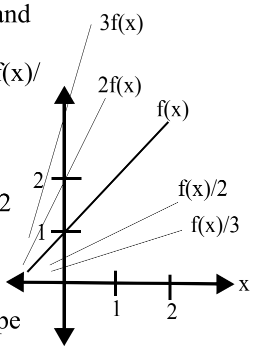


Solution:
See the figure.

- **$af(x)$ multiplies $f(x)$ at all points by a. Note that, y-intercept is also multiplied by a.**

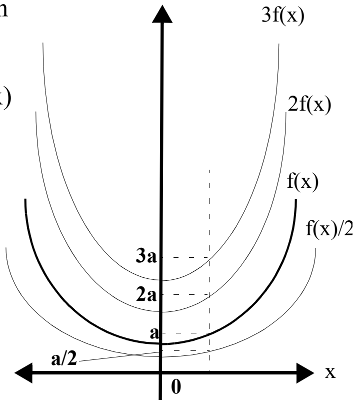
Examples:

1. (Medium)
$f(x) = x + 1$, with slope 1 and y-intercept 1.
Display $f(x)$, $2f(x)$, $3f(x)$, $f(x)/2$ and $f(x)/3$ on the same figure.



Solution:
$2f(x) = 2x + 2$, with slope 2 and y-intercept 2.
$3f(x) = 3x+ 3$with slope 3 and y-intercept 3.
$f(x)/2 = x/2 + 1/2$ with slope 1/2 and y-intercept 1/2.
$f(x)/3 = x/3 + 1/3$ with slope 1/3 and y-intercept 1/3.

2. (Hard)
If $f(x)$ is a function displayed in the below figure, display $2f(x)$, $3f(x)$ and $f(x)/2$ on the same figure.

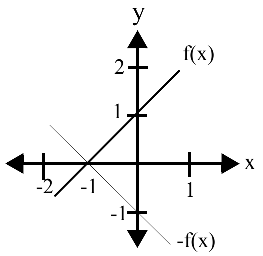


Solution:
See the figure. Note that you don't need to know the exact formula of $f(x)$ to answer the question.

- **$-f(x)$ is mirror image of $f(x)$ along x- axis.**

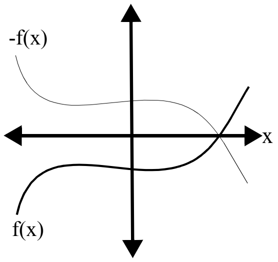
Examples:

1. (Medium)
In the figure, $f(x) = x + 1$ and $-f(x) = -x - 1$ are displayed. As you see, they are the mirror images of each other along the x-axis.



2. (Medium)
In the figure, f(x) and -f(x) are displayed. As you see, they are the mirror images of each other along the x-axis.

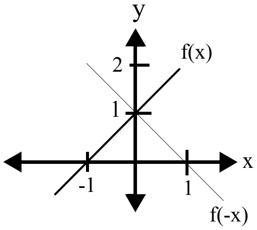
This exercise shows that, you don't need to know the exact formula of f(x) to display -f(x).



- **f(-x) is mirror image of f(x) along y- axis.**

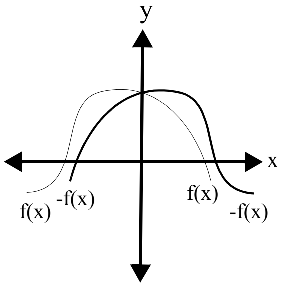
Examples:

1. (Medium)
In the figure, f(x) = x + 1 and f(-x) = -x + 1are displayed. As you can see, they are the mirror images of each other along the y-axis.



2. (Medium)
In the figure, f(x) and f(-x) are displayed. As you can see, they are the mirror images of each other along the y-axis.

This exercise shows that, you don't need to know the exact formula of f(x) to display f(-x).



Practice Exercises:

1. (Medium)
If f(x) = -x - 3, draw

a. f(x)

b. f(x) - 3

c. -3f(x)

d. f(x - 3)

e. f(-3x)

2. (Hard)
f(x) = 3 for x < 1
f(x) = 2 for 1 ≤ x ≤ 2
f(x) = x for x > 2

Draw:

a. f(x)

b. f(x) - 1

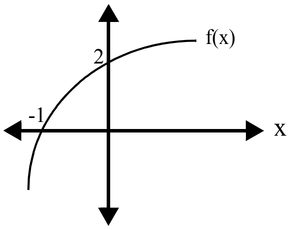
c. -2f(x)

d. f(x - 3)

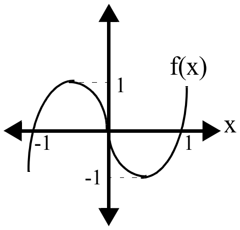
e. f(-2x)

f. f(x/2)

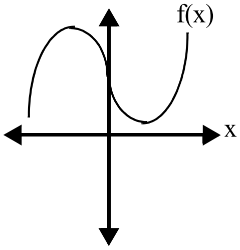
3. (Hard)
f(x) is shown in the figure. Draw f(x) - 2 and f(x - 2) on the same graph.



4. (Hard)
f(x) is shown in the figure. Draw 2f(x), f(2x) and f(x/2) on the same graph.

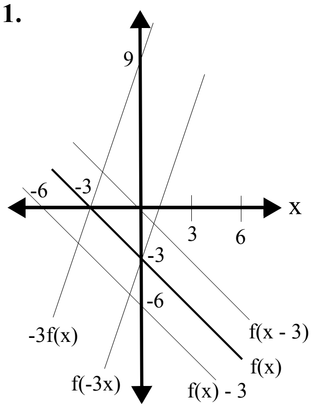


5. (Hard)
f(x) is shown in the figure. Draw -f(x) and f(-x) on the same graph.

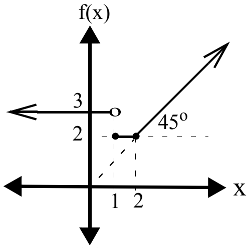


Answers:

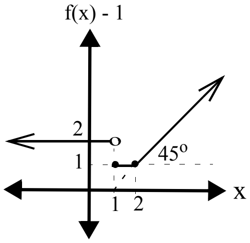
1.



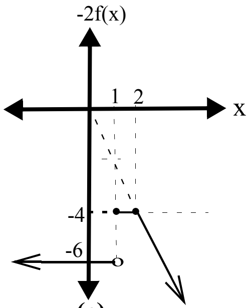
2.



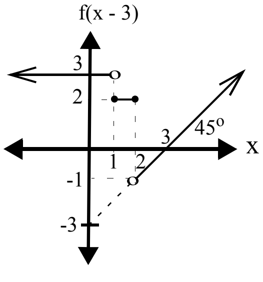
(a)



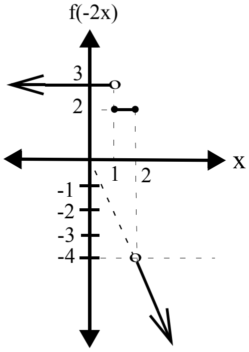
(b)



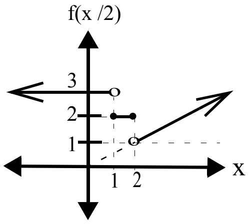
(c)



(d)



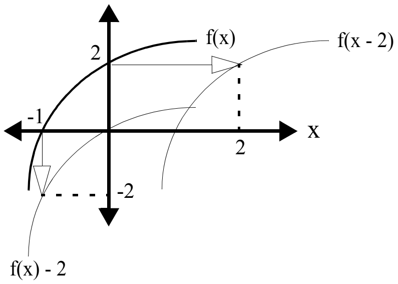
(e)



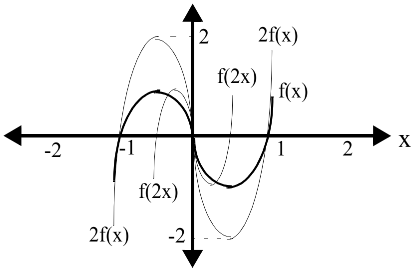
(f)

• - Points that are included
o - Points that are excluded

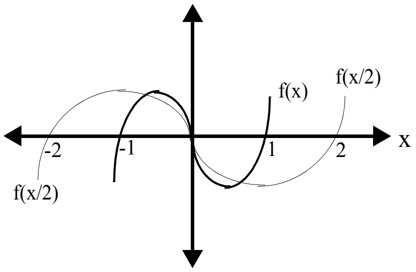
3.



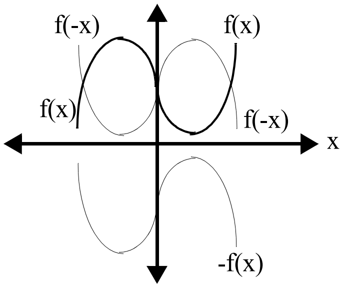
4.



4.



5.



Linear Functions

A function, $f(x)$, is linear if $f(x) = ax + b$, where the exponent of x is 1 and a and b are constants.

Examples:

1. (Easy)
$g(x) = 2x - 3$ is a linear function.

2. (Easy)
$g(x) = 2(x - 4) + 7$ is a linear function.
Note that $g(x) = 2(x - 4) + 7 = 2x - 1$ is linear.

3. (Medium)
$f(x) = \frac{x^2 - 2}{x + 2}$ is a linear function. Note that

$$f(x) = \frac{x^2 - 4}{x + 2} = \frac{(x + 2)(x - 2)}{x + 2} = x - 2$$

is linear.

4. $h(y) = y^{1/2}$ is not a linear function, because the exponent of y is 1/2.

5. $h(y) = y^2 - 3$ is not a linear function, because the exponent of y is 2.

Linear functions represent lines. You have seen several examples and exercises in Chapter 6, "Points, Lines and Angles" and "Lines on x-y Plane" sections. **It is very important that you read this section. There are several questions in SAT about them.** Here we will only provide one example to refresh your memory.

Example:

In the figure, if line k has a slope of -1, what is the y-intercept of k?

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

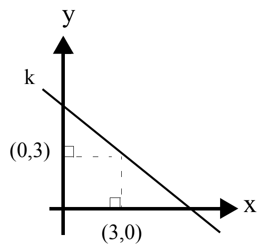


Figure is not drawn to scale.

Solution:

Line k has equation $y = mx + b$ where m and b are constants. Since line k has slope -1, the value of m is -1. From the information in the graph, it is clear that the point (3, 3) is on line k. This means that $(x, y) = (3, 3)$ satisfies the equation $y = -x + b$ ➔ $3 = -3 + b$ ➔ $b = 6$, which is the y-intercept of line k. The answer is (A).

Practice Exercises:

1. (Easy)
Which of the functions below are linear?

a. $f(s) = s^2 - s$
b. $f(s) = 2^s - s$
c. $f(s) = - s + 2^s$
d. $f(s) = - s + 2^7$
e. $f(x) = 1 - x$
f. $f(x) = \frac{x^2 + 1}{3}$
g. $f(x) = \frac{x^2 + 1}{3x}$
h. $f(x) = \frac{x^2 + x}{3x}$

2. (Medium)
$g(x)$ is a linear function with $g(-1) = 3$ and slope $= -3$. Graph $g(x)$. What are

a. $g(5)$?
b. x-intercept?
c. y-intercept?
d. the value of x when $g(x) = 4$?

3. (Medium)
$g(x)$ is a linear function with $g(-1) = 3$ and $g(1) = -3$. Graph $g(x)$. What are the following:

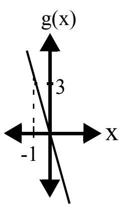
a. the slope?
b. x-intercept?
c. y-intercept?

4. (Medium)
If $f(x) = - x + 4$, then draw the following:

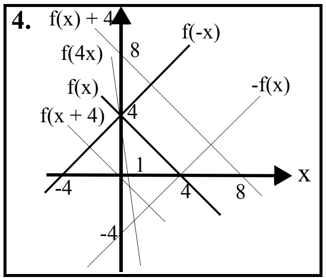
a. $f(x + 4)$
b. $f(4x)$
c. $f(x) + 4$
d. $-f(x)$
e. $f(-x)$

Answers: 1. d, e, h; **2. a.** -15**; 2. b.** 0; **2. c.** 0; **2. d.** -4/3; **3.a.** -3; **3.b.** 0; **3.c.** 0

Graphs for Exercises **2** and **3** are the same. The figure displays g(x) for both exercises.



4.



Quadratic Functions

Definition

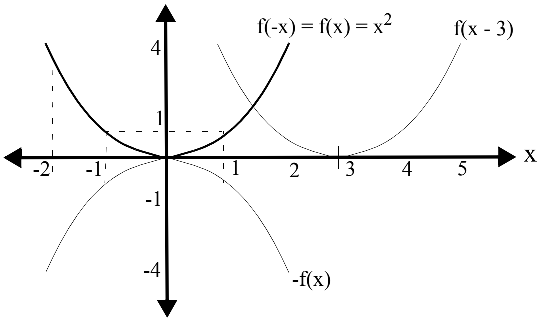
Quadratic functions are second degree polynomials where the highest exponent of the unknown is 2.

Examples:

1. (Easy)
$f(x) = x^2$, $f(x) = (x - a)^2$ and $f(x) = ax^2 + bx + c$ are all quadratic functions because the highest exponent is 2 for all of them.

2. (Medium)
$f(x) = x^2$
Draw the graphs of f(x), -f(x), f(-x) and f(x - 3)

Solution:



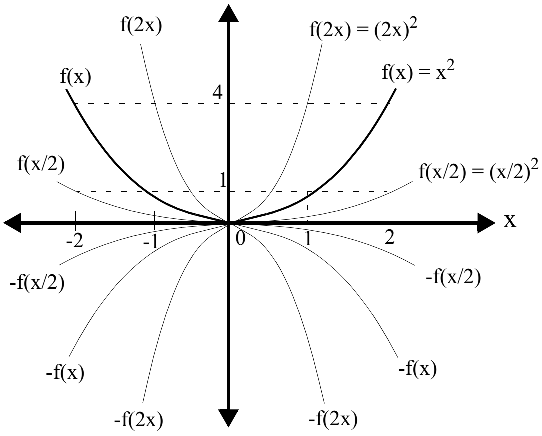
Notice that $f(x) = x^2$ is symmetrical around the y-axis. Therefore, f(x) = f(-x).

Also notice that -f(x) is the mirror image of f(x) about the x-axis.

$f(x - 3) = (x - 3)^2 = x^2 - 6x + 9$, is a version of x^2 shifted by 3 units to the right on the x axis.

3. (Medium)
$f(x) = x^2$
Draw the graphs of f(x), f(ax), f(x/a), -f(x), -f(ax) and -f(x/a) for a = 2.

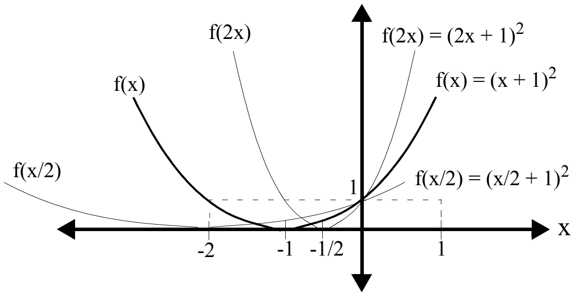
Solution:



Notice that for a = 2, which is greater than 1, f(2x) is the "squeezed" form of f(x). For b = 1/a, which is less than 1, f(bx) = f(x/2) is the "expanded" form of f(x).

4. (Hard)
$f(x) = (x + 1)^2$
Draw the graphs of f(x), f(2x) and f(x/2).

Solution:



Note that $f(x) = (x + 1)^2$ is x^2, shifted to the left by 1.

Notice that f(2x) is the "squeezed" form of f(x), with the same y-interface and f(x/2) is "expanded" form of f(x) with the same y-interface.

Practice Exercises:

1. (Easy)
Which of the following functions represent a quadratic function?

a. $f(s) = s^2 - s$

b. $f(s) = 2^s - s$

c. $f(s) = s^2 - s + 2^s$

d. $f(x) = \frac{x^2 + 1}{3}$

e. $f(x) = \dfrac{x^2 + 1}{3x}$

f. $f(x) = \dfrac{x^3 + x}{3x}$

2. If $f(x) = x^2 - 2x + 4$, then find

a. (Easy)
$f(x) + 3$

b. (Easy)
$3f(x)$

c. (Easy)
$f(3x)$

d. (Medium)
$f(x + 3)$

3. If $f(x) = x^2$, draw

a. (Easy)
$f(x)$

b. (Easy)
$f(x) - 3$

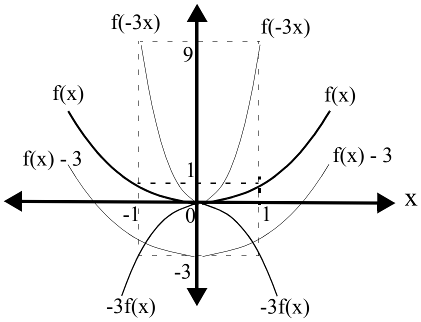
c. (Easy)
$-3f(x)$

d. (Medium)
$f(-3x)$

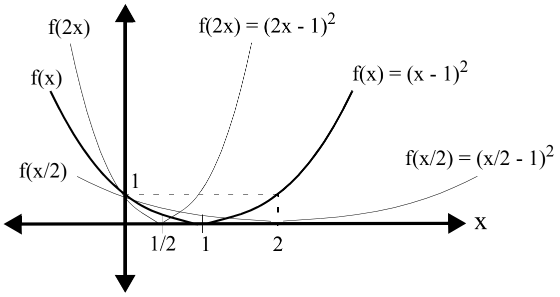
4. (Hard)
$f(x) = x^2 - 2x + 1$
Draw the graphs of f(x), f(2x), f(x/2).

Answers: 1. a, d and f; **2. a.** $x^2 - 2x + 7$; **b.** $3x^2 - 6x + 12$ **c.** $9x^2 - 6x + 4$; **d.** $x^2 + 4x + 7$;

3.



4. $f(x) = x^2 - 2x + 1 = (x - 1)^2$



Exercises

Simple Algebra

One Variable Simple Equations

1. (Easy)
If $33 = 12 + x/3$, then $x = ?$

2. (Easy)
If $2h - 4 = 8 + 4h$, then $h = ?$

3. (Easy)
If $4(a - 3) = 16$, then $a = ?$

4. (Easy)
If $-5(3 - y) = 3(y + 2)$, then $y = ?$

5. (Easy)
If $7 - 3(3n - 7) = -9$, then $n = ?$

6. (Easy)
If $-7 - 3(-3n - 7) = -9$, then $n = ?$

7. (Medium)
If $x^2 = 25$, then $x = ?$

8. (Medium)
Display the solutions to questions 1 to 6 on a number line.

Inequalities

1. (Easy)
If $33 < 12 + x/3$, solve for x.

2. (Easy)
If $2h - 4 < 8 + 4h$, solve for h.

3. (Easy)
If $4(a - 3) > 16$, solve for a.

4. (Easy)
If $-5(3 - y) \le 3(y + 2)$, solve for y.

5. (Easy)
If $7 - 3(3n - 7) \ge -9$, solve for n.

6. (Easy)
If $-7 - 3(-3n - 7) > -9$, solve for n.

7. (Medium)
On a number line, A, B and C are three distinct points with coordinates of 3, 8 and $2x + 1$ respectively. If C is between A and B, how many integer values x can have?

8. (Medium)
If $x^2 < 25$, solve for x.

9. (Medium)
If $x^2 > 25$, solve for x.

10. (Medium)
If $x^2 < 1/25$, solve for x.

11. (Medium)
If $x^2 > 1/25$, solve for x.

12. (Medium)
If $(x - 2)^2 > 25$, solve for x.

13. (Medium)
If $(x + 2)^2 - 4x < 20$, solve for x.

14. (Medium)
Display the solutions to questions 8 to 13 on a number line.

Equations with Multiple Unknowns

1. (Easy)
If your Math grade, m, is 55 points more than 1/2 of your English grade, e,

- **a.** Write an expression that represents your math grade.
- **b.** If your Math grade is 95, what is your English grade?

2. (Easy)
A telephone company charges $5.50 per month for service fees and $0.15 for each call.

a. Write an expression that represents the total monthly charge, T, if you make n calls in one month.

b. If you have made 200 calls in one month, what is you total payment?

c. If your total payment is $25.00 in one month, how many calls you have made in that month?

3. (Medium)
Consider the sequence: -14, -11, -8,...

a. Write an expression the the nth term.

b. What is the value of the 20th term?

4. (Medium)
Consider the sequence: 1, 2, 5, 10, 17,...

a. Write an expression the the nth term.

b. What is the value of the 20th term?

c. Which term has a value of 50?

Equations with Powers

1. (Easy)
If $q^3 = -27$, then q = ?

(A) 3 only
(B) -3 only
(C) 9 only
(D) 3 or -3
(E) 9 or -3

2. (Easy)
If $q^{-6} = 8$, then q^2 = ?

(A) 1/2 only
(B) -1/2 only
(C) 2 only
(D) 1/2 or -1/2
(E) 2 or -2

3. (Medium)
If $q^{1/5} + 1 = 3$, then q = ?

4. (Medium)
If $q^{-1/5} - 2 = 1$, then q = ?

5. (Medium)
If $2^{-b-5} = 4^{3b/4}$, then b = ?

6. (Medium)
If $x^{-5/2} \cdot x^3 = 4$, then x = ?

(A) 2
(B) 4
(C) 16
(D) -16
(E) -2

7. (Medium)
If $(x^{3/2})^{-3/2} + 8 = 24$, then x = ?

(A) $\dfrac{1}{2 \cdot 2^{7/9}}$
(B) 1/4
(C) $2^{17/9}$
(D) 4
(E) $4^{8/9}$

8. (Medium)

If $\dfrac{h^{-\frac{1}{2}} \cdot h^{4-\frac{1}{2}}}{h^2} - 3h = 8$, then h = ?

Radical Equations

1. (Medium)
If $\sqrt[4/3]{16 \cdot x} = 4$, then $-15/16 - x^3$ = ?

2. (Medium)
If 5th root of cube of m equals to 5th power of 2, then m = ?

3. (Medium)
If $\sqrt[-4/3]{81 \cdot x} = 6^{-3}$, then $\sqrt[x]{256} = ?$

4. (Medium)
If $\sqrt[-4/x]{16} = 8^{x+1}$, then x = ?

5. (Medium)
The relationship between the radius, r, of a sphere and its volume, V, is $r = c(\sqrt[3]{V})$, where c is a constant. If the volume of a sphere is V_o when its radius is r_o, what is the volume, V_f, of a sphere when r_o is doubled? Express your answer in terms of V_o.

Absolute Value

1. If p < q < 0, then

a. (Easy)
$|q| = ?$

b. (Medium)
Which of the following could possibly be the value of $|p + q| - |p - q|$?

(A) 2p
(B) -2p
(C) 2q
(D) -2q
(E) 0

c. (Medium)
$|(p - q)^2| - (p - q)^2 = ?$

d. (Medium)
Which of the following could possiblybe the value of $|(p - q)^3| - (p - q)^3$?

(A) $(q - p)^3$
(B) $2(q - p)^3$
(C) $(p - q)^3$
(D) $2(p - q)^3$
(E) $6(p^2q - q^2p)$

e. (Medium)
$\left|\frac{p}{q}\right| \cdot \left(-\frac{q}{p}\right) = ?$

2. If p < 0 < q, then

a. (Easy)
$|q| = ?$

b. (Medium)
$\left|\frac{p}{q}\right| \cdot \left(-\frac{q}{p}\right) = ?$

c. (Medium)
$|(p - q)^2| - (p - q)^2 = ?$

d. (Medium)
Which of the following could possibly be the value of $|(p - q)^3| - (p - q)^3$?

(A) $(q - p)^3$
(B) $2(q - p)^3$
(C) $(p - q)^3$
(D) $2(p - q)^3$
(E) $6(p^2q - q^2p)$

e. (Hard)
Which of the following could possibly be the value of $|p + q| - |p - q|$?

I. 2q
II. 2p
III. -2q

(A) I only
(B) II only
(C) III only
(D) I and II
(E) II and III

3. (Medium)
$|b - 10|^2 - (10 - b)^2 - 2b = 2$, then b = ?

4. (Medium)
On a number line, the coordinates of two points, A and B, are -1 and 5, respectively. If x is the coordinate of the midpoint, C, which of the following is correct?

I. $x + 1 = x - 5$
II. $|x + 1| = |x - 5|$
III. $(x + 1)^2 = (x - 5)^2$

(A) I only
(B) II only
(C) III only
(D) II and III
(E) I and II and III

5. (Medium)
On a number line, the coordinates of A, B, C and D are -2, -8, 1 and x respectively. If AB = CD, x = ?

(A) -7
(B) -5
(C) -3
(D) 3
(E) 5

6. (Medium)
On a number line, x and y are the coordinates of A and B, respectively. If $|x - y| + 2|3y - 3x| = 8$, AB = ?

7. (Medium)
If $||x| + 3| -2|x| - |2x| = 8$, then x = ?

I. 5/3
II. -5/3
III. -8/3

(A) I only
(B) II only
(C) III only
(D) II and III
(E) None

8. (Medium)
True or false: If $x \neq 0$, $|x| = -x$ and $y = -2x$, $x - y < 0$

9. (Medium)
If $x < y < 0$, simplify $|x + |x + y|| + |x - y|$

10. (Medium)
If $|x^2 + 6x + 9| = 16$, x = ?

11. (Hard)
If $0 < a/b < 1$ and $\frac{|a^2 - b^2|}{a - b} = 7$, then a + b = ?

12. (Hard)
If $|x - 3| - |x + 1| = 0$, then x = ?

13. (Very Hard)
If $|2x -1| - |3 + 4x| = 5$, then x = ?

Inequalities With Absolute Value

1. (Medium)
If $|x^2 + 6x + 9| > 16$, then x = ?

2. (Hard)
What is the minimum integer solution to the inequality $-5 < |x+1| < 3$?

3. (Hard)
What is the maximum value of x that satisfies the inequality $-5 < ||x| +1| - x < 3$?

4. (Hard)
If $\frac{|n-3|-|3n-9|}{3+|3-n|} \le 3$, then n = ?

5. (Hard)
If $|x - y| + 2|3y - 3x| < 8$ and $y = 2$, then x = ?

6. (Hard)
If $||x| + 3| -2|x| - |2x| > 8$, then x = ?

7. (Hard)
If $|x - 3| - |x + 1| < 0$, then x = ?

8. (Hard)
If $0 < a/b < 1$ and $\frac{|a^2-b^2|}{a-b} \le 7$, then a + b = ?

9. (Very Hard)
If $|2x -1| - |3 + 4x| > 5$, then x = ?

Proportionality

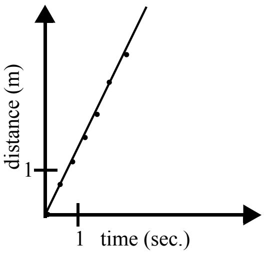
Direct Proportionality

1. (Medium)
a + 4 is proportional to b - 2, and b = 8 when a = 1. What is the value of b when a = 3?

2. (Medium)
a + 4 is proportional to b^2, and b = 8 when a = 2. What is the value of b when a = 1?

3. (Medium)
If it takes 2 gallons of paint to paint a 1000 square foot wall, how much paint is necessary to paint a 2500 square foot wall?

4. (Medium)
In a physics lab, students are asked to measure the distance a cart travels versus time. Their results are displayed in the figure.



Which of the below conclusions is correct?

I. The distance travelled is proportional to time.
II. The motion of the cart can be written as $d = k \cdot t$, where d is the distance traveled, t is the time and k is a constant.
III. The velocity, the distance traveled per second, of the cart is about 1.5 m/sec.

(A) I only
(B) II only
(C) II only
(D) I and II
(E) I, II and III

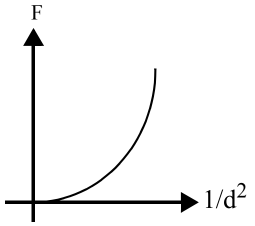
5. (Hard)
a, b and c are proportional to k, with proportionality constants 2, 3 and 4 respectively. If a + b + c = 9, then c = ?

Inverse Proportionality

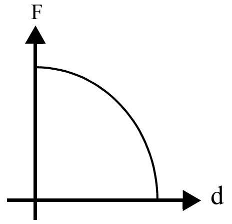
1. (Medium)
a + 4 is inversely proportional to b - 2, and
b = 8 when a = 1.
What is the value of b when a = 3?

2. (Medium)
The gravitational force, F, between two objects is inversely proportional to the square of the distance, d, between the objects. Which of the following graphs represents the relationship between the gravitational force and the distance?

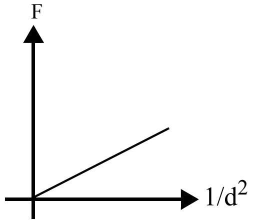
(A)



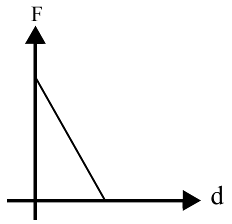
(B)



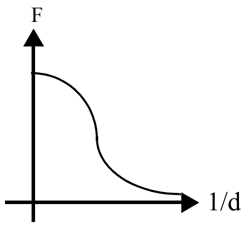
(C)



(D)



(E)



3. (Hard)
a, b and c are inversely proportional to k, with proportionality constants 2, 3 and 4 respectively.
If a + b + c = 9, then c = ?

Mixed Proportionality

1. (Medium)
Two workers build a fence in 10 hours. How many hours will it take to build a fence twice as long with three workers? Assume that all the workers work at equal speed and quality.

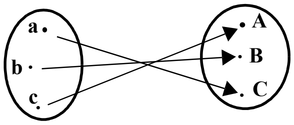
2. (Medium)
a + 4 is inversely proportional to b - 2 and directly proportional to c. When a = 1, b = 8 and c = 4. What is the value of b when a = 3 and c = 6?

Advanced Algebra

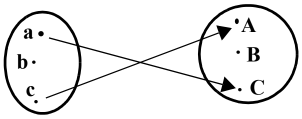
Functions

1. (Medium)
The relationship between x and y is given below. Which of these relationships are functions? If one is a function, give its domain, range, x-intercept and y-intercept whenever appropriate. If not, define a domain so that the relationship becomes a function. In your determination, consider all real values of x whenever applicable.

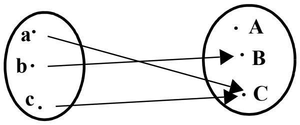
a.



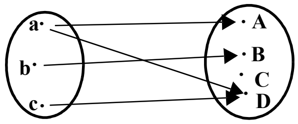
b.



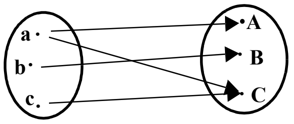
c.



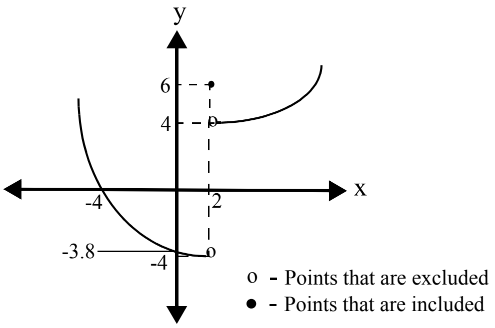
d.



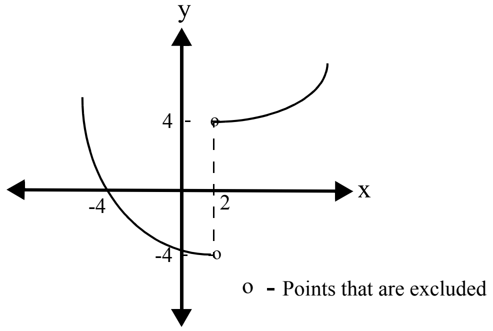
e.



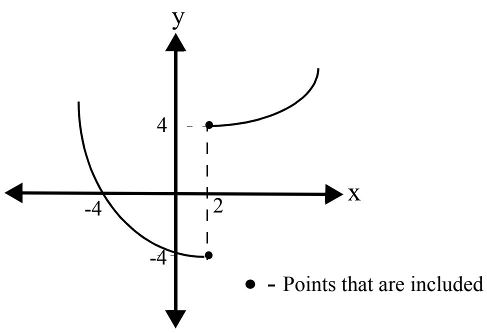
f.



g.



h.



i. f(x) = 2 for x < -1
f(x) = 1/x for x > -1

j. f(x) = 2 for x < -1
f(x) = 1/x for x > -1
f(x) = 1 for x = -1

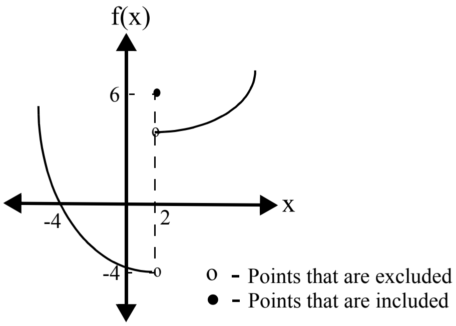
k. f(x) = 2 for x < 1
f(x) = 1/x for x > 1
f(x) = 1 for x = 1

2. Find f(2) for the below functions:

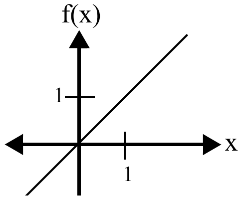
a. (Easy)
$f(x) = x + 2^x$

b. (Medium)
$f(x) = x^2 - 1/x + 2^{-2x} - 3$

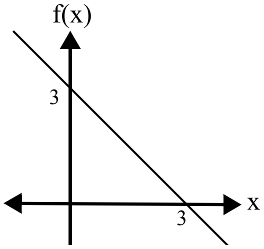
c. (Medium)



d. (Medium)



e. (Medium)



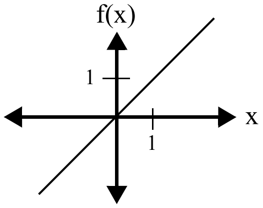
3. If f(x) = 10, then find x for the following functions.

a. (Medium)
$f(x) = 2$ for $x < -1$
$f(x) = 1/x$ for $x > -1$
$f(x) = 1$ for $x = -1$

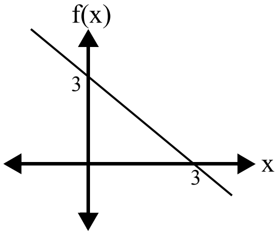
b. (Medium)

$$f(x) = \frac{1}{10x^2 - \frac{18}{5}}$$

c. (Medium)



d. (Medium)



e. (Medium)

$$f(x) = \frac{1}{10x^2 + 2x + \frac{1}{5}}$$

f. (Medium)

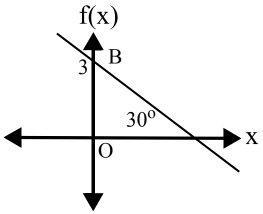
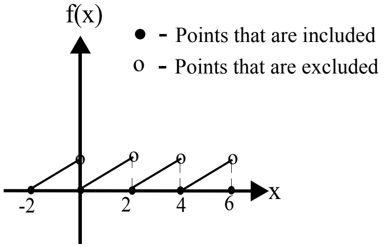


Figure is not drawn to scale.

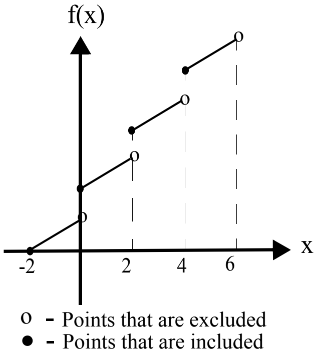
4. (Hard)
If f(x) = f(x + 2) for all real values of x, which of the following can be f(x)?

I.



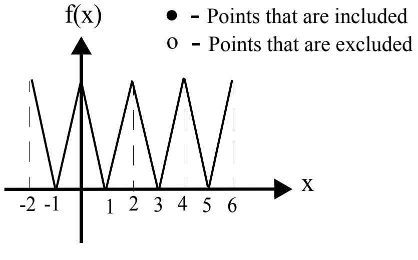
Pattern continues for all the real values of x.

II.



Pattern continues for all the real values of x.

III.



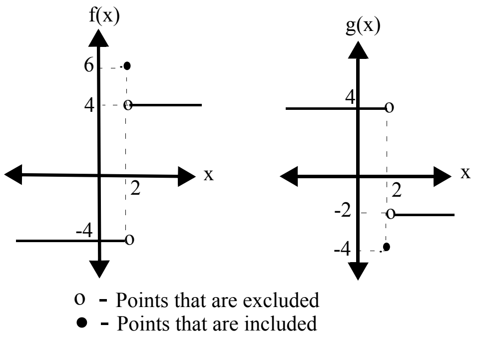
Pattern continues for all the real values of x.

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) I and II and III

Addition and Subtraction of Functions

1. (Easy)
$f(x) = 5x - 4$ and $g(x) = -x + 1$.
If $r(x) = f(x) + g(x) + h(x) = 9x$, what is $h(x)$?

2. (Medium)



f(x) and g(x) are two functions shown above. Which of the below functions is $h(x) = f(x) - g(x)$?

(A) $h(x) = 0$ for $x < 2$
$h(x) = 2$ for $x \geq 2$

(B) $h(x) = -8$ for $x < 2$
$h(x) = 6$ for $x > 2$
$h(x) = 10$ for $x = 2$

(C) $h(x) = 0$ for $x \leq 2$
$h(x) = 2$ for $x > 2$

(D) $h(x) = -8$ for $x < 2$
$h(x) = 6$ for $x \geq 2$

(E) $h(x) = -8$ for $x < 2$
$h(x) = 6$ for $x > 2$
$h(x) = 4$ for $x = 2$

3. (Hard)
$f(x) = -8$ for $x < -3$
$f(x) = x$ for $-3 < x < 2$
$f(x) = 4$ for $x = 2$
$f(x) = -1$ for $x = -3$
$f(x) = -x - 2$ for $x > 2$

$g(x) = x - 1$ for $x \leq 3$
$g(x) = 2$ for $x > 3$

Calculate and draw the graph for $h(x) = f(x) + g(x)$?

Multiplication and Division of Functions

1. (Medium)
$$\frac{x^2 + 4x + 4}{x + 2} = ?$$

2. (Medium)
$$\frac{9x^2 - 6x + 1}{1 - 3x} = ?$$

3. (Medium)
$$\frac{4x^2 + 12xa + 9a^2}{2x + 3a} = ?$$

4. (Medium)
$$\frac{3x - 2a}{9x^2 - 12xa + 4a^2} = ?$$

5. (Medium)
$$\frac{16a^2 - 144x^2}{12x - 4a} = ?$$

6. (Medium)
$$\frac{24x + 8a}{144x^2 - 16a^2} = ?$$

7. (Hard)
$$\frac{6x^2 - 3x - 9}{x + 1} = ?$$

Linear Functions

1. (Medium)
f(x) and g(x) are two linear functions. Which of the following must be linear?

I. $f(x) - g(x)$

II. $r \cdot f(x) + s \cdot g(x)$, where r and s are constants.

III. $f(x) \cdot g(x)$

(A) I only
(B) II only
(C) III only
(D) I and II
(E) I and II and III

2. (Medium)
f(x) is a linear function and r is a constant. Which of the following must be linear?

- **(A)** f(rx)
- **(B)** $r \cdot f(x)$
- **(C)** f(x)/r
- **(D)** f(x - r)
- **(E)** All of the above.

3. (Medium)
If f(x) is a linear function and if f(6) = 7 and f(8) = 12, what is f(4)?

4. (Medium)
f(x) is a linear function and $r \neq 1$ is a non-zero constant.
Which of the following has the same y-intercept?

- **(A)** f(x) + r and $r \cdot f(x)$
- **(B)** f(x) + r and $f(rx)$
- **(C)** $r \cdot f(x)$ and $f(rx)$
- **(D)** f(x) and $r \cdot f(x)$
- **(E)** f(x) and $f(rx)$

5. (Medium)
f(x) is a linear function and $r \neq 1$ is a non-zero constant. Which of the following has the same slope?

- **(A)** f(x) + r and $r \cdot f(x)$
- **(B)** f(x) + r and $f(rx)$
- **(C)** $r \cdot f(x)$ and $f(rx)$
- **(D)** f(x) and $r \cdot f(x)$
- **(E)** f(x) and $f(rx)$

6. (Medium)
f(x) is a linear function. and r is a constant. For which values of r do f(x) and f(rx) have the same slope?

7. (Hard)
f(x) is a linear function and r is a constant. Under which of the following conditions is the slope of f(x) greater than the slope of f(rx)?

- **I.** f(x) is ascending and r < 1
- **II.** f(x) is descending and $0 \leq r < 1$
- **III.** f(x) is descending and r > 1

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I and II
- **(E)** I and III

8. (Hard)
f(x) is a linear function and r is a constant. Under which conditions is the slope of f(x) smaller than the slope of f(rx)?

9. (Hard)
f(x) is a linear function and r is a constant. Under which conditions is the y-intercept of f(x) less than the y-intercept of f(x + r)?

Quadratic Functions

1. (Medium)
f(x) and g(x) are two quadratic functions.
$f(x) = ax^2 + bx + c$ and $g(x) = dx^2 + ex + f$,
Under what conditions the following functions are linear?

- **a.** f(x) + g(x)
- **b.** $r \cdot f(x) + q \cdot g(x)$, where r and s are two non-zero constants.

2. (Medium)
f(x) is a quadratic function and r is a non-zero constant. Which of the following must be quadratic?

(A) f(rx)
(B) $r \cdot f(x)$
(C) f(x)/r
(D) f(x - r)
(E) All of the above

3. (Medium)
f(x) is a quadratic function and $r \neq 1$ is a non-zero constant. Which of the following has the same y-intercept as f(x)?

I. f(x) + r
II. $r \cdot f(x)$
III. $f(rx)$

(A) II only
(B) III only
(C) II and III
(D) I and II
(E) I and III

4. (Medium)
f(x) and g(x) are linear functions. Which of the following functions is quadratic?

I. $(f(x) + g(x))^2$
II. $f(x) \cdot g(x)$
III. $(f(x))^2 + (g(x))^2$

(A) I only
(B) II only
(C) III only
(D) I and II
(E) I and II and III

5. (Medium)
If f(x) is a quadratic function and if f(0) = 5, f(6) = 7 and f(8) = 13, what is f(4)?

Answers

Simple Algebra

One Variable Simple Equations

1. 63
2. -6
3. 7
4. 21/2
5. 37/9
6. -23/9
7. 5 or -5
8. See Solutions

One Variable Inequalities

1. $x > 63$
2. $h > -6$
3. $a > 7$
4. $y \leq 21/2$
5. $n \leq 37/9$
6. $n > -23/9$
7. 2
8. $-5 < x < 5$
9. $x < -5$ or $x > 5$
10. $-1/5 < x < 1/5$
11. $x < -1/5$ or $x > 1/5$
12. $x < -3$ or $x > 7$
13. $-4 < x < 4$
14. See Solutions

Equations with Multiple Unknowns

1.
   a. $m = e/2 + 55$
   b. 80
2.
   a. $T = 0.15n + 5.5$
   b. \$35.50
   c. 130
3.
   a. $t_n = -17 + 3n$
   b. 43
4.
   a. $t_n = (n-1)^2 + 1$
   b. 362
   c. 8th term

Equations with Powers

1. (B)
2. 1/2
3. 32
4. 1/243
5. -2
6. (C)
7. (A)
8. -4

Radical Equations

1. -1
2. $2^{25/3}$
3. $\sqrt{2}$ or $-\sqrt{2}$
4. -3/4
5. $8V_o$

Absolute Value

1.
   a. $-q$
   b. (D)
   c. 0
   d. (B)
   e. -1
2.
   a. q
   b. 1
   c. 0
   d. (B)
   e. (E)
3. -1
4. (D)
5. 7 or -5
6. 8/7
7. (E)
8. True
9. $-x$
10. 1 or -7
11. -7
12. 1
13. No solution

Inequalities With Absolute Value

1. $x > 1$ or $x < -7$
2. -3
3. ∞
4. All real values of n.
5. $6/7 < x < 22/7$
6. No solution
7. $x > 1$
8. $a + b \geq -7$
9. No solution

Direct Proportionality

1. 52/5
2. $4\sqrt{\frac{10}{3}}$ or $-4\sqrt{\frac{10}{3}}$
3. 5 gallons
4. (E)
5. 4

Inverse Proportionality

1. 44/7
2. (C)
3. 4

Mixed Proportionality

1. 40/3 hours
2. 59/7

Advanced Algebra

Functions

1. See Solutions
2.
   a. 6
   b. 0.5625
   c. 6
   d. 2
   e. 1
3.
   a. 0.1
   b. $\frac{\sqrt{37}}{10}, -\frac{\sqrt{37}}{10}$
   c. 10
   d. -7
   e. -0.1
   f. $-7\sqrt{3}$
4. (D)

Addition and Subtraction of Functions

1. $5x + 3$
2. (B)
3. See Solutions

Multiplication and Division of Functions

1. $x + 2$
2. $1 - 3x$
3. $2x + 3a$
4. $1/(3x - 2a)$
5. $-12x - 4a$
6. $1/(6x - 2a)$
7. $6x - 9$

Linear Functions

1. (D)
2. (E)
3. 2
4. (E)
5. (C)
6. 1
7. (E)
8. f(x) is ascending, $r > 1$ or f(x) is descending, $r < 1$
9. f(x) is ascending, $r > 0$ or f(x) is descending, $r < 0$

Quadratic Functions

1.
   a. $a = -d$
   b. $ar = -qd$
2. (E)
3. (B)
4. (E)
5. 11/3

Solutions

Simple Algebra

One Variable Simple Equations

1. Answer: 63
$33 = 12 + x/3 \Rightarrow x/3 = 33 - 12 = 21 \Rightarrow$
$x = 3 \cdot 21 = 63$

2. Answer: -6
$2h - 4 = 8 + 4h \Rightarrow 2h - 4h = 8 + 4 \Rightarrow -2h = 12 \Rightarrow$
$h = 12/(-2) = -6$

3. Answer: 7
$4(a - 3) = 16 \Rightarrow a - 3 = 16/4 = 4 \Rightarrow a = 4 + 3 = 7$

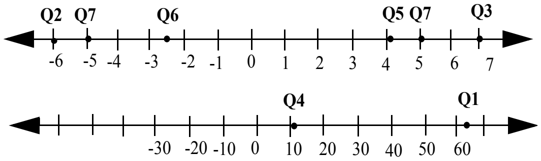
4. Answer: 21/2
$-5(3 - y) = 3(y + 2) \Rightarrow -15 + 5y = 3y + 6 \Rightarrow$
$5y - 3y = 6 + 15 \Rightarrow 2y = 21 \Rightarrow y = 21/2$

5. Answer: 37/9
$7 - 3(3n - 7) = -9 \Rightarrow 7 - 9n + 21 = -9 \Rightarrow$
$9n = 7 + 21 + 9 = 37 \Rightarrow n = 37/9$

6. Answer: -23/9
$-7 - 3(-3n - 7) = -9 \Rightarrow -7 + 9n + 21 = -9 \Rightarrow$
$9n = -9 + 7 - 21 = -23 \Rightarrow n = -23/9$

7. Answer: 5 or -5
$x^2 = 25 \Rightarrow x = 5$ or $x = -5$. Squares of both 5 and -5 are 25.

8.



Question numbers are on top of the number lines in bold.

Inequalities

1. Answer: $x > 63$
$33 < 12 + x/3 \Rightarrow x/3 > 33 - 12 = 21 \Rightarrow$
$x > 3 \cdot 21 = 63 \Rightarrow x > 63$

2. Answer: $h > -6$
$2h - 4 < 8 + 4h \Rightarrow 2h - 4h < 8 + 4 \Rightarrow -2h < 12 \Rightarrow$
$2h > -12 \Rightarrow h > -6$

3. Answer: $a > 7$
$4(a - 3) > 16 \Rightarrow a - 3 > 16/4 = 4 \Rightarrow$
$a > 4 + 3 = 7 \Rightarrow a > 7$

4. Answer: $y \le 21/2$
$-5(3 - y) \le 3(y + 2) \Rightarrow -15 + 5y \le 3y + 6 \Rightarrow$
$5y - 3y \le 6 + 15 \Rightarrow 2y \le 21 \Rightarrow y \le 21/2$

5. Answer: $n \le 37/9$
$7 - 3(3n - 7) \ge -9 \Rightarrow 7 - 9n + 21 \ge -9 \Rightarrow$
$-9n \ge -9 - 7 - 21 = -37 \Rightarrow 9n \le 37 \Rightarrow$
$n \le 37/9$

6. Answer: $n > -23/9$
$-7 - 3(-3n - 7) > -9 \Rightarrow -7 + 9n + 21 > -9 \Rightarrow$
$9n > -9 + 7 - 21 = -23 \Rightarrow n > -23/9$

7. Answer: 2
If x is an integer, then the coordinate of C is also an integer. Since C is in between A and B, C's coordinate can be 4, 5, 6 or 7. On the other hand, since C's coordinate = 2x + 1, which is an odd number, it can only be 5 and 7. So the answer is 2. Let's find these two values of x:
C's coordinate = 5 $\Rightarrow 2x + 1 = 5 \Rightarrow x = 2$
C's coordinate = 7 $\Rightarrow 2x + 1 = 7 \Rightarrow x = 3$

8. Answer: $-5 < x < 5$
$x^2 < 25 \Rightarrow -5 < x < 5$

9. Answer: $x < -5$ or $x > 5$
$x^2 > 25 \Rightarrow x < -5$ or $x > 5$

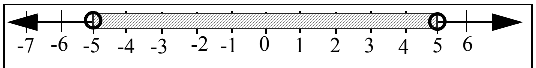
10. Answer: $-1/5 < x < 1/5$
$x^2 < 1/25 \Rightarrow -1/5 < x < 1/5$

11. Answer: $x < -1/5$ or $x > 1/5$
$x^2 > 1/25 \Rightarrow x < -1/5$ or $x > 1/5$

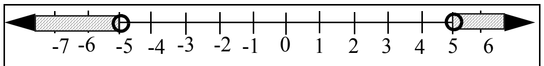
12. Answer: $x < -3$ or $x > 7$
$(x - 2)^2 > 25 \Rightarrow$
$x - 2 > 5 \Rightarrow x > 7$ or
$x - 2 < -5 \Rightarrow x < -3$

13. Answer: $-4 < x < 4$
$(x + 2)^2 - 4x < 20 \Rightarrow x^2 + 4x + 4 - 4x < 20 \Rightarrow$
$x^2 + 4 < 20 \Rightarrow x^2 < 16 \Rightarrow -4 < x < 4$

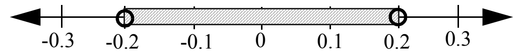
14.



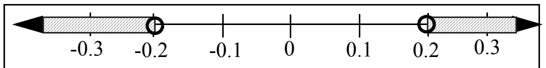
Question 8. Note that -5 and 5 are not included.



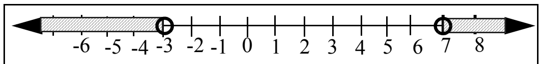
Question 9. Note that -5 and 5 are not included.



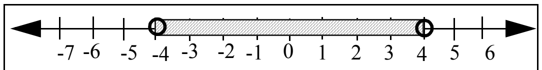
Question 10. Note that -0.2 = -1/5 and 0.2 = 1/5 are included.



Question 11. Note that -0.2 and 0.2 are not included.



Question 12. Note that -3 and 7 are not included.



Question 13. Note that -4 and 4 are not included.

Equations with Multiple Unknowns

1. Answer: a. $m = e/2 + 55$; b. 80

a. $m = e/2 + 55$, e and m are English and Math grades, respectively.

b. $m = 95 \Rightarrow 95 = e/2 + 55 \Rightarrow$
$e/2 = 95 - 55 = 40 \Rightarrow e = 80$

2. Answer: a. $T = 0.15n + 5.5$; b. $35.50; c. 130

a. $T = 0.15n + 5.5$

b. $T = 0.15 \times 200 + 5.5 = \$35.50$

c. $T = 25 \Rightarrow 25 = 0.15n + 5.5 \Rightarrow$
$0.15n = 25 - 5.5 = 19.5 \Rightarrow$
$n = 19.5/0.15 = 130$

3. Answer: a. $t_n = -17 + 3n$; b. 43

a. In the sequence-14, -11, -8,..., each term is 3 more than the previous one.
Hence the expression that gives the nth term is
$t_n = -17 + 3n$, where $n \geq 1$

b. The value of 20th term =
$t_{20} = -17 + 3 \cdot 20 = 43$

4. Answer: a. $t_n = (n - 1)^2 + 1$; b. 362; c. 8th term

a. In the sequence1, 2, 5, 10, 17,..., each term is one more than the square of integers, 0, 1, 2, 3... etc.

So the expression that gives the nth term is
$t_n = (n - 1)^2 + 1$, where $n \geq 1$

b. $t_{20} = (20 - 1)^2 + 1 = 362$

c. nth term is 50 $\Rightarrow t_n = (n - 1)^2 + 1 = 50 \Rightarrow$
$(n - 1)^2 = 50 - 1 = 49 \Rightarrow n - 1 = 7 \Rightarrow n = 8$

The 8th term is 50.

Equations with Powers

1. Answer: (B)
$q^3 = -27 \Rightarrow q = (-27)^{1/3} = -3$
Note that the cube of 3 is 27, not -27.

2. Answer: 1/2
$q^{-6} = 8 \Rightarrow q^6 = 1/8 \Rightarrow q = (1/8)^{1/6} = 1/8^{1/6} \Rightarrow$
$q^2 = (1/8^{1/6})^2 = 1/8^{2/6} = 1/8^{1/3} = 1/2$

3. Answer: 32
$q^{1/5} + 1 = 3 \Rightarrow q^{1/5} = 3 - 1 = 2 \Rightarrow$
$q = 2^5 = 32$

4. Answer: 1/243
$q^{-1/5} - 2 = 1 \Rightarrow q^{-1/5} = 1 + 2 = 3 \Rightarrow$
$q^{1/5} = 1/3 \Rightarrow q = (1/3)^5 = 1/3^5 = 1/243$

5. Answer: -2

$$2^{-b-5} = 4^{\frac{3b}{4}} \Rightarrow$$

$$2^{-b-5} = 4^{\frac{3b}{4}} = (2^2)^{\frac{3b}{4}} = 2^{2 \cdot \frac{3b}{4}} = 2^{\frac{3b}{2}} \Rightarrow$$

$$-b - 5 = \frac{3}{2}b \Rightarrow \frac{3}{2}b + b = -5 \Rightarrow \frac{5}{2}b = -5 \Rightarrow b = -2$$

6. Answer: (C)

$$x^{-5/2} \cdot x^3 = 4 \Rightarrow x^{-\frac{5}{2}+3} = 4 \Rightarrow x^{1/2} = 4 \Rightarrow$$

$$x = 4^2 \Rightarrow x = 16$$

7. Answer: (A)

$$(x^{3/2})^{-3/2} + 8 = 24 \Rightarrow (x)^{-9/4} = 16 \Rightarrow$$

$$(x)^{9/4} = \frac{1}{16} \Rightarrow$$

$$x = \frac{1}{16^{4/9}} \Rightarrow x = \frac{1}{(2^4)^{4/9}} \Rightarrow x = \frac{1}{2^{16/9}}$$

$$x = \frac{1}{(2^{9+7})^{1/9}} \Rightarrow x = \frac{1}{2 \cdot 2^{7/9}}$$

The answer is (A).

8. Answer: $h = -4$

$$\frac{h^{-\frac{1}{2}} \cdot h^{4-\frac{1}{2}}}{h^2} - 3h = 8 \Rightarrow \frac{h^{-\frac{1}{2}+4-\frac{1}{2}}}{h^2} - 3h = 8 \Rightarrow$$

$$\frac{h^3}{h^2} - 3h = 8 \Rightarrow h - 3h = 8 \Rightarrow h = -4$$

Radical Equations

1. Answer: -1

$$\sqrt[4/3]{16 \cdot x} = 4 \Rightarrow (16x)^{3/4} = 4 \Rightarrow$$

$$(2^4 x)^{3/4} = 4 \Rightarrow 2^3 \cdot x^{3/4} = 4 \Rightarrow$$

$$x^{3/4} = 4/8 = 1/2 \Rightarrow x^3 = \frac{1}{2^4} = \frac{1}{16} \Rightarrow$$

$-15/16 - x^3 = -1$

2. Answer: $2^{25/3}$
5th root of cube of m equals 5th power of 2 $\Rightarrow$

$$\sqrt[5]{m^3} = 2^5 \Rightarrow (m^3)^{1/5} = 2^5 \Rightarrow$$

$$m^{3/5} = 2^5 \Rightarrow m = (2^5)^{5/3} \Rightarrow m = 2^{25/3}$$

3. Answer: $\sqrt{2}$ or $-\sqrt{2}$

$$\sqrt[-4/3]{81 \cdot x} = 6^{-3} \Rightarrow (81x)^{-3/4} = 6^{-3} \Rightarrow$$

$$(3^4 x)^{-3/4} = (2 \cdot 3)^{-3} \Rightarrow 3^{-3} \cdot x^{-3/4} = 2^{-3}3^{-3}$$

$$\Rightarrow x^{-3/4} = 2^{-3} \Rightarrow x = 2^4 = 16 \Rightarrow$$

$$\sqrt[x]{256} = (2^8)^{1/16} = 2^{1/2} = \sqrt{2} \text{ or } -\sqrt{2}$$

4. Answer: -3/4

$$\sqrt[-4/x]{16} = 8^{x+1} \Rightarrow (2^4)^{-x/4} = (2^3)^{x+1} \Rightarrow$$

$$2^{-x} = 2^{3x+3} \Rightarrow -x = 3x + 3 \Rightarrow x = -3/4$$

5. Answer: $8V_o$

The volume is V_o when the radius is $r_o \Rightarrow$

$$r_o = c(\sqrt[3]{V_o}) \Rightarrow r_o = cV_o^{1/3} \Rightarrow V_o = \left(\frac{r_o}{c}\right)^3$$

When $r = 2r_o$, $V_f = \left(\frac{2r_o}{c}\right)^3 = 2^3\left(\frac{r_o}{c}\right)^3 = 8V_o$

Absolute Value

1. Answers: **a.** -q; **b.** (D); **c.** 0; **d.** (B) **e.** -1

a. $q < 0 \Rightarrow |q| = -q$

b. $p < q < 0 \Rightarrow$
$|p + q| = -p - q$ and $|p - q| = q - p \Rightarrow$
$|p + q| - |p - q| = -p - q + p - q = -2q$
So the answer is (D).

c. $(p - q)^2$ is square of a non-zero number. So it is always positive. Therefore,
$|(p - q)^2| - (p - q)^2 = (p - q)^2 - (p - q)^2 = 0$

d. $p < q \Rightarrow (p - q)^3 < 0 \Rightarrow$
$|(p - q)^3| = (q - p)^3$ and $(p - q)^3 = -(q - p)^3 \Rightarrow$
$|(p - q)^3| - (p - q)^3 = (q - p)^3 + (q - p)^3 =$
$2(q - p)^3$
The answer is (B).

e. $p < q < 0 \Rightarrow \left|\frac{p}{q}\right| = \frac{p}{q} \Rightarrow$

$$\left|\frac{p}{q}\right| \cdot \left(-\frac{q}{p}\right) = \left(\frac{p}{q}\right) \cdot \left(-\frac{q}{p}\right) = -1$$

2. Answers: **a.** q; **b.** 1; **c.** 0; **d.** (B); **e.** (E)

a. $0 < q \Rightarrow |q| = q$

b. $p < 0 < q \Rightarrow \frac{p}{q} < 0 \Rightarrow \left|\frac{p}{q}\right| = -\frac{p}{q} \Rightarrow$

$$\left|\frac{p}{q}\right| \cdot \left(-\frac{q}{p}\right) = \left(-\frac{p}{q}\right) \cdot \left(-\frac{q}{p}\right) = 1$$

c. $(p - q)^2$ is square of a non-zero number. So it is always positive. Therefore,

$|(p - q)^2| - (p - q)^2 = (p - q)^2 - (p - q)^2 = 0$

d. $p < 0 < q \Rightarrow p - q < 0 \Rightarrow$

$|(p - q)^3| = -(p - q)^3 \Rightarrow$

$|(p - q)^3| - (p - q)^3 = -(p - q)^3 - (p - q)^3 =$

$-2(p - q)^3 = 2(q - p)^3$

The answer is (B).

e. $p < 0 < q \Rightarrow p - q < 0 \Rightarrow |p - q| = q - p$

$|p + q|$ depends on the values of p and q.
For $|p| > q$
$p + q < 0 \Rightarrow |p + q| = -p - q \Rightarrow$
$|p + q| - |p - q| = -p - q + p - q = -2q$
So case III can be correct.

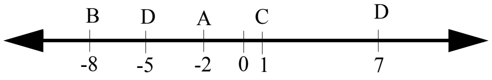
For $|p| < q$
$p + q > 0 \Rightarrow |p + q| = p + q \Rightarrow$
$|p + q| - |p - q| = p + q + p - q = 2p$
So case II can also be correct.

For $|p| = q$
$p + q = 0 \Rightarrow |p + q| = 0 \Rightarrow$
$|p + q| - |p - q| = p - q = p - |p| = p + p =$
$2p = -2q$
Case II and III are the same and they both can be correct.
The answer is (E).

3. Answer: -1
Regardless of the value of b, the values of $|b - 10|^2$ and $(10 - b)^2$ are the same because both of these terms are positive. Hence they cancel each other. You don't have to calculate them. Once these terms are eliminated, the equation becomes $-2b = 2 \Rightarrow b = -1$

4. Answer: (D)
$AC = |x - (-1)| = |x + 1|$ and $BC = |x - 5|$
C is the mid-point between A and B $\Rightarrow AC = BC \Rightarrow$
$|x + 1| = |x - 5|$ and $(x + 1)^2 = (x + 5)^2$. So case II and III are both correct.
Case I is not correct because while x + 1 is the distance between x and -1, hence a positive number, x - 5 is a negative number.
If case I were x + 1 = 5 - x, then it would have been correct.
The answer is (D).

5. Answer: 7 or -5
$AB = |-8 - (-2)| = |-8 + 2| = 6 \Rightarrow CD = |x - 1| = 6 \Rightarrow$
$x - 1 = 6 \Rightarrow x = 7$ or
$1 - x = 6 \Rightarrow x = -5$
These points are displayed in the figure below.



6. Answer: 8/7
$|x - y| + 2|3y - 3x| = 8 \Rightarrow |x - y| + 6|y - x| = 8$

Let d be the distance between x and y. $\Rightarrow$
$d = |x - y| = |y - x| \Rightarrow$
$|x - y| + 6|y - x| = d + 6d = 8 \Rightarrow 7d = 8 \Rightarrow d = 8/7$

7. Answer: (E)
Let $d = |x| \geq 0 \Rightarrow$
$||x| + 3| -2|x| - |2x| = |d + 3| - 2d - 2d = d + 3 - 2d - 2d =$
$3 - 3d = 8 \Rightarrow d = -5/3$
Since $d \geq 0$, $d = -5/3$ is not a solution.

Therefore the answer is (E).

8. Answer: True
$|x| = -x \Rightarrow x < 0$ and $y = -2x \Rightarrow y > 0 \Rightarrow x - y < 0$

9. Answer: -x
$x < y < 0 \Rightarrow |x + y| = -x - y \Rightarrow$
$|x + |x + y|| = |x - x - y| = |-y| = -y$ and $|x - y| = y - x \Rightarrow$
$|x + |x + y|| + |x - y| = -y + y - x = -x$

10. Answer: 1 or -7
$|x^2 + 6x + 9| = |(x + 3)^2| = (x + 3)^2 = 16 \Rightarrow$
$x + 3 = 4$ or $x + 3 = -4$
$x + 3 = 4 \Rightarrow x = 1$
$x + 3 = -4 \Rightarrow x = -7$

11. Answer: -7
$0 < a/b \Rightarrow$
a and b are both positive or they are both negative.

a and b are both positive and $a/b < 1 \Rightarrow$
$0 < a < b \Rightarrow |a^2 - b^2| = -(a^2 - b^2) = -(a + b)(a - b)$

a and b are both negative and $a/b < 1 \Rightarrow$
$b < a < 0 \Rightarrow |a^2 - b^2| = -(a^2 - b^2) = -(a + b)(a - b)$

Therefore for both cases:

$$\frac{|a^2 - b^2|}{a - b} = \frac{-(a + b)(a - b)}{a - b} = -(a + b) = 7 \Rightarrow$$

$a + b = -7$

12. Answer: 1
For $x < -1$
$|x - 3| = 3 - x$ and $|x + 1| = -x - 1 \Rightarrow$
$|x - 3| - |x + 1| = 3 - x + x + 1 = 4 \neq 0$
There is no solution in this range.

For $-1 \le x < 3$
$|x - 3| = 3 - x$ and $|x + 1| = x + 1 \Rightarrow$
$|x - 3| - |x + 1| = 3 - x - x - 1 = 2(1 - x) = 0 \Rightarrow x = 1$

For $x \ge 3$
$|x - 3| = x - 3$and $|x + 1| = x + 1 \Rightarrow$
$|x - 3| - |x + 1| = x - 3 - x - 1 = -4 \neq 0$
There is no solution in this range.

So the answer is 1.

13. Answer: No solution
For $3 + 4x \ge 0$
$x \ge -3/4$ and $|3 + 4x| = 3 + 4x$

For $3 + 4x < 0$
$x < -3/4$ and $|3 + 4x| = -3 - 4x$

For $2x - 1 \ge 0$
$x \ge 1/2$
$|2x -1| = 2x - 1$ and $|3 + 4x| = 3 + 4x \Rightarrow$
$|2x -1| - |3 + 4x| = 2x - 1 - 3 - 4x = -2x - 4 = 5 \Rightarrow$
$x = -9/2$
Since -9/2 is not greater than 1/2, there is no solution in this interval.

For $2x - 1 < 0$
$x < 1/2$ and $|2x -1| = 1 - 2x$
If $-3/4 \le x < 1/2$, then $|3 + 4x| = 3 + 4x \Rightarrow$
$|2x -1| - |3 + 4x| = 1 - 2x - 3 - 4x = -6x - 2 = 5 \Rightarrow$
$x = -7/6$
Since -7/6 is not in between -3/4 and 1/2, there is no solution in this interval.
If $x < -3/4$, then $|3 + 4x| = -3 - 4x \Rightarrow$
$|2x -1| - |3 + 4x| = 1 - 2x + 3 + 4x = 2x + 4 = 5 \Rightarrow$
$x = 1/2$
Since 1/2 is not less than -3/4, there is no solution in this interval.

Thus there is no solution to this equation.

Inequalities With Absolute Value

1. Answer: $x > 1$ or $x < -7$
$|x^2 + 6x + 9| = |(x + 3)^2| = (x + 3)^2 > 16 \Rightarrow$
$x + 3 > 4$ or $x + 3 < -4$
$x + 3 > 4 \Rightarrow x > 1$
$x + 3 < -4 \Rightarrow x < -7$

2. Answer: -3
Since $|x + 1| \ge 0$,
$-5 < |x + 1| < 3$ becomes $0 \le |x + 1| < 3$

$|x + 1|$ is a distance between two points with coordinates x and -1. If this distance is less than 3, x has to be between $-1 - 3 = -4$ and $-1 + 3 = 2$ (Note that both -4 and 2 are excluded.)

So the minimum integer value that x can have is $-4 + 1 = -3$.

Alternate Solution:
If $x + 1 \ge 0$, then $|x + 1| = x + 1 \Rightarrow$
$0 \le x + 1 < 3 \Rightarrow -1 \le x$ and $x < 3 - 1 = 2$

If $x + 1 < 0$, $|x + 1| = -x - 1 \Rightarrow$
$0 \le -x - 1 < 3 \Rightarrow x > -3 - 1 = -4$ and $x \le -1$

The minimum of these solutions is $x > -4$
Therefore the minimum integer solution is -3.

3. Answer: ∞
For $x \ge 0$, $|x| = x \Rightarrow$
$||x| + 1| - x = |x + 1| - x = x + 1 - x = 1 < 3$
Since the inequality is satisfied regardless of the value of x, as long as $x \ge 0$, we have a solution to the inequality.

So the answer is ∞.

4. Answer: All real values of n.
Let $d = |n - 3| = |3 - n|$

$$\frac{|n - 3| - |3n - 9|}{3 + |3 - n|} = \frac{d - 3d}{3 + d} \le 3 \Rightarrow$$

$d - 3d \le 9 + 3d \Rightarrow -2d \le 9 + 3d \Rightarrow 5d \ge -9 \Rightarrow$
$d \ge -\frac{9}{5}$

Since d is the distance between two points on the number line, with coordinates n and 3, it is always a non-negative number.

Hence $d \ge -\frac{9}{5}$ is always true for all values on n.

5. Answer: $6/7 < x < 22/7$
$|x - y| + 2|3y - 3x| < 8 \Rightarrow |x - y| + 6|y - x| < 8$

Let $d = |x - y| = |y - x| \Rightarrow$
$|x - y| + 6|y - x| = d + 6d < 8 \Rightarrow 7d < 8 \Rightarrow d < 8/7$

$y = 2 \Rightarrow d = |x - 2| < 8/7$

Since d is the distance between two points on the number line, with coordinates x and 2, when $d < 8/7$, x is in between
$2 - \frac{8}{7} = \frac{6}{7}$ and $2 + \frac{8}{7} = \frac{22}{7}$

6. Answer: No solution
Let $d = |x| \geq 0 \Rightarrow$
$||x| + 3| -2|x| - |2x| = |d + 3| - 2d - 2d = d + 3 - 2d - 2d =$
$3 - 3d > 8 \Rightarrow d < -5/3$
Since $xd \geq 0$, $d = -5/3$ is not a solution.

So, no real value of x can satisfy this inequality.

7. Answer: $x > 1$
For $-1 < x$
$|x - 3| = 3 - x$ and $|x + 1| = -x - 1$
$|x - 3| - |x + 1| = 3 - x + x + 1 = 4$
Since 4 is not less than 0, there is no solution in this range.

For $-1 \leq x < 3$
$|x - 3| = 3 - x$ and $|x + 1| = x + 1$
$|x - 3| - |x + 1| = 3 - x - x - 1 = 2 - 2x < 0 \Rightarrow x > 1$
Since the range of x is limited by 3, the solution to the inequality in this range is $1 < x < 3$

For $x \geq 3$
$|x - 3| = x - 3$and $|x + 1| = x + 1 \Rightarrow$
$|x - 3| - |x + 1| = x - 3 - x - 1 = -4$
Since -4 is always less than 0, the inequality is satisfied for all $x \geq 3$.

All the ranges combined yields the result: The inequality is satisfied for $x > 1$

8. Answer: $a + b \geq -7$
$0 < a/b \Rightarrow$ a and b are either both positive or both negative.

a and b are both positive and $a/b < 1 \Rightarrow$
$0 < a < b \Rightarrow |a^2 - b^2| = -(a^2 - b^2) = -(a + b)(a - b)$

a and b are both negative and $a/b < 1 \Rightarrow$
$b < a < 0 \Rightarrow |a^2 - b^2| = -(a^2 - b^2) = -(a + b)(a - b)$

Therefore for both cases:
$$\frac{|a^2 - b^2|}{a - b} = \frac{-(a + b)(a - b)}{a - b} = -(a + b) \leq 7 \Rightarrow$$
$a + b \geq -7$

9. Answer: No solution
For $3 + 4x \geq 0$
$x \geq -3/4$ and $|3 + 4x| = 3 + 4x$

For $3 + 4x < 0$
$x < -3/4$ and $|3 + 4x| = -3 - 4x$

For $2x - 1 \geq 0$
$x \geq 1/2$
$|2x - 1| = 2x - 1$ and $|3 + 4x| = 3 + 4x \Rightarrow$
$|2x - 1| - |3 + 4x| = 2x - 1 -3 - 4x = -2x - 4 > 5 \Rightarrow$
$x < -9/2$
Since the numbers less than -9/2 are not greater than 1/2, there is no solution in this interval.

For $2x - 1 < 0$
$x < 1/2$ and $|2x -1| = 1 - 2x$
If $-3/4 \leq x < 1/2$, then $|3 + 4x| = 3 + 4x \Rightarrow$
$|2x -1| - |3 + 4x| = 1 - 2x - 3 - 4x = -6x - 2 > 5 \Rightarrow$
$x < -7/6$
Since the numbers less than -7/6 are not in between -3/4 and 1/2, there is no solution to the inequality in this range.

If $x < -3/4$, then $|3 + 4x| = -3 - 4x \Rightarrow$
$|2x -1| - |3 + 4x| = 1 - 2x + 3 + 4x = 2x + 4 > 5 \Rightarrow$
$x > 1/2$
Since the numbers more than 1/2 are not less than -3/4, there is no solution in this interval.

Hence, there is no solution to this inequality for any value of x.

Proportionality

Direct Proportionality

1. Answer: 52/5
$(a + 4)\alpha(b - 2) \Rightarrow a + 4 = k(b - 2)$, where k is a constant.
$b = 8$ when $a = 1 \Rightarrow 1 + 4 = k(8 - 2) \Rightarrow k = 5/6$

If a = 3, then
$$3 + 4 = \frac{5}{6}(b - 2) \Rightarrow 7 \times \frac{6}{5} = b - 2 \Rightarrow$$
$$b = \frac{42}{5} + 2 = \frac{52}{5}$$

2. Answer: $4\sqrt{\frac{10}{3}}$ or $-4\sqrt{\frac{10}{3}}$

$(a + 4)\alpha b^2 \Rightarrow a + 4 = kb^2$, where k is a constant.

$b = 8$ when $a = 2 \Rightarrow 2 + 4 = k8^2 \Rightarrow k = 3/32$

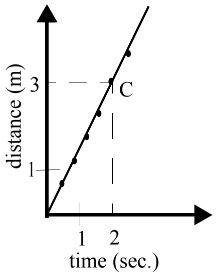
If a = 1, then
$$1 + 4 = \frac{3}{32}b^2 \Rightarrow 5 \times \frac{32}{3} = b^2 \Rightarrow$$
$$b = \sqrt{\frac{160}{3}} = 4\sqrt{\frac{10}{3}} \text{ or } b = -\sqrt{\frac{160}{3}} = -4\sqrt{\frac{10}{3}}$$

3. Answer: 5 gallons

The amount of paint, p, necessary is directly proportional to the area, a, to be painted.
$p \propto a$ ➔ $p = ka$, where k is a constant.
Since 2 gallons of paint is used to paint 1000 square foot, $2 = 1000k$ ➔ $k = 2/1000$
To paint 2500 square feet, you need

$$p = ka = \frac{2}{1000} \cdot 2500 = 5 \text{ gallons of paint.}$$

4. Answer: (E)



The graph is a line graph and it shows the proportionality between distance and time. Case I is correct.

If the distance is proportional to the time, then $\text{distance} = k \cdot \text{time}$
Case II is correct.

To calculate the speed, you need to identify the coordinates of another point on the line.

In the figure, the coordinates of point C is shown as 2 and 3. These values are obtained by using the unit length and unit time given in the original figure.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \text{slope} = \frac{3}{2} = 1.5 \text{ m/sec.}$$

Case III is also correct.

Therefore the answer is (E).

5. Answer: 4

$a = 2k$, $b = 3k$ and $c = 4k$

Since you need to find the value of c, you should express a and b in terms of c. Here is how:

$c = 4k$ and $b = 3k$ ➔ $\frac{b}{c} = \frac{3}{4}$ ➔ $b = \frac{3c}{4}$

$c = 4k$ and $a = 2k$ ➔ $\frac{a}{c} = \frac{2}{4} = \frac{1}{2}$ ➔ $a = \frac{c}{2}$

Substitute these values into $a + b + c = 9$:

$\frac{c}{2} + \frac{3c}{4} + c = 9$ ➔ $\frac{9}{4}c = 9$ ➔ $c = 4$

Inverse Proportionality

1. Answer: 44/7

$a + 4$ is inversely proportional to $b - 2$ ➔
$a + 4 = k/(b - 2)$, where k is a constant.

$b = 8$ when $a = 1$ ➔ $1 + 4 = \frac{k}{8-2}$ ➔ $k = 30$
If $a = 3$, then

$3 + 4 = \frac{30}{b-2}$ ➔ $b - 2 = \frac{30}{7}$ ➔ $b = \frac{44}{7}$

2. Answer: (C)

Let $u = 1/d^2$. If force F is inversely proportional to the square of the distance, then

$F \propto \frac{1}{d^2}$ ➔ $F \propto u$

The only answer choice that shows the proportionality between F and u (or $1/d^2$) is (C). The answer is (C).

3. Answer: 4

$a = 2/k$, $b = 3/k$ and $c = 4/k$

Since you need to find the value of c, you should express a and b in terms of c. Here is how:

$c = 4/k$ and $b = 3/k$ ➔ $\frac{b}{c} = \frac{3}{4}$ ➔ $b = \frac{3c}{4}$

$c = 4/k$ and $a = 2/k$ ➔ $\frac{a}{c} = \frac{2}{4} = \frac{1}{2}$ ➔ $a = c/2$

Substitute these values into $a + b + c = 9$:

$\frac{c}{2} + \frac{3c}{4} + c = 9$ ➔ $\frac{9}{4}c = 9$ ➔ $c = 4$

Mixed Proportionality

1. Answer: $\frac{40}{3}$ hours.

The length, L, of the fence to be built is proportional to the number of workers, n, and the duration of time, t, they work on it. ➔ $L \propto n \times t$
➔ $L = kn \times t$, where k is a constant.

If it takes 10 hours to build a fence of length L_2 for two workers, than $L_2 = 2 \times 10 \times k$ ➔
$k = \frac{L_2}{20}$
If 3 workers are working on a fence of length $2L_2$, then

$2L_2 = \frac{L_2}{20} \times 3 \times t$ ➔ $t = \frac{40}{3}$ hours.

2. Answer: 59/7

$(a + 4) = k\frac{c}{b-2}$, where k is a constant.
When $a = 1$, $b = 8$ and $c = 4$ ➔

$1 + 4 = k\frac{4}{8-2} = k\frac{4}{6} = k\frac{2}{3}$ ➔ $k = 15/2$

If $a = 3$ and $c = 6$, then

$3 + 4 = \frac{15 \times 6}{2(b-2)}$ ➔ $\frac{45}{7} = b - 2$ ➔ $b = \frac{59}{7}$

Advanced Algebra

Functions

1.

a. Function. Domain is a, b, c; Range is A, B, C.

b. Not a function, because point b is not mapped. If b is excluded from the domain, it becomes a function.

c. Function, Domain is a, b, c. Range is B, C

d. Not a function, because point a is mapped to more than one value. It will be a function if point a is mapped to only A or D, not both.

e. Not a function, because point a is mapped to more than one value. It will be a function if point a is mapped to only A or C, not both.

f. Function. Domain is all real values of x.
Range is y > -4. x-intercept = -4,
y-intercept = -3.8

g. Not a function, because it is not defined for x = 2. It is a function for all values of x, but 2.

h. Not a function, because it has 2 values of y, 4 and -4, for x = 2. It will be a function if x = 2 is excluded from its domain.

i. Not a function, because it is not defined for x = -1. It will be a function if x = -1 is excluded from its domain.

j. Function. Domain is all real values of x.
Range is all real values of $y \geq 0$ and $y \leq -1$
x-intercept: none. y-intercept: none

k. Function. Domain: all real values of x.
Range: $0 \leq y \leq 1$ and y = 2
x-intercept: none.
y-intercept: 2

2.

a. Answer: 6
$f(2) = 2 + 2^2 = 6$

b. Answer: 0.5625
$f(2) = 2^2 - 1/2 + 2^{-4} - 3 =$
$4 - 1/2 + 1/16 - 3 = 9/16 = 0.5625$

c. Answer: 6
From the graph: f(2) = 6

d. Answer: 2
Since the figure is a graph of a line with slope 1 and y-intercept 0, it is the equation of a line f(x) = x ➔ f(2) = 2

e. Answer: 1
Since the figure is a graph of a line with slope -1 and y-intercept 3, it is the equation of the line f(x) = -x + 3 ➔ f(2) = -2 + 3 = 1

3. If f(x) = 10, then find x for the following functions.

a. Answer: 0.1
f(x) = 1/x = 10 ➔ x = 1/10 = 0.1

b. Answer: $x = \frac{\sqrt{37}}{10}$ or $x = -\frac{\sqrt{37}}{10}$

$f(x) = \frac{1}{10x^2 - \frac{18}{5}} = 10$ ➔

$10x^2 - \frac{18}{5} = \frac{1}{10}$ ➔ $x^2 = 37/100$ ➔

$x = \frac{\sqrt{37}}{10}$ or $x = -\frac{\sqrt{37}}{10}$

c. Answer: 10
Since the figure is a graph of a line with slope 1 and y-intercept 0, it is f(x) = x
f(x) = 10 ➔ x = 10

d. Answer: -7
Since the figure is a graph of a line with slope -1 and y-intercept 3, it is f(x) = -x + 3
f(x) = -x + 3 = 10 ➔ x = -7

e. Answer: -0.1

$f(x) = \frac{1}{10x^2 + 2x + \frac{1}{5}} = 10$ ➔

$100x^2 + 20x + 2 = 1$ ➔

$100x^2 + 20x + 1 = 0$ ➔ $(10x + 1)^2 = 0$ ➔

10x + 1 = 0 ➔ x = -1/10 = -0.1

f. Answer: $-7\sqrt{3}$
The slope of the line in the figure is
$-\tan(30) = -1/(\sqrt{3})$ and the y-intercept is 3.
Hence the equation of the line is
$f(x) = -x/(\sqrt{3}) + 3$
$f(x) = -x/(\sqrt{3}) + 3 = 10$ ➔ $x = -7\sqrt{3}$

4. Answer: (D)
f(x + 2) is the same as f(x) except it is shifted by 2 to the left. Thus, if f(x) = f(x + 2), you should be able to get the same function if you shift f(x) by 2 to the left. Among the answer choices, I and III satisfy this condition. The answer is (D).

Addition and Subtraction of Functions

1. Answer: 5x + 3
$r(x) = f(x) + g(x) + h(x) = 5x - 4 - x + 1 + h(x) =$
$4x - 3 + h(x) = 9x$ ➔ $h(x) = 9x - 4x + 3 = 5x + 3$

2. Answer: (B)
For $x < 2$, $h(x) = f(x) - g(x) = -4 - 4 = -8$
For $x > 2$, $h(x) = f(x) - g(x) = 4 - (-2) = 4 + 2 = 6$
For $x = 2$, $h(x) = f(x) - g(x) = 6 - (-4) = 6 + 4 = 10$

So the answer is (B)

3.

For $x < -3$,
$h(x) = f(x) + g(x) =$
$-8 + x - 1 = x - 9$

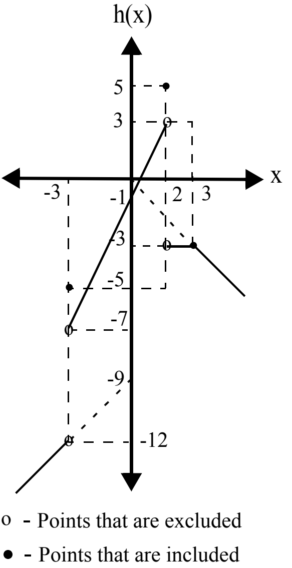
For $x = -3$,
$h(x) = f(x) + g(x) =$
$-1 + x - 1 =$
$-1 - 3 - 1 = -5$

For $-3 < x < 2$,
$h(x) = f(x) + g(x) =$
$x + x - 1 = 2x - 1$

For $x = 2$,
$h(x) = f(x) + g(x) =$
$4 + x - 1 = 4 + 2 - 1 = 5$

For $2 < x \leq 3$,
$h(x) = f(x) + g(x) =$
$-x - 2 + x - 1 = - 3$

For $x > 3$,
$h(x) = f(x) + g(x) =$
$-x - 2 + 2 = - x$



Multiplication and Division of Functions

1. Answer: x + 2

$$\frac{x^2 + 4x + 4}{x + 2} = \frac{(x + 2)^2}{x + 2} = x + 2$$

2. Answer: 1 - 3x

$$\frac{9x^2 - 6x + 1}{1 - 3x} = \frac{(3x - 1)^2}{-(3x - 1)} = 1 - 3x$$

3. Answer: 2x + 3a

$$\frac{4x^2 + 12xa + 9a^2}{2x + 3a} = \frac{(2x + 3a)^2}{2x + 3a} = 2x + 3a$$

4. Answer: 1/(3x - 2a)

$$\frac{3x - 2a}{9x^2 - 12xa + 4a^2} = \frac{3x - 2a}{(3x - 2a)^2} = \frac{1}{3x - 2a}$$

5. Answer: -12x - 4a

$$\frac{16a^2 - 144x^2}{12x - 4a} = \frac{(4a - 12x) \cdot (4a + 12x)}{-(4a - 12x)} =$$

$$-12x - 4a$$

6. Answer: 1/(6x - 2a)

$$\frac{24x + 8a}{144x^2 - 16a^2} = \frac{2(12x + 4a)}{(12x - 4a) \cdot (12x + 4a)} =$$

$$\frac{2}{12x - 4a} = \frac{1}{6x - 2a}$$

7. Answer: 6x - 9

$$\frac{6x^2 - 3x - 9}{x + 1} = \frac{3(2x^2 - x - 3)}{x + 1} =$$

$$\frac{3(x + 1)(2x - 3)}{x + 1} = 6x - 9$$

Linear Functions:

1. Answer: (D)
f(x) is linear ➔ $f(x) = ax + b$ and g(x) is linear ➔ $g(x) = cx + d$, where a, b, c, d are constants.

$f(x) - g(x) = ax + b - cx - d = (a - c)x + b - d =$ $px + q$, where p and q are constants: $p = a - c$ and $q = b - d$ ➔ $f(x) - g(x)$ is linear.

$r \cdot f(x) + s \cdot g(x) = r(ax + b) + s(cx + d) =$ $(ra + sc)x + (rb + sd) = px + q$, where p and q are constants: $p = ra + sc$ and $q = rb + sd$ ➔ $r \cdot f(x) + s \cdot g(x)$ is linear.

$f(x) \cdot g(x) = (ax + b)(cx + d) =$ $acx^2 + (ad + bc)x + bd$. This is not linear because of the existence of the first term, acx^2.

The answer is (D)

2. Answer: (E)
f(x) is linear ➔ $f(x) = ax + b$, where a and b are constants. ➔

$f(rx) = a(rx) + b = arx + b = px + b$, where p is a constant: $p = ar$ ➔ f(rx) is linear.

$r \cdot f(x) = r(ax + b) = rax + rb = px + q$, where p and q are constants: $p = ra$ and $q = rb$ ➔ $r \cdot f(x)$ is linear.

$f(x)/r = (ax + b)/r = (a/r)x + b/r = px + q$, where p and q are constants: $p = a/r$ and $q = b/r$ ➔ f(x)/r is linear.

$f(x - r) = a(x - r) + b = ax + (b - ar) = ax + q$, where a and q are constants: $q = b - ar$ ➔ f(x)/r is linear.

The answer is (E)

3. Answer: 2
If f(x) is a linear function and f(6) = 7 and f(8) = 12, then f(x) represents a line with points (6,7) and (8,12)

are on it. The point $(4,y)$ where $y = f(4)$ is also on the line. Since the slope of a line can be calculated by using any two points on the line, we can calculate y by calculating the slope of the line.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12 - 7}{8 - 6} = \frac{5}{2} = \frac{y - 12}{4 - 8} \Rightarrow$$

$$2(y - 12) = 5(4 - 8) = -20 \Rightarrow$$

$y = -20/2 + 12 = 2$

Alternate Solution:

$f(x)$ is a linear ➔ $f(x) = ax + b$, where a is the slope and b is the y-intercept.

$$\left.\begin{array}{l} f(6) = 7 \Rightarrow 6x + b = 7 \\ f(8) = 12 \Rightarrow 8x + b = 12 \end{array}\right\} \Rightarrow 2a = 5 \Rightarrow a = 5/2$$

Substituting $a = 5/2$ into the first equation yields $15 + b = 7$ ➔ $b = -8$

Hence, $f(x) = 5x/2 - 8$ ➔ $f(4) = 20/2 - 8 = 2$

4. Answer: (E)
$f(x)$ is linear ➔ $f(x) = ax + b$, where a is the slope and b is the y-intercept. ➔

$f(x) + r = ax + b + r$ ➔ y-intercept $= b + r$

$r \cdot f(x) = r(ax + b) = rax + rb$ ➔ y-intercept $= rb$

$f(rx) = a(rx) + b = arx + b$ ➔ y-intercept $= b$, which is the same as y-intercept of $f(x)$.

So the answer is (E).

5. Answer: (C)
$f(x)$ is linear ➔ $f(x) = ax + b$, where a is the slope and b is the y-intercept. ➔

$f(x) + r = ax + b + r$ ➔ slope $= a$, which is the same as the slope of $f(x)$. However, this option is not one of the answer choices.

$r \cdot f(x) = r(ax + b) = rax + rb$ ➔ slope $= ra$

$f(rx) = a(rx) + b = arx + b$ ➔ slope $= ar$, which is the same as the slope of $r \cdot f(x)$.

So the answer is (C).

6. Answer: 1
$f(x)$ is linear ➔ $f(x) = ax + b$, where a, the slope and b are constants.

$f(rx) = a(rx) + b = arx + b$ ➔ slope $= ar$
$ar = a$ ➔ $r = 1$

7. Answer: (E)
Let's examine each case:

Case I.
$f(x)$ is ascending ➔ $f(x)$ has a positive slope ➔
$f(x) = ax + b$, where the slope $= a > 0$
$f(rx) = a(rx) + b = arx + b$ ➔ slope $= ar$
$r < 1$ ➔ $a > ar$ ➔ the slope of $f(x)$ is larger than the slope $f(rx)$.

Case II.
$f(x)$ is descending ➔ $f(x)$ has a negative slope ➔
$f(x) = -ax + b$, where the slope $= -a$ and $a > 0$
$f(rx) = -a(rx) + b = -arx + b$ ➔ slope $= -ar$
$0 \le r < 1$ ➔ $a > ar$ ➔ $-a < -ar$ ➔ the slope of $f(x)$ is less than the slope $f(rx)$

Case III.
$f(x)$ is descending ➔ $f(x)$ has a negative slope ➔
$f(x) = -ax + b$, where the slope $= -a$ and $a > 0$
$f(rx) = -a(rx) + b = -arx + b$ ➔ slope $= -ar$
$r > 1$ ➔ $a < ar$ ➔ $-a > -ar$ ➔ the slope of $f(x)$ is greater than the slope $f(rx)$

So the answer is (E).

8. Answer: $f(x)$ is ascending and $r > 1$ or $f(x)$ is descending and $r < 1$

Let $f(x) = ax + b$ ➔ the slope of $f(x) = a$ and
$f(rx) = a(rx) + b = arx + b$ ➔ slope of $f(rx) = ar$

The slope of $f(x)$ is less than the slope of $f(rx)$ ➔
$a < ar$ ➔ $a > 0, r > 1$ or $a < 0, r < 1$➔

Under two conditions the slope of $f(x)$ is less than the slope of $f(rx)$. They are:
$f(x)$ is ascending $(a > 0)$ and $r > 1$ or
$f(x)$ is descending $(a < 0)$ and $r < 1$

9. Answer: $f(x)$ is ascending and $r > 0$ or $f(x)$ is descending and $r < 0$

Let $f(x) = ax + b$ ➔
the y-intercept of $f(x) = b$ and
$f(x + r) = a(x + r) + b = ax + ar + b$ ➔
y-intercept of $f(x + r) = ar + b$

y-intercept of $f(x)$ is less than the y-intercept of $f(x + r)$ ➔ $b < ar + b$ ➔ $0 < ar$ ➔
$a > 0, r > 0$ or $a < 0, r < 0$
Under two conditions the y-intercept of $f(x)$ is less than the y-intercept of $f(x + r)$.
$f(x)$ is ascending $(a > 0)$ and $r > 0$ or
$f(x)$ is descending $(a < 0)$ and $r < 0$

Quadratic Functions:

1.

a. Answer: $a = -d$
$f(x) + g(x) = ax^2 + bx + c + dx^2 + ex + f =$
$(a + d)x^2 + (b + e)x + c + f = px^2 + qx + r$
where p, q and r are constants:
$p = a + d$, $q = b + e$ and $r = c + f$ ➔
$f(x) + g(x)$ is linear only if
$p = 0$ ➔

$p = a + d = 0$ ➔ $a = -d$

b. Answer: $ar = -qd$

$r \cdot f(x) + q \cdot g(x) =$

$r(ax^2 + bx + c) + q(dx^2 + ex + f) =$
$(rax^2 + rbx + rc) + (qdx^2 + qex + qf) =$
$(ra + qd)x^2 + (rb + qe)x + (rc + qf) =$
$tx^2 + hx + k$
where t, h and k are constants: $t = ra + qd$, $h = rb + qe$ and $k = rc + qf$ ➔

$r \cdot f(x) + q \cdot g(x)$ is linear only if $t = 0$ ➔
$t = 0$ ➔ $t = ar + qd = 0$ ➔ $ar = -qd$

2. Answer: (E)
f(x) is a quadratic function. ➔ $f(x) = ax^2 + bx + c$, where a, b and c are constants. ➔

$f(rx) = a(rx)^2 + b(rx) + c = ar^2x^2 + brx + c = px^2 + qx + c$, where p and q are constants: $p = ar^2$ and $q = br$ ➔ f(rx) is quadratic.

$r \cdot f(x) = r(ax^2 + bx + c) = rax^2 + rbx + rc = px^2 + qx + t$, where p and q are constants: $p = ra$, $q = rb$ and $t = rc$ ➔ $r \cdot f(x)$ is quadratic.

$f(x)/r = (ax^2 + bx + c)/r = (a/r)x^2 + (b/r)x + c/r = px^2 + qx + t$, where p, q and t are constants: $p = a/r$ $q = b/r$ and $t = c/r$ ➔ f(x)/r is quadratic.

$f(x - r) = a(x - r)^2 + b(x - r) + c =$
$ax^2 + (b - 2ar)x + ar^2 - br + c = ax^2 + px + q$
where a, p and q are constants: $p = b - 2ar$ and $q = ar^2 - br + c$ ➔ f(x)/r is quadratic.

The answer is (E).

3. Answer: (B)
f(x) is a quadratic function ➔ $f(x) = ax^2 + bx + c$, where a, b and the y-intercept, c are constants. ➔

Case I:
$f(x) + r = ax^2 + bx + c + r$ ➔
y-intercept = c + r

Case II:
$r \cdot f(x) = r(ax^2 + bx + c) =$

$rax^2 + rbx + rc$ ➔ y-intercept = rc

Case III:
$f(rx) = a(rx)^2 + brx + c =$

$ar^2x^2 + brx + c$ ➔ y-intercept = c

Since the y-intercept of the function in Case III is the same as the y-intercept of f(x), the answer is (B).

4. Answer: (E)
f(x) is a linear ➔ $f(x) = ax + b$, where a and b are constants.
g(x) is a linear ➔ $g(x) = cx + d$, where c and d are constants.

Case I:
$(f(x) + g(x))^2 = (ax + b + cx + d)^2 =$

$((a + c)x + (b + d))^2$ ➔ $(px + q)^2 =$

$p^2x^2 + 2pqx + q^2 = mx^2 + nx + s$, where p, q, m, n and s are constants. ➔

$(f(x) + g(x))^2$ is quadratic.

Case II:
$f(x) \cdot g(x) = (ax + b)(cx + d) =$

$acx^2 + (ad + bc)x + bd = mx^2 + nx + s$, where m, n and s are constants. ➔ $f(x) \cdot g(x)$ is quadratic.

Case III:
$(f(x))^2 + (g(x))^2 =$

$(ax + b)^2 + (cx + d)^2 =$

$(a^2 + c^2)x^2 + (2ab + 2cd)x + b^2 + d^2 =$

$mx^2 + nx + s$, where m, n and s are constants. ➔

$(f(x))^2 + (g(x))^2$ is quadratic.

The answer is (E)

5. Answer: 11/3
f(x) is a quadratic ➔ $f(x) = ax^2 + bx + c$, where a, b and c are constants.

$f(0) = 5$ ➔ $c = 5$

$f(6) = 7$ ➔ $6^2a + 6b + 5 = 7$ ➔

$36a + 6b = 2$ ➔ $b = (1 - 18a)/3 = 1/3 - 6a$

$f(8) = 13$ ➔ $8^2a + 8b + 5 = 13$ ➔ $8^2a + 8b = 8$ ➔

$8a + 1/3 - 6a = 1$ ➔ $2a + 1/3 = 1$ ➔ $a = 1/3$ ➔

$b = 1/3 - 6a = 1/3 - 2 = -5/3$

Substituting the values of a, b and c into f(x):

$f(x) = ax^2 + bx + c = (1/3)x^2 - (5/3)x + 5$ ➔

$f(4) = 16/3 - 20/3 + 5 = 11/3$

8

OTHERS

In this chapter, you will find several different topics which are in SAT but can not be classified as arithmetic, geometry or algebra. They are:

Rounding
Data Representation: Tables, PieCharts & Graphs
Sets
Defined Operators
Logic
Statistics
Sequences
Sums
Counting
  Basic Counting
  Advance Counting Techniques
    Combinations
    Permutations
    Mutually Exclusive Events
    Independent Events
Probability

25% the SAT math questions are in these topics. Their difficulty levels vary from Easy to Hard.

Rounding

Rounding numbers

Some questions in SAT require you to give the answer in rounded numbers.

When you round (approximate) a number, round it to the closer of the lower or upper limits.

Examples:

1. (Easy)
Round (approximate) the below numbers to the nearest integer.

a. 1.8

b. 11.1

c. -7.9

<u>Solution:</u>

a. The upper integer limit (lowest integer higher than 1.8) of 1.8 is 2 and its lower integer limit (highest integer lower than 1.8) is 1. In other words, 1.8 is in between 1 and 2. Since 1.8 is closer to 2 than 1, it is rounded up to 2.

b. The upper integer limit of 11.1 is 12 and its lower integer limit is 11. Since 11.1 is closer to 11 than 12, it is rounded down to 11.

c. The upper integer limit of -7.9 is -7 and its lower integer limit is -8. Since -7.9 is closer to -8 than -7, it is rounded down to -8.

2. (Easy)
Round (approximate) the below numbers to the nearest hundreds digit.

a. 1851

b. 199

c. -87783

<u>Solution:</u>

a. The upper limit (lowest hundreds higher than 1851) of 1851 is 1900 and its lower limit (highest hundreds lower than 1851) is 1800. Since 1851 is closer to 1900 than 1800, it is rounded up to 1900.

b. The upper limit of 199 is 200 and its lower limit of 100. Since 199 is closer to 200 than 100, it is rounded up to 200.

c. The upper limit of -87783 is -87700 and its lower limit is -87800. Since -87783 is closer to -87800 than -87700, it is rounded down to -87800.

3. (Easy)
Round (approximate) the below numbers to the nearest tenth digit.

a. 1851.28

b. 0.1178

c. -8.7783

<u>Solution:</u>

a. The upper limit of 1851.28 is 1851.3 and its lower limit is 1851.2. Since 1851.28 is closer to 1851.3 than 1851.2, it is rounded up to 1851.3.

b. The upper limit of 0.1178 is 0.2 and its lower limit is 0.1. Since 0.1178 is closer to 0.1 than 0.2, it is rounded down to 0.1.

c. The upper limit of -8.7783 is -8.7 and its lower limit is -8.8. Since -8.7783 is closer to -8.8 than -8.7, it is rounded down to -8.8.

Practice Exercises:

1. (Easy)
Round (approximate) 1128 to the nearest hundreds digit.

2. (Easy)
Round (approximate) -123.167 to the nearest tenth digit.

3. (Easy)
A = 1.8, B = 5.0. What is the average of A and B? Approximate your answer to the nearest whole number.

4. (Easy)
Kim's yearly income is $33,500. What is her monthly income? Round your answer to the nearest cent.

5. (Medium)
David's height increased 10% between 2004 and 2005. If his height is 1.40 meters in 2005, what was his height in 2004? Round your answer to the nearest centimeter. Note that 1 meter = 100 centimeters.

Answers:
1. 1100; **2.** -123.2; **3.** 3; **4.** $2791.67; **5.** 1.27 meters

Rounding data in Pie Charts and Graphs

Some of the questions in data representation requires you to round charts and graphs visually and numerically. We will study the data representation in detail in the next section. If you have difficulty understanding the examples here, study these two sections together.

Examples:

Pie chart in the figure represents the land use in Fox county.

In the chart:
I - Industrial
C - Commercial
PL - Park Land
MFR - Multi Family Residential
SFR - Single Family Residential

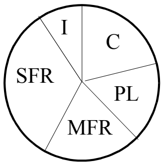


Figure 1

Use this chart for Example 1 below and Practice Exercise 1 at the end of this section.

1. (Medium)
In figure 1, which of the following represents best the percentage of the commercial land in Fox county?

(A) 20%
(B) 25%
(C) 45%
(D) 80%
(E) 90%

Solution:

As shown in the figure, 1/4 of the pie (90^o out of 360^o) is 25% of the total land use.

Since commercial land is less than 1/4 of the total land, the answer is (A), 20%.

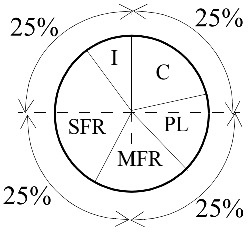


Figure shows the production of ABC company between 2002 and 2005.

Use this graph for the example below and to answer the Practice Exercise 2, in this section.

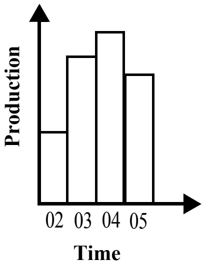


Figure 2

2. (Medium)
In Figure 2, what is the percentage increase in production between 2002 and 2003?

(A) 50%
(B) 100%
(C) 150%
(D) 200%
(E) There is not enough information provided to answer the question.

Solution:

2003 production is about twice as much as 2002 production. Let the 2002 production be x. Then, 2003 production is 2x. The percentage increase is (2x - x)100/x = 100%. Hence the answer is (B).

As you can see in this example, you don't need the actual numbers to answer the question.

Practice Exercises:

1. (Medium)
In Figure 1, which of the following represents best the percentage of the industrial and commercial land combined?

(A) 20%
(B) 30%
(C) 40%
(D) 50%
(E) 60%

Hint: If you need to, you can use the techniques provided in Appendix B, to create the desired angles.

2. (Medium)
In Figure 2, what is the ratio of the production in 2005 to the production in 2002?

(A) 2
(B) 1.8
(C) 1.2
(D) 0.8
(E) 0.5

Answers: 1. (B); **2.** (B)

Data Representation: Tables, Pie Charts & Graphs

In each SAT there are questions about data representation of various kind. They are visual representation of a survey, a company or census data, etc. You can find these questions in SAT mostly at Easy level.

Tables

Tables are one of the ways to display the data. Usually tables display the actual raw data in an organized form, hence provide the least visual aid to the user.

In most cases, the user must create other tables, pie charts and/or graphs to understand and interpret the data better.

Examples:

1. (Easy)
Sales revenue of ABC company from company's four different projects is displayed in Table A.

Table A

	Revenue
Project A	\$39,995
Project B	\$82,003
Project C	\$21,189
Project D	\$59,780

It is not very difficult to see several facts (such as Project B creates the most revenue, Project A creates about twice the revenue as Project C, etc.) by just looking at these figures. However, it is not very intuitive.

For example, if you want to know approximately what percent of the total sales are generated by project C, you need to take the following steps:

Total revenue:
39,995 + 82,003 + 21,189 + 59,780 = \$202,967

Percentage of sales from Project C:

$$100 \times \frac{21189}{202967} \cong 10.44\,\%$$

Note that the " $\cong$ " symbol indicates that the result is rounded.

2. (Easy)
You can create another table to display the same data, which will make it very easy to understand some facts about the revenue of the company. Here is how:

Let's represent each \$10,000 by the "**\$**" sign and display the same data in a new table, Table B.

Table B

	Revenue
Project A	**\$\$\$\$**
Project B	**\$\$\$\$\$\$\$\$**
Project C	**\$\$**
Project D	**\$\$\$\$\$\$**

In this new table, all the revenue figures are approximated to the nearest \$10,000. In Table B, you can readily see several facts without making any effort. Of course you lose some of the details in this representation, but sometimes you are not interested in details. All you want to see is the big picture.

Let's calculate the percentage of sales from Project C one more time by using Table B.

All you need to do is count the number of "**\$**" signs to get the total revenue.
It is 4 + 8 + 2 + 6 = 20, corresponds to a total revenue of \$200,000. Similarly, Project C has 2 "\$" signs and brings \$20,000.

Therefore the percentage of sales from Project C:

$$100 \times \frac{2}{20} = 10\,\% \text{ or } 100 \times \frac{20000}{200000} = 10\,\%$$

The answer is 10%, approximately the same as the previous result of 10.44%.

3. (Medium)
In Table A of Example 1, what percentage of sales are generated by project A?

- **(A)** 10.44%
- **(B)** 19.71%
- **(C)** 29.45%
- **(D)** 40.40%
- **(E)** 51.03%

Solution:
You don't need to calculate the exact percentages. Approximately, the total revenue is \$200,000. Project A's revenue is about \$40,000. Hence the answer is roughly

$$100 \times \frac{40000}{200000} = 20\,\%.$$

The closest answer is (B). Note that the rest of the answer choices are so far removed from 20% that we can safely eliminate them.

Practice Exercises:

1. Answer the below questions by using Table A above. Don't use your calculator.

a. (Easy)
What is the total sales in dollars?
Round your answer to the nearest thousands digit.

b. (Medium)
What percentage of sales is generated by Project D?

(A) 10.44%
(B) 19.71%
(C) 29.45%
(D) 40.40%
(E) 51.03%

c. (Medium)
Approximately 30% of all sales generated by which of the following projects?

(A) Projects A and D, combined.
(B) Projects B and C, combined.
(C) Projects A, B and C, combined.
(D) Projects A and C, combined.
(E) Projects A, C and D, combined.

2. Answer the below questions by using Table B above.

a. (Easy)
What is the total sales in dollars?

b. (Medium)
What percentage of sales is generated by Project B?

c. (Medium)
60% of all sales generated by which of the following combinations?

(A) Projects A and D, combined.
(B) Projects B and C, combined.
(C) Projects A, B and C, combined.
(D) Projects A and C, combined.
(E) Projects A, C and D, combined.

Answers:
1.a. \$203,000; **1.b.** (C); **1.c.** (D);
2.a. \$200,000; **2.b.** 40%; **2.c.** (E)

Pie Charts

In a pie chart, the whole is represented by a circle (the whole pie). Each part of the whole is represented by a piece of the pie.

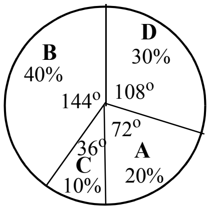
To explain the relationship between the actual data and a pie chart better, let's convert the data in Table B (or Table A) in the previous section to a pie chart:

Percentages of the revenue and the piece of the pie that represents each project is as follows:

	Percentage Revenue	Piece of the pie in degrees
Project A	20%	$360 \cdot 20/100 = 72°$
Project B	40%	$360 \cdot 40/100 = 144°$
Project C	10%	$360 \cdot 10/100 = 36°$
Project D	30%	$360 \cdot 30/100 = 108°$
Total	100%	360^o

As you can see in the last row of the table, 100% of the revenue is represented by 360^o. Each piece of the pie is calculated as a percentage of 360^o:

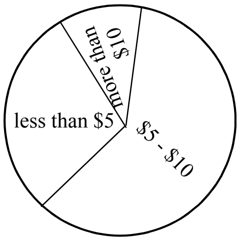
$72^o = 20\%$ of 360^o
$144^o = 40\%$ of 360^o
$36^o = 10\%$ of 360^o
$108^o = 30\%$ of 360^o



The figure is the pie chart representing the data in the table above.

Example 1:

a. (Medium)
The figure shows the percentage of different price ranges of goods in a store. What percent of the goods cost more that \$10?



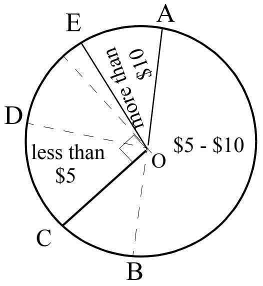
(A) 6
(B) 12
(C) 22
(D) 45
(E) 55

b. (Medium)
If there are 360 items in the store cost less than \$5, how many items there are with sale price more than \$10?

(A) 100
(B) 140
(C) 180
(D) 200
(E) 240

Solution:

a. As shown in the above figure, 25% of the pie (1/4 of the pie), is defined by $\angle AQD$ and arc AD.



Since $\overline{OE}$ roughly bisects $\angle AOD$, the answer is around 25/2 = 12.5%. The closest answer is (B).

b. In the figure, $\angle EOC$ is a little bit bigger than 90^{o}. Thus, the items with sale price less than \$5 is slightly more than 25% of the total number of items. Let's assume that they are 30% of the total. If 30% of the goods is 360, 12% (items with sales price more than \$10) of the goods is

$$\frac{360}{30} \times 12 = 144$$

The closest answer is (B).

Practice Exercises:

1. (Medium)

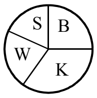
Kingston county has total of four high schools. Table displays the number of students graduated from these high schools in 2005.

High School	**Number of Graduates**
Blair	208
Sonoma	150
Kent	270
Washington	179

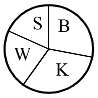
Which of the following pie charts represents the school graduation data best.

In the answer choices, each school is represented by the first letter of its name.

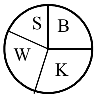
(A)



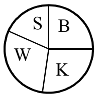
(B)



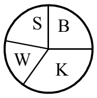
(C)



(D)



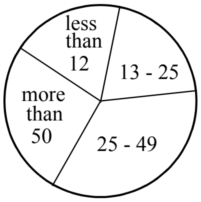
(E)



Hint: Wrong answers have incorrect proportions that you can identify easily.

2. (Medium)

Figure shows the age distribution of the artists in a theater.



a. Which of the following best represents the percentage of the players between 13 to 25 years old?

(A) 10%

(B) 18%

(C) 25%

(D) 60%

(E) 70%

b. If there are 8 artists between 13 to 25 years old, which of the following best provides the total number of artists in the theater?

(A) 100

(B) 80

(C) 45

(D) 25

(E) 15

Answer: 1. (A); **2.a.** (B); **2.b.** (C)

Graphs

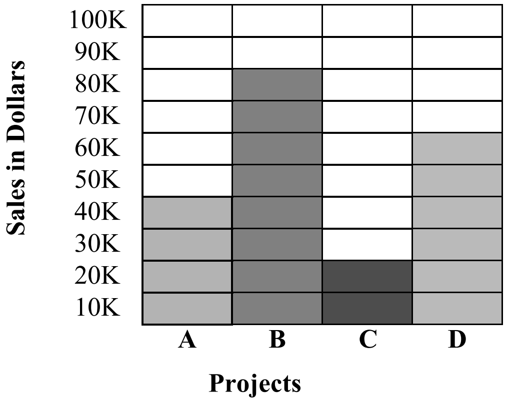
Graphs are another way of displaying data.

There are 3 types of graphs in the SAT: bar graph, histograms and scattered graphs.

Bar Graphs

Below is a bar graph of the data presented in Tables A or B in the "Tables" section above.

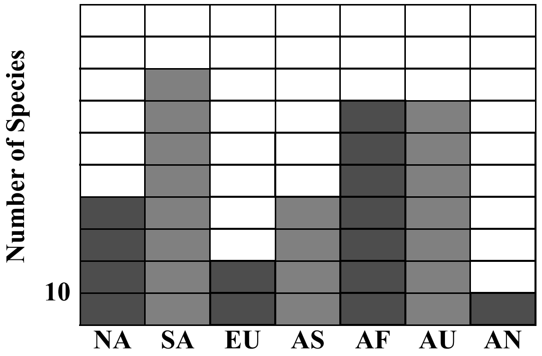
Company ABC's Sales Record in Dollars



As you can see, a bar graph provides a clear visual aid and renders the data easy to compare. In the above graph, you can clearly see the relative sales volumes between the 4 projects.

Below is a graph of the number of new species found in different continents in the year 2000.

NA - North America, SA - South America, EU - Europe, AS - Asia, AF - Africa, AU - Australia, AN - Antarctica



Use this bar graph for the examples and Practice Exercises below.

Examples:

1. (Easy)
How many species are found in Australia?

Solution:
In the graph, each rectangle represents 10 species. So there are 70 species found in Australia.

2. (Easy)
What is the total number of species found in the year 2000?

Solution:
40 + 80 + 20 + 40 + 70 + 70 + 10 = 330 species.

3. (Medium)
What percentage of the new species is found in Europe?

Solution:

$$\frac{20}{330} \times 100 = 6.06\,\%$$

4. (Medium)
In which continent approximately 25% of the species is found?

Solution:
25% of 330 is 82.5. With 80 new species found, South America comes closest to 82.5. So the answer is South America.

Practice Exercises:

Use the above bar graph to answer the following questions.

1. (Easy)
How many species are found in North and South America combined?

.

2. (Medium)
What is the percentage of the new species found in Africa? Approximate your answer to the nearest whole number.

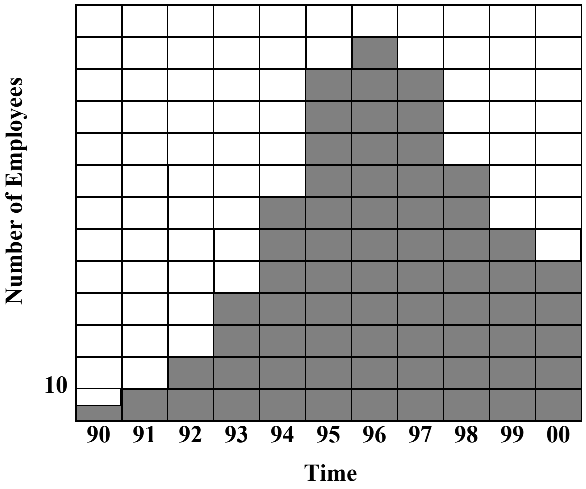
3. (Medium)
In which continent approximately 12% of the species is found?

Answers:
1. 120; **2.** 21%; **3.** NA (North America) and AS (Asia)

Histograms

Histograms display data that change over **time**. Examples are: a company's profit over time, population change over time, number of tourists that visit Washington DC over time, etc.

Below is the graph of the number of employees in the ABC Inc. between 1990 - 2000. It is used in the examples and Practical Exercises below.



Examples:

1. (Easy)
What is the smallest increase in the number of employees between 1990 and 1996?

Solution:
10 - 5 = 5, from 5 employees in 1990 to 10 employees in 1991.

2. (Easy)
What is the largest decrease in the number of employees between 1996 and 2000?

Solution:

110 - 80 = 30, from 110 employees in 1997 to 80 employees in 1998.

3. (Easy)
What is the increase in the number of employees between 1990 and 2000?

Solution:

50 - 5 = 45, from 5 employees in 1990 to 50 employees in 2000.

4. (Medium)
What is the smallest percent increase in the number of employees between 1990 and 1996?

Solution:

The smallest percent increase is

$$\frac{120-110}{110} \times 100 = 9.1\,\% \text{ in 1996.}$$

Note that the percent change is not the same as the change in the number of employees. For example, in 1991, the number of employees increased only by 5, but since the company had only 5 employees in 1990, it corresponds to a 100% increase.

5. (Medium)
What is the largest percent decrease in the number of employees between 1996 and 2000?

Solution:

The largest percent decrease is

$$\frac{110-80}{110} \times 100 = 27.3\,\% \text{ in 1998.}$$

Note that the largest percentage decrease and the largest decrease are both in 1998. This is only a coincidence.

6. (Medium)
What is the percent increase in the number of employees between 1990 and 2000?

Solution:

The percent increase between 1990 and 2000 is

$$\frac{50-5}{5} \times 100 = 900\,\%.$$

Practical Exercises:

1. (Medium)
Prepare a table displaying the absolute and percentage increases/decreases in the number of employees of ABC Inc. for each consecutive year from 1990 to 2000. Notice that the absolute change does not correspond to the percentage change. Round your answers to the nearest integer.

2. (Easy)
What is the biggest increase in the number of employees between 1990 and 2000?

3. (Medium)
What is the largest percent increase in the number of employees between 1990 and 2000?

Answers:
1.

Year	Absolute Change	Percentage Change
1991	5	100%
1992	10	100%
1993	20	100%
1994	30	75%
1995	40	57%
1996	10	9%
1997	-10	-8%
1998	-30	-27%
1999	-20	-25%
2000	-10	-17%

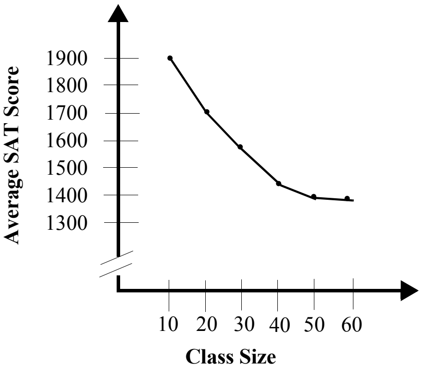
2. 40 in 1995; **3.** 100% in years 1991, 1992 and 1993

Scattered Graphs

When two variables are related, you can plot one of them against the other to show the relationship. You can have a number of conclusions by just looking at the graph.

Examples:

Below graph shows average SAT score versus average high school class size.

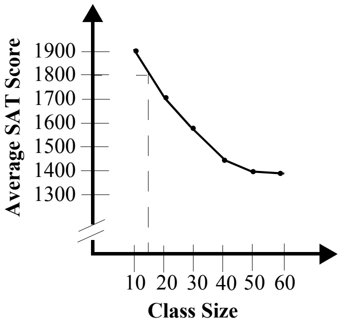


Use this graph in the Examples and the Practice Exercises below.

1. (Easy)
What is the approximate SAT score for the class size of 15?

Solution:

Although you don't have the direct data for the SAT score for class size 15, you can read it from the graph as shown in the figure. SAT score is 1800 for this class size.

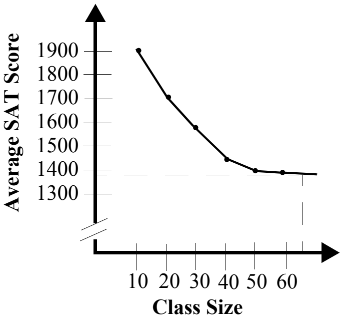


This technique is called the **interpolation**. In this method, read the result in between the data points by using a graph.

2. (Medium)
What is the approximate SAT score for the class size of 65?

Solution:

Although you don't have the direct data for the SAT score for class size 65, you can read it from the graph as shown in the figure. SAT score is around 1370 for this class size.



This technique is called the **extrapolation**. In this method, read the result beyond the data points by extending the graph out of its domain.

3. (Medium)
Consider the graph above. Which of the below statements is correct?

I. Average SAT score drops as the class size increases.

II. Average SAT score increases as the class size decreases.

III. Rate of change in the SAT score is constant for the class sizes displayed in the figure.

(A) I only

(B) II only

(C) III only

(D) I and II

(E) I and III

Solution:

I and II are stating the same fact that as class sizes increase, SAT score drops. They are both correct.

The rate of change is the slope of the curve in the figure. As you can see, it is not constant. For the slope to be constant, the graph had to be a straight line. Hence III is wrong.

The answer is (D).

4. (Medium)
Which of the following best represents the rate of drop in SAT score at class size 25?

(A) 5

(B) 15

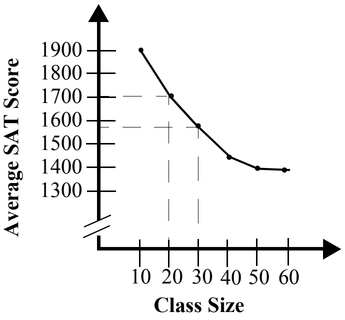
(C) 25

(D) 35

(E) Can not be estimated by using the data provided.

Solution:

The rate of drop is the slope of the curve at class size 25. As shown in the figure, you can use upper and lower values around the class size 25 to calculate the slope.



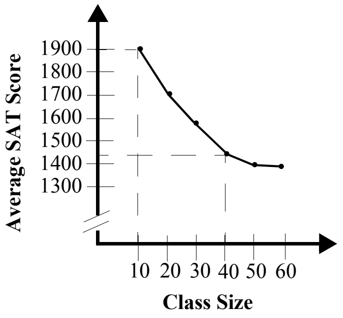
It is $\frac{1700 - 1570}{30 - 20} = 13$ points per student.
15 points per student is close enough to our answer. In fact we could get 15 easily if we read the data slightly differently. Hence the correct answer is (B).

5. (Medium)
What is the average rate of decrease in the SAT score between the class sizes 10 and 40?

Solution:

The average rate of the drop is: $\frac{1900 - 1440}{40 - 10} =$ 15 points per student.

The values in this expression is read from the graph.



Practice Exercises:

Use the SAT Score vs. Class Size data presented above to answer the below exercises.

1. (Easy)
What is the approximate SAT score for the class size of 35?

2. (Medium)
What is the approximate SAT score for the class size of 5?

3. (Medium)
Which of the following represents the rate of drop in SAT score best at class size 15?

(A) 5
(B) 10
(C) 20
(D) 30
(E) Can not be estimated by using the data provided.

4. (Medium)
What is the <u>average</u> rate of decrease in the SAT score between the class sizes 30 and 60?

5. (Medium)
Which of the following statements is true?

I. Reducing the class size is most effective in increasing the SAT score for already smaller classes.

II. Reducing the class size is most effective in increasing the SAT score for class sizes more than 40.

III. Reducing the class size is very effective in increasing the SAT score for all the class sizes.

(A) I only
(B) II only
(C) III only
(D) I and III
(E) I and II and III

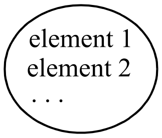
Answers:
1. 1500; **2.** 2030; **3.** (C); **4.** 6 points/student; **5.** (A)

Sets

A **set** is a collection of **elements**. Elements of a set can be anything: numbers, points, animals, colors, geometric shapes, plants, etc. A set is represented as follows.

Set A = (Describe the elements) or
Set A = {element 1, element 2, ...} or

Set A

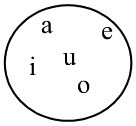


Examples:

1. (Easy)
Set of all vowels in the English alphabet can be represented by

Set A = Vowels in the English alphabet
or
Set A = {a, e, i, o, u}
or

Set A



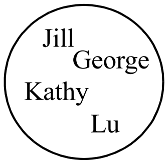
2. (Easy)
The set of odd numbers between -8 and 8 is
{-7, -5, -3, -1, 1, 3, 5, 7}

3. (Easy)
The set of positive integers less than 6 is
{1, 2, 3, 4, 5}

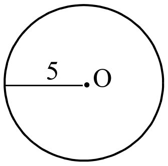
4. (Easy)
The set of countries in North America is
{USA, Canada}

5. (Easy)
The figure represents the set of students in Drama Club, born in August.

It can also be expressed as:
{Jill, George, Kathy, Lu}

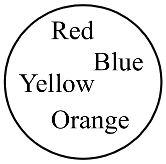


6. (Medium)
The set of points at a distance 5, from the point O is Circle O with radius 5.



Practice Exercises:

1. (Easy)
What are the elements of the set in the figure?



2. (Easy)
Form a set of 3 female and 3 male names.

3. (Easy)
Form a set that contains the squares of integers between (and including) -5 and 5.

4. (Medium)
Let k be a line and A and B the two points on the line k. What is the set of points that are at equal distance from A and B?

Answers:
1. {Red, Blue, Yellow, Orange};
2. {Jane, Jo, Karen, Joe, Mark, Richard};
3. {0, 1, 4, 9, 16, 25};
4. A line perpendicular to line k and passing through the mid point between the points A and B.

Union of Sets

The union of two sets is another set that contains all the elements of both sets.

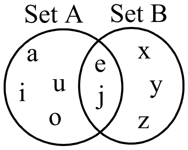
Examples:

1. (Easy)
Set A = {1, 2, 3} and Set B = {-1, 0}.
Union of Set A and Set B = {-1, 0, 1, 2, 3}

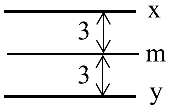
2. (Easy)
Set Z = The set of odd numbers between -6 and 8 = {-5, -3, -1, 1, 3, 5, 7}
Set X = A set of positive integers less than 6 = {1, 2, 3, 4, 5}
Union of Set Z and Set X = {-5, -3, -1, 1, 2, 3, 4, 5, 7}

3. (Easy)
Set A = the first 5 letters of the alphabet.
Set B = {c, d, k, u, x}
Union of Set A and Set B = {a, b, c, d, e, k, u, x}

4. (Easy)
In the figure,
Set A = {a, e, i, j, o, u} and
Set B = e, j, x, y, z}
The union of Set A and Set B = {a, e, i, j, o, u, x, y, z}



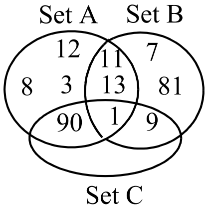
5. (Medium)
Set A is the collection of points on line m.
Set B is the collection of points that are at a distance of 3 from line m.
In the figure,
Set A = {line m}
Set B = {line x, line y}
The union of Set A and Set B = {line m, line x, line y}



Practice Exercises:

1. (Easy)
In the figure, what are the elements of

a. Set A
b. Set B
c. Set C
d. The union of Set A, Set B and Set C



2. (Easy)
Let Set A = {1, 2, 3}, Set B = {-1, 0, 1, 2},
Set C = {2, 4, 6}.

What is the union of Set A, Set B and Set C?

3. (Easy)
Set A = The collection of odd numbers between -6 and -2
Set B = The values of m satisfying $m = n^2 - 1$, where n is an integer and $-6 \le n \le 2$.
Set C = The values of m satisfying $m = n^2$, where n is an integer and $-6 \le n \le 2$.

What is the union of Set A, Set B and Set C?

4. (Medium)
Consider two concentric circles. If Set A is the points on and inside of the smaller circle and the Set B is the points in between the two circles including the circles, what is the union of Set A and Set B?

Answers:
1.a. Set A = {1, 3, 8, 11, 12, 13, 90},
1.b. Set B = {1, 7, 9, 11, 13, 81}, **1.c.** Set C = {1, 9, 90},
1.d. The union of Set A, Set B and Set C =
{1, 3, 7, 8, 9, 11, 12, 13, 81, 90};
2. {-1, 0, 1, 2, 3, 4, 6};
3. {-5, -3, -1, 0, 1, 3, 4, 8, 9, 15, 16, 24, 25, 35, 36};
4. All the points in and on of the larger circle.

Intersection of Sets

Intersection of two sets is another set that contains all the elements common to both sets.

If there are no common elements, then the intersection of the sets is a **null** set with no elements and it is represented as: Set A = ∅

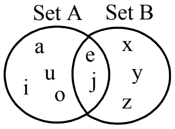
Examples:

1. (Easy)
Let Set A = {1, 2, 3} and Set B = {-1, 0, 1, 2}
Intersection of Set A and Set B = {1, 2}

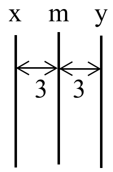
2. (Easy)
Set A = The set of odd numbers between -6 and 8 = {-5, -3, -1, 1, 3, 5, 7}
Set B = The set of positive integers less than 6 = {1, 2, 3, 4, 5}
Intersection of Set A and Set B = {1, 3, 5}

3. (Easy)
Set A = The first 5 letters of the alphabet.
Set B = {c, d, k, u, x}
Intersection of Set A and Set B= {c, d}

4. (Easy)
In the figure, the intersection of Set A and Set B = {e, j}



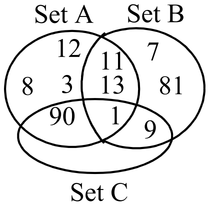
5. (Medium)
Set A is the collection of points on line m.
B is the collection of points that are at distance 3 from line m.
In the figure, Set A = {line m}
Set B = {line x, line y}
Since both line x and line y are parallel to line m, they don't cross.
Hence the intersection of Set A and Set B = ∅



Practice Exercises:

1. (Easy)
What is the intersection of

a. Set A and Set B
b. Set B and Set C
c. Set A and Set C
d. Set A, Set B and Set C



2. (Easy)
Let Set A = {1, 2, 3}, Set B = {-1, 0, 1, 2}, Set C = {2, 4, 6}. What is the intersection of Set A, Set B and Set C?

3. (Medium)
Set A = The set of odd numbers between -6 and -2.
Set B = The values of m satisfying $m = n^2 - 1$, where n is an integer between -6 and 2.
Set C = The values of m satisfying $m = n^2$, where n is an integer between -6 and 2.
What is the intersection of

a. Set A and Set B?
b. Set B and Set C?
c. Set A and Set C
d. Set A, Set B and Set C?

4. (Medium)
Consider two concentric circles. If Set A is the points on and inside of the smaller circle and Set B is the points in between the two circles including the circles, what is the intersection of Set A and Set B?

Answers:
1.a. {1, 11, 13}; **1.b** {1, 9}; **1.c** {1, 90}; **1.d.** {1};
2. {2}; **3. a.** ∅ ; **b.** ∅ ; **c.** ∅ ; **d.** ∅ ;
4. Small circle (inside is not included).

Defined Operators

Each SAT has one or two questions in which the numbers and/or operators are represented by unusual symbols. These operators are defined in the question. You need to understand this definition to be able to answer the question. Questions involving defined operators can be at any level.

Examples:

1. (Easy)
The number of "@"s and "O"s in a diagram represent the units digit and the 10s digit of a number, respectively.
For example: $\boxed{@@@OO}$ = 23, because there are three @s and two Os in the figure.

If a = $\boxed{OOO}$ and b = $\boxed{@@@}$, what is $a \cdot b$?

Solution:
$a = 30$ and $b = 3$, $a \cdot b = 90$
Note that in the first figure, there are no @ symbol. So the units digit of a is 0 (zero).

2. (Medium)
If a|b is (the square of the first number, a^2) over (second number minus two, b - 2), what is 4|7?

Solution:

$$4|7 = \frac{4^2}{7-2} = 3.2$$

Practice Exercises:

1. (Medium)
The number of "@"s and "O"s in a diagram represent the units digit and the 10s digit of a number respectively.
For example: $\boxed{@@@OO}$ = 23

If a = $\boxed{OOO}$ and a - b = $\boxed{@@@}$, then what is ab?

2. (Medium)
a|b is (the square of the first number, a^2) over (second number minus two, b - 2). If x|y = 9 and x and y are integers, what are the minimum positive values of x and y?

3. (Medium)
R is an operator that reverses the digits of an integer. For example R652 = 256.

S is an operator that subtracts 1 from each digit of an integer with non-zero digits. For example, S758 = 647

a. a, b and c are three positive integers. Let c, b and a be the unit, tens and hundreds digits of an integer. If R(Sabc) = 178, what is the original integer, abc?

b. x, y, x + y and $x \cdot y$ are positive integers with non-zero digits. Which of the below cases are true?

I. R(Sx) = S(Rx)
II. R(S(x + y)) = R(Sx) + R(Sy)
III. $R(Sx \cdot y) = (R(Sx)) \cdot (R(Sy))$

(A) I only
(B) II only
(C) III only
(D) II and III
(E) None

Answers: **1.** 810; **2.** x = 3, y = 3; **3.a.** 982; **3.b.** (A)

Logic

Logic questions measure your ability of drawing the correct conclusions from a set of given facts.

Read the question very carefully. If necessary, draw charts, tables, etc. Pay special attention to "and", "or", "must", "may", "all", "some", "not", "except", etc. You need to pay attention not only to what is said in the question, but also what is <u>not</u> said.

For example: "At least 50 students will go to the school concert" does <u>not</u> mean that some students will not go to the school concert. We only know that 50 students will definitely go to the concert, but we don't know the others. They may or may not go to the concert.

Examples:

1. (Easy)
In John's closet, at least two of his outfits are not red and some of his outfits are blue. Which of the following statements is correct?

(A) John does not have red outfits.
(B) Most of John's outfits are blue.
(C) John has at least one green outfit.
(D) John has at least one red outfit.
(E) None of the above.

<u>Solution:</u>

Case (A) and case (D) are not correct because the statement only says that two of his outfits are not red. He may or may not have red outfit.

Case (B) is wrong because the statement only indicates that he has some blue outfits, nothing is said about the quantity of blue outfits.

Case (C) is wrong because nothing is said about the green outfits.

So the answer is (E), "None of the above".

2. (Medium)
Nancy has her piano lessons on either Saturday or Sunday, but not both. She has soccer practice every day except Fridays and Sundays. In addition, she is in the swim team and practice swimming on Mondays, Wednesdays and Fridays. She also wants to be a member of the cinema club and attend their meetings. If she can only do a maximum of two extra curriculum activities a day, what days of the week she can make a 100% commitment to the cinema club?

<u>Solution:</u>

Sunday	P
Monday	S, Sw
Tuesday	S
Wednesday	S, Sw
Thursday	S
Friday	Sw
Saturday	P, S

P - Piano lessons
S - Succer practice
Sw - Swimming

The chart shows Nancy's daily activities.

Note that you have to count both Saturday and Sunday for the piano lessons even though on one of these two days she will be free.

It is clear that Nancy can make 100% commitment to Cinema Club meetings on Sundays, Tuesdays, Thursdays and Fridays.

3. (Hard)
Ellen is traveling on foot. The figure is Ellen's map, showing the available roads and the distances between the towns. All distances shown in the map are in miles.

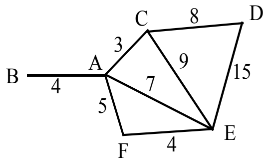


Figure not drawn to scale.

Ellen decides that she will stay in the same town for the night if she wakes up later then 9:00 AM. Otherwise, she will move to the closest town that she has not been at yet and stay there for the night. To go to the closest town, she is allowed to pass by the towns that she has already stayed in but she can not stay in the same town more than once. On Sunday morning, she wakes up at 8:30 AM at her hometown A. Starting with the first day of her trip, her wake-up times for the 7 days are as follows:

Sunday – 8:30 AM
Monday – 8:00 AM
Tuesday – 9:15 AM
Wednesday – 9:30 AM
Thursday – 8:45 AM
Friday – 7:30 AM
Saturday – 7:00 AM

On the 8th day (2nd Sunday), when she wakes up, where is she and how far did she travel?

<u>Solution:</u>

She will be in town D and have traveled 38 miles.

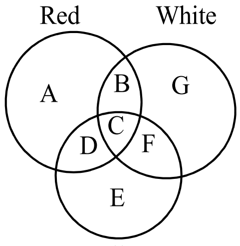
Here are the towns she stayed each night and the distances she traveled each day.

Day	**Town**	**Distance**
Sunday Morning	A	0
Sunday Night	C	3
Monday	B (passing through A)	7 = 3 + 4
Tuesday	B (She was late)	0
Wednesday	B (She was late)	0
Thursday	F (passing through A)	9 = 4 + 5
Friday	E	4
Saturday	D	15
	Total	38 miles

Note that she is allowed to pass through the towns where she has already stayed in to end up with the shortest travel distance.

Practice Exercises:

1. (Easy)
Figure shows sets of flags in red, white, yellow and any combination of these three colors. Which of the following regions represents the flags that has at least two colors on them?



(A) B, D and F only
(B) A, E and G only
(C) C only
(D) B, C, D and F only
(E) All but C

2. (Medium)
John is 5'8". Peter is taller than John. Bill is shorter than Peter. Richard is taller that Peter. Kevin's height is different than Peter's height. Which of the following is true?

- **(A)** John is the shortest man.
- **(B)** Richard is the tallest man.
- **(C)** 3 of the 5 men can be at the same height.
- **(D)** None of these men can be at the same height.
- **(E)** None of the above.

3. (Medium)
Sue is making a necklace from black and white beads. If 2 or more of the same color beads are not allowed to be next to each other, which of the following can not be the number of beads used to make the necklace?

- **(A)** 12
- **(B)** 21
- **(C)** 54
- **(D)** 56
- **(E)** 60

4. (Medium)
Sue is making a necklace from red, yellow and purple beads. If the order of the beads are red, yellow and purple throughout the necklace, which of the following can not be the number of beads used to make the necklace?

- **(A)** 12
- **(B)** 21
- **(C)** 54
- **(D)** 56
- **(E)** 60

5. (Hard)
In a parking lot:

8 cars are blue and 2-Door
94 cars are not blue
14 cars are black
30 cars are 2-Door

Which of the following best represents the numbers of cars which are not blue, not black and not 2-Door?

- **(A)** 72
- **(B)** 58 or less
- **(C)** more than 49, less than 65
- **(D)** more than 57, less than 73
- **(E)** 72 or less

Hint: First assume that all the black cars are 2-Door.
Then assume that none of the black cars are 2-Door.

Answers: 1. (D); **2.** (C); **3.** (B); **4.** (D); **5.** (D)

Statistics

There are three concepts that you need to know about statistics. They are:

Average or arithmetic mean
Median
Mode

Average or Arithmetic Mean

Average or arithmetic mean of a set of n numbers is their total sum divided by n: The average of $\{a_1, a_2, a_3, .., a_n\}$ is $(a_1 + a_2 + a_3 + ... + a_n)/n$

Examples:

1. (Easy)
What is the average of {3, 5, 6, 1}?

Solution:
The average is (3 + 5 + 6 + 1) / 4 = 3.75

2. (Easy)
If the average of 8 numbers is 7, what is the addition of these numbers?

Solution:
The addition is $7 \cdot 8 = 56$

3. (Hard)
What is the average of the consecutive integers between 56 and 200, inclusive?

Solution:

The set of numbers are:
{56, 57, 58, ...198, 199, 200}

If you add the first and the last numbers, second and the second from the last numbers, third and third from the last numbers, etc. you will have the same total of 256. Here is how:
$56 + 200 = 57 + 199 = 58 + 198 = ... = 127 + 129 = 256$

Because there are $200 - 56 + 1 = 145$ numbers in the above set, you can only pair 144 of them and get $144/2 = 72$ pairs.

The number in the middle of the set is $(200 + 56)/2 = 128$. 128 can not be paired with any other member to give the total of 256.

Hence the addition of all the numbers in the above set is $72 \cdot 256 + 128 = 18560$

The average of all the numbers is $18560/145 = 128$

The average of an odd number of consecutive integers is the number in the middle of the set.

The average of an even number of consecutive integers is the average of the two numbers in the middle of the set.

Practice Exercises:

1. (Easy)
What is the average of {-3, 5, 6, 0}?

2. (Easy)
If the average of 8 numbers is -3, what is the addition of these numbers?

3. (Medium)
The average of three numbers $\{1, 2x, x^2\}$ is 300. What is the value of x?

4. (Medium)
The average of a set of 3 numbers is 22 and the average of a second set of 5 numbers is 6. What is the average of all the numbers combined?

5. (Hard)
What is the average of the consecutive integers between 55 and 200?

Answers: **1.** 2; **2.** -24; **3.** 29 or -31; **4.** 12; **5.** 127.5

Median

In a given set of numbers, median is the mid value that splits the set into two. The number of values greater than the median is equal to the number of values less than the median.

The easiest way to find the median of a set is to sort the elements in an increasing order.

Examples:

1. (Easy)
What is the median of the integers {3, 5, 6, 1, 8}?

Solution:

First, let's rewrite the numbers in the set from smallest to biggest: {1, 3, 5, 6, 8}

You now can easily see that the median is 5, the number in the middle. There are two numbers greater than 5 (6 and 8), and there are two numbers smaller than 5 (1 and 3).

2. (Medium)
The median of the five distinct positive integers, {1, 3, x, 6, 8}, is 3.

What is the value of x?

Solution:

There are already 2 numbers, 6 and 8, which are greater than the median, 3. Hence there has to be two numbers which are less than 3. One of these numbers is 1 and the other is x. Since x has to be positive and different than the other numbers in the set, x = 2.

3. (Hard)
If $-1 < x < 0$, what is the median of $\{x, 2x, x^2, 2x + 1, 3x\}$

(A) x

(B) 2x

(C) x^2

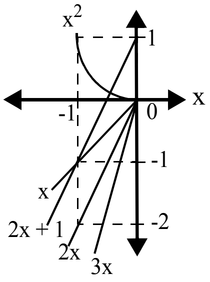
(D) 2x + 1

(E) The median can not be determined from the information given.

Solution:

Figure displays x, 2x, x^2, 2x + 1, 3x for $-1 < x < 0$.

As you can see, between -1 and 0, the values of 3x and 2x are the smallest two and the values of x^2 and 2x + 1 are the biggest two of these 5 functions.



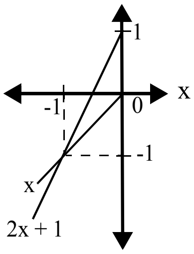
Hence the median is x. So the answer is (A).

Alternate Solution:

You don't have to graph all the possible solutions. Since $x < 0$, you know that 3x and 2x are the smallest two terms of the set. Thus, neither 3x nor 2x may be the median. The median is the next lowest element of the set.

Since x^2 is always positive, $x < x^2$, for all negative values of x. So x^2 may not be the median.

In the figure, x and 2x + 1 are displayed. In the interval of $-1 < x < 0$, x is lower than 2x + 1. Hence in this interval, x is the median of the set. So the answer is (A).



Alternate Solution: Solution by example:

Let x = -1/2 ➔

$2x = -1$, $3x = -3/2$, $x^2 = 1/4$, $2x + 1 = 0$

Let's put these numbers in ascending order:

{-3/2, -1, -1/2, 0, 1/4} or $\{3x, 2x, x, 2x + 1, x^2\}$

x (-1/2) is the median. So the answer is (A).

Practice Exercises:

1. (Easy)
What is the median of the integers {3, -5, 6, 1, -8}?

2. (Medium)
The median of five distinct integers, {1, 4, x, 6, 8}, is 2x. What is the value of x?

3. (Hard)
If $1 < x < 2$, what is the median of $\{x, 2x, x^2, 2x + 1, 3x\}$?

- **(A)** x
- **(B)** 2x
- **(C)** x^2
- **(D)** 2x + 1
- **(E)** The median can not be determined from the information given.

Answers: 1. 1; **2.** 2; **3.** (B)

Median Facts

- If there are an even number of distinct numbers, the median can not be one of those numbers. The median is the average of the two numbers in the middle. For example the median of {1,3,5,6} is (3 + 5) / 2 = 4
- Median of a set remains the same if you increase the value of the biggest element or if you decrease the value of the smallest element. For example, the median values of the following 4 sets are all 5.

  {1, 3, 5, 6, 8}, {-1, 3, 5, 6, 8}, {1, 3, 5, 6, 18}, {-10, 3, 5, 6, 28}
- In general, the median and the average are not the same value. However, if the elements of the set are uniformly increasing or decreasing, average is equal to the median, which is the number in the middle. For example, the elements of set of {5,6,7,8,9} are uniformly increasing. Each element is one larger than the previous element. The median and the average of this set are both 7.

Examples:

1. (Medium)
{a, b, c, d} is a set of numbers with $a < b < c < d < 8$. If you add 10 to the group, both the median and the average of the new group becomes 6. What is the average of the original numbers?

Solution:

The new group is {a, b, c, d, 10}. Its median is c = 6. Its average is

$$\frac{a+b+c+d+10}{5} = \frac{a+b+6+d+10}{5} =$$

$$\frac{a+b+d+16}{5} = 6 \Rightarrow a + b + d = 14$$

The average of the original group is

$$\frac{a+b+c+d}{4} = \frac{14+6}{4} = 5$$

2. (Hard)
If the median of a group of consecutive integers is 29.5, which of the following may be addition of these numbers?

- **(A)** 377
- **(B)** 296
- **(C)** 206.5
- **(D)** 147.5
- **(E)** 118

Solution:

Since the median is not an integer, there must be an even number of integers in the group.
The number of integers in the group = 2n, where n is a positive integer.

On the other hand, the group of integers are consecutive, so their average is equal to their median, which is 29.5.

The addition of the numbers is their average, 29.5, multiplied by their number, 2n, which is $29.5 \times 2n = 59n$. Hence the addition must be a multiple of 59. The only answer choice that is the multiple of 59 is 118. The answer is (E).

Alternate Solution:

Since the median is not an integer, there must be an even number of integers in the group. The average of the two integers in the middle is 29.5. These two integers are 29 and 30. You can start adding the consecutive integers around 29 and 30 until you find one of the answers. Here is how:
29 + 30 = 59, which is not one of the answers.
28 + 29 + 30 + 31 = 118, which is answer (E).

Faster Alternate Solution:

The addition of the elements has to be divisible by 29.5, the average.

377 and 296 are not divisible by 29.5. 206.5 and 147.5 are divisible by 29.5, but they are decimal numbers. Addition of integers can not be decimal. 118 is both divisible by 29.5 and an integer. The answer is (E)

3. (Hard)
What is the average of the consecutive integers between 56 and 200?

Solution:

You can find the same example in the previous section, but here you can use the new knowledge and solve it with less effort.

The set of numbers are:
{56, 57, 58, ...198, 199, 200}

The median of the set = $\frac{56 + 200}{2}$ = 128

Since the integers are consecutive, the average is equal to the median, which is 128.

4. (Medium)
If the median of a set of consecutive odd integers is 29, which of the following may be the addition of these numbers?

(A) 348
(B) 296
(C) 206
(D) 145
(E) 116

Solution:

The addition of the elements has to be divisible by 29, the average. Among the answer choices, 348 and 145 are divisible by 29.

Since the median of consecutive odd integers is 29, an odd number, there has to be an odd number of integers in the set. The addition of an odd number of odd integers is odd. Hence 348, an even number, can not be the answer. The answer is (D), 145.

Practice Exercises:

1. (Easy)
What is the average and the median of {-3, 1, -8, 7}?

2. (Medium)
Which of the following sets has the average and median of {8, 12, 15, 28, 30}?

(A) {6, 12, 15, 28, 34}
(B) {4, 12, 15, 28, 30}
(C) {8, 12, 15, 28, 50}
(D) {6, 14, 15, 28, 30}
(E) {6, 12, 15, 28, 32}

Hint: Don't calculate the medians and/or the averages of all the sets. Notice that the middle three numbers are the same. So the medians of all these sets are 15. To keep the average the same, the addition of first and last numbers in the answer must be 30 + 8 = 38

3. (Hard)
If the median of a set of consecutive even integers is 29, which of the following may be the addition of these numbers?

(A) 377
(B) 290
(C) 206
(D) 148
(E) 87

Hint: If the median of the consecutive even integers is 29, an odd number, there has to be an even number of integers in the set.

Answers:
1. Median = -1, Average = -0.75; **2.** (E); **4.** (B)

Mode

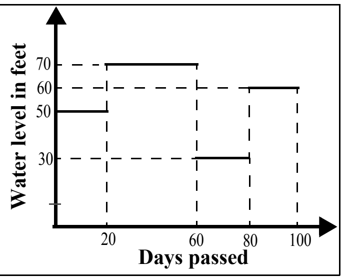
Mode of a set of numbers is the most frequently repeated number in the set. For example the mode of set {3, 5, 6, 1, 1} is 1, because there are two 1's in the set. All other numbers are repeated only once.

If a set has all identical elements, the mode, the median and the average are the same. For example, the set {3,3,3,3} has 3 as its average, median and mode.

Examples:

The graph is showing the water level in a dam.

Use the graph for the questions 1 - 3



1. (Medium)
What is the mode for the daily water level?

Solution:
For the longest amount of time, 40 days between the days 20 to 60, the water level stays the same at 70 feet. So the mode is 70.

2. (Medium)
What is the median daily water level?

Solution:
There are 40 days (between 20 and 60) on which the water level is highest (70 feet) and higher than 60 feet.

There are a total of 40 days (days: 0 - 20 and 60 - 80) on which the water level is lower than 60 feet. Hence the median water level is 60 feet.

3. (Hard)
What is the average daily water level?

Solution:
From the beginning to the end, the water stayed at:
50 feet for 20 days
70 feet for 60 - 20 = 40 days.
30 feet for 80 - 60 = 20 days.
60 feet for 100 - 80 = 20 days.

Hence the average water level =

$$\frac{50 \cdot 20 + 70 \cdot 40 + 30 \cdot 20 + 60 \cdot 20}{20 + 40 + 20 + 20} = 56 \text{ feet}$$

Practice Exercises:

1. (Easy)
What are the average, median and the mode of {3, 3, 8, 1, 5, 1, 1}?

2. (Median)
What is the average size of the inner angles of a 9 sided polygon?

3. (Median)
What is the mode of the inner angles of a 9 sided regular polygon?

4. (Medium)
Mode of the 6 numbers shown in the figure is 3. What is the mean of these six numbers?

	2	4	4		

Answers:
1. Average = 22/7, Median = 3, Mode = 1;
2. 140°; **3.** 140°; **4.** 19/6

Sums

Sum of Elements

Consider a set of numbers. Let a_n be the nth element of the set. The sum of the elements of the set from n = m (minimum limit) to n = h (maximum limit) is:

$$S = a_m + a_{(m+1)} + a_{(m+2)} + ... + a_{(h-1)} + a_h$$

Sum of Constants

If $a_n = c$, a constant, for all values of n, then the above sum becomes:
$S = c + c + c + ... + c + c = (h - m + 1)c$

Note that in the above equation, the constant c is added as the index increases from m to h. Since the number of integers between m and h is (h - m + 1), the sum becomes (h - m + 1)c.

The number of the consecutive integers between two integers is one of the classic questions in SAT. Make sure to remember that number of the integers between "integer low" and "integer high" is:

"integer high" - "integer low" + 1

Examples:

1. (Medium)
How many integers are there between -5 and 97?

Solution:
97 - (-5) + 1 = 103

2. (Medium)
Which one is bigger, the number of integers between -5 and 97, or , 23 and 125?

Solution:

The number of integers between -5 and 97 =
$97 - (-5) + 1 = 103$

The number of integers between 23 and 125 =
$125 - 23 + 1 = 103$

So, they are equal.

Sum of Consecutive Integers

The sum of consecutive integers, represented by n, from m (minimum limit) to h (maximum limit) is:

$S = m + (m + 1) + (m + 2) + ... + (h - 1) + h$

Note that both “m” and “h” are included in the sum.

The best way to calculate the above sum is to write it twice, one in the way it is written above and the other by reversing the terms, starting from the last term and ending with the first term:

$S = m + (m + 1) + (m + 2) + ... + (h - 2) + (h - 1) + h$

$S = h + (h - 1) + (h - 2) + ... + (m + 2) + (m + 1) + m$

Note that the second equation is the reverse of the first equation. Both expressions express the same sum, S. If you add the two equations term by term from the beginning, you get:

$2S = (m + h) + (m + 1 + h - 1) + (m + 2 + h - 2) + ... +$
$(h - 2 + m + 2) + (h - 1 + m + 1) + (h + m) =$

$(m + h) + (m + h) + (m + h) + ... + (m + h) + (m + h) =$
(number of integers between “m” and “h”)(m + h)

The number of integers between “m” and “h” =
$h - m + 1$ ➔
$2S = (h - m + 1)(m + h)$ ➔
$S = (h - m + 1)(m + h)/2$

Note that the information given in this section will also help you in preparing for the other parts of the SAT. For example, the sum of integers between the two given integers is a classic question in SAT. Hence study this section and understand the examples below very well.

Examples:

1. (Medium)
What is the sum of integers from 1 to 100?

Solution:

$S = (h - m + 1)(m + h)/2$, where $h = 100$, $m = 1$
$S = (100 - 1 + 1)(1 + 100)/2 = 10100/2 = 5050$

Alternate Solution:

You don’t have to remember the formula for the sum. You can use the same method and calculate the sum quickly.
$S = 1 + 2 + 3 + ... + 98 + 99 + 100$
$S = 100 + 99 + 98 + ... + 3 + 2 + 1$

Add the two expressions together:
$2S = 101 + 101 + ... + 101 + 101 + 101 =$
$101(100 - 1 + 1) = 10100$ ➔ $S = 10100/2 = 5050$

2. (Medium)
What is the sum of integers from -14 to 77?

Solution:

$S = (h - m + 1)(m + h)/2$, where $h = 77$, $m = -14$
$S = (77 - (-14) + 1)(-14 + 77)/2 =$
$(92 \cdot 63)/2 = 2898$

Alternate Solution:

$S = -14 - 13 - 12 -... + 75 + 76 + 77$
$S = 77 + 76 + 75 +... - 12 - 13 - 14$

Add the two expressions together:
$2S = 63 + 63 + ... + 63 + 63 + 63 =$
$63(77 - (-14) + 1) = 63 \times 92 = 5796$ ➔
$S = 8796/2 = 2898$

Practice Exercises:

1. (Medium)
How many integers are there between -56 and -9?

2. (Medium)
What is the sum of the integers between -124 and -142?

Answers: 1. 48; **2.** -2527

Sequences

A sequence is an ordered list of numbers. These numbers are called the elements of the sequence. For example, {1, 2, 3, 4, ... } and {1, 2, 4, 8, ... } are two sequences. The first sequence starts with 1 and each element is one more than the previous element. The second sequence starts with 1 and each element is twice as big as the previous element.

The questions about sequences require you to figure out the logic of the order of the numbers in the sequence.

Examples:

1. (Easy)
What is the fifth element in the sequence
{-1, 2, -3, 4, ... }?

Solution:

In this sequence, the value of the nth element is n if n is even, -n if n is odd. Hence the 5th term is -5.

2. (Easy)
What is the fifth element in the sequence
{1, 4, 9, 16, ... }

Solution:

In this sequence, the value of the nth element is n^2. Hence the 5th element is 25.

3. (Medium)
What is the 6th element in the sequence
{1, 2, 3, 5, 8, ... }?

Solution:

In this sequence, each element is the addition of the previous two elements. Hence the 6th element is 13.

4. (Medium)
What is the 6th element in the sequence
{0, 1, 2, 5, 26, ... }?

Solution:

In this sequence, each element is one more than the square of the previous element. Hence the 6th element is $26^2 + 1 = 677$

5. (Medium)
What is the 6th element in the sequence
{-10, -4, -1, 0.5, 1.25, ... }?

Solution:

In this sequence, each element is one more than the half of the previous element. Hence the 6th element is 1.625.

6. (Hard)
What is the 7th element of the sequence
{0, 1, 1, 2, 5, 29, ... }?

Solution:

In this sequence, each element is the addition of the squares of the previous two terms. Hence the 7th element is $5^2 + 29^2 = 866$

7. (Hard)
What is the 7th element of the sequence
{1, 2, 1, 1, 0, 1, ... }?

Solution:

In this sequence, each element is the square of the subtraction of the previous two elements. Hence the 7th element is $(1 - 0)^2 = 1$

Practice Exercises:

1. (Easy)
What is the 5th element in the sequence
{2, 4, 6, ... }?

2. (Easy)
What is the 5th element in the sequence
{1, 2, 4, 8, ... }?

3. (Medium)
What is the 6th element in the sequence
{-1, 4, -9, 16, ... }?

4. (Medium)
What is the 6th element in the sequence
{1, 4, 19, 94, ... }?

5. (Medium)
What is the 6th element in the sequence
{-10, -6, -4, -3, ... }?

6. (Hard)
What is the 7th element of the sequence
{-1, 4, -2, 3, -3, 2, ... }?

Answers: 1. 10; **2.** 16; **3.** 36; **4.** 469; **5.** -2.25; **6.** -4

There are two types of special sequences in SAT - Arithmetic and Geometric Sequences. They are explained in the next two sections. Studying these sections will also help you answer some of the counting and sum questions correctly.

Arithmetic Sequences

In an arithmetic sequence the difference between the two consecutive elements is constant. The general formula for the nth element is a + d(n - 1), where a is the first element of the sequence. d is the difference between the two consecutive elements and $n \geq 1$. Note that both a and d can be positive or negative.

Examples:

1. (Easy)
{1, 3, 5, 7, ... } is a sequence of positive odd numbers. It starts with 1 and each element is 2 more than the previous one.

2. (Medium)
Express the nth element in the sequence {1, 3, 5, 7... }

Solution:

The nth element is 1 + 2(n - 1) = 2n - 1

3. (Medium)
The nth element of a sequence is -3n + 5.

a. What is the first element of the sequence?

b. What is the difference between the consecutive elements?

Solution:

a. For n = 1, -3n + 5 = -3 + 5 = 2. So the first element is 2.

b. For n = 2, -3n + 5 = -6 + 5 = -1. The difference between the first and the second terms is -1 - 2 = -3. So the difference between the consecutive terms is -3.

Here we used 1st and 2nd terms. In an arithmetic sequence, the difference, d, remains the same for all terms. You can prove that d is the same for all values of n as follows:

The difference between element n + 1 and element n is (-3(n + 1) + 5) - (-3n + 5) = -3n -3 + 5 + 3n - 5 = -3

4. (Medium)
If the nth element of a sequence is -3(n - 1) -1, write the first 4 elements of the sequence.

Solution:
-3(n - 1) -1 ➔
The first element is -1 and the elements are decreasing by 3 ➔
First 4 elements of the sequence is -1, -4, -7, -10

5. (Medium)
In a sequence each element is 3 more than the previous element. If the sequence starts with number 7, express the nth element in the sequence.

Solution:
7 + 3(n - 1) = 3n + 4

6. (Medium)
If the 6th element of an arithmetic sequence is -12, and the 10th element is -32, what is the value of the 1st element?

Solution:
Between the 6th and 10th elements, the value drops by -12 - (-32) - 20. Since in an arithmetic sequence the difference between the consecutive elements remains constant, the elements are decreasing by
20/(10 - 6) = 5 for each element of the sequence. ➔ d = -5

6th element is 12 ➔
a - 5(6 - 1) = -12, where a is the first element. ➔ a = 13

Practice Exercises:

1. (Medium)
Express the nth element in the sequence
2, 4, 6, 8, ...

2. (Medium)
In a sequence each element is 3 less than the previous element. If the sequence starts with number 5, express the nth element in the sequence.

3. (Medium)
If the 5th element of an arithmetic sequence is -12, and the 10th element is -32, what is the 19th element?

4. (Medium)
If the elements of an arithmetic series increases by 7 and if the 56th element is 0, write a general expression for the nth element.

Answers:
1. 2n; **2.** -3n + 8; **3.** -68; **4.** 7n - 392

The Sum and Average of Elements of Arithmetic Sequences

Some questions in SAT involves the sum and/or the average of some of the elements of the arithmetic sequences.

If you know how to calculate the sum of constants and the sum of consecutive integers, you can calculate the sum of elements of an arithmetic sequence.

General formula for the nth element of an arithmetic sequence is a + d(n - 1) = (a - d) + dn, where a is the first element and d is the difference between the consecutive elements. Let a - d = c v nth term of the arithmetic sequence is c + dn.

The sum of all the elements between n = m (minimum limit) and n = h (maximum limit) is:
$S = S_c + dS_n$
where S_c is the sum of (h - m + 1) constants and S_n is the sum of consecutive integers from m to h. ➔

$$S = (a-d)(h-m+1) + d(h+m)\frac{(h-m+1)}{2}$$

The above sum formula is simplified if you want to calculate the sum of the first h elements. For m = 1, it is:

$$h(a-d) + \left(\frac{dh}{2}(h+1)\right) = h\left(a + \frac{d}{2}(h-1)\right)$$

Both of these formulas are long and difficult to memorize. However if you understand how they are derived, you can create your own formula for each specific case. You will find examples of how this is done in the following examples.

Examples:

1. (Hard)
What is the sum of the first 10 elements of the sequence {1, 3, 5, 7, ... }?

Solution:
nth element of the sequence is 1 + 2(n - 1) = 2n - 1

To calculate the sum, you can use the general formula with a = 1, d = 2, m = 1 and h = 10

S = (1 - 2)(10 - 1 + 1) + (2(10 + 1)(10 - 1 + 1)/2) = -10 + 110 = 100

Alternate Solution:

Use the simplified formula for m = 1:

$$S = h\left(a + \frac{d}{2}(h - 1)\right) = 10\left(1 + \frac{2}{2}(10 - 1)\right) = 100$$

Another Alternate Solution:

Drive your own formula for the sum:
nth element of the sequence is 1 + 2(n - 1) = 2n - 1

The sum of the first 10 elements

S = 2(addition of integers from 1 to 10) - (addition of 10 "ones") =

$$2 \cdot (10 + 1) \cdot \frac{10 - 1 + 1}{2} - 10 = 100$$

Smart Alternate Solution:

S = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19
S = 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1

Add the two expression together:
2S =
20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 = 200 ➔ S = 200/2 = 100

Even Smarter Alternate Solution:

The elements of the sequence in question is regularly increasing. Hence the median of the elements between 1st and 10th elements is also the average of these elements.

There are 10 - 1 + 1 = 10 elements in this interval. So the median is the average of the two middle elements. Since (10 + 1)/2 = 5.5, the two middle elements are the 5th and 6th elements. The values of these elements are 9 and 11. The average = The median = (9 + 11)/2 = 10
The sum, S = Average multiplied by the number of elements = 10 · 10 = 100

Another Alternate Solution:

Since the numbers are small, you can use your calculator to add them up:
1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100

2. (Medium)
What is the average of the first 10 elements of the sequence given in the previous question?

Solution:

The sum, S, of the first 10 elements is 100. Hence the average is 100/10 = 10

Smart Alternate Solution:

You don't have to calculate the sum of the terms to calculate the average of the elements in an arithmetic sequence. Since the elements of an arithmetic sequence are regularly increasing or decreasing, the median is always equal to the average.

The median of the sequence
1, 3, 5, 7, 9, 11, 13, 15, 17, 19 is (9 + 11)/2 = 10
So the average is also 10.

3. (Hard)
The nth element of an arithmetic sequence is -3(n - 1) -1. What is the average of the elements between 11th and 200th of it?

Solution:

nth element of the sequence is -3(n - 1) -1 ➔ first element, a = -1 and the difference, d, of the consecutive terms is -3.

To calculate the sum, you can use the general formula with a = -1, d = -3, m = 11and h = 200

$$S = (a - d)(h - m + 1) + \left(d(h + m)\frac{(h - m + 1)}{2}\right)$$

S = (-1 + 3)(200 - 11 + 1) +
(- 3(200 + 11)(200 - 11 + 1)/2) =
380 - 60135 = -59755

The average = S/(200 - 11 + 1) =
-59755/190 = -314.5

Smart Alternate Solution:

You don't have to calculate the sum of the terms to calculate the average of the elements in an arithmetic sequence. Since the elements of an arithmetic sequence are regularly increasing or decreasing, the median is always equal to the average.

There are 200 - 11 + 1 = 190 terms between the elements 11th and 200th. Since the number of elements is an even number, the medium is the average of the two elements in the middle.

Because (11 + 200)/2 = 105.5, these two elements are 105th and 106th elements.

The values of these elements are
-3(105 - 1) - 1= -313 and -316 respectively.
Hence the median and the average in question is (-313 - 316) = 314.5

Practice Exercises:

1. (Medium)
The elements of a sequence are all -12. What is the sum of the elements between the 8th and 30th elements?

2. (Medium)
The elements of a sequence are all 6. What is the value of S_1 - S_2, where S_1 is the sum of the elements between 1st and 20th elements and S_2 is the sum of the elements between 111st and 130th elements?

3. (Hard)
What is the sum of the elements between 50th and 150th of the sequence if the nth element of the sequence is -2n + 10. Use the method applied in the first "*Smart Alternate Solution*" part of the Example 1 above.

4. (Hard)
What is the average of the elements between 50th and 200th of the sequence if the nth element of the sequence is 4n + 3? Use the method applied in the first "*Smart Alternate Solution*" part of the Example 3above.

Answers: 1. -276; **2.** 0 ; **3.** -19190; **4.** 403

Geometric Sequences

In a geometric sequence the ratio of the two consecutive elements is a constant. The general formula for the nth element is $ar^{(n - 1)}$, where "a" is the first element of the sequence and "r" is the ratio of the two consecutive elements. Note that both a and r can be positive or negative, and $n \geq 1$.

Examples:

1. (Easy)
1, 2, 4, 8, ... is a sequence of positive numbers that starts with 1 and each element is twice as big as the previous one.

2. (Medium)
Express the nth element in the sequence 1, 2, 4, 8, ...

Solution:
The first element is 1 and the ratio of the two consecutive elements is 2. Hence the nth element is $2^{(n - 1)}$, where $n \geq 1$

3. (Medium)
If the nth element of a sequence is $-3^{(n - 1)}$, write the first 4 elements of the sequence.

Solution:
nth element of a sequence is $-3^{(n - 1)}$ ➔
The first element is -1 and the ratio of the consecutive elements is 3 ➔
The first 4 elements of the sequence is
-1, -3, -9, -27

4. (Medium)
If the nth element of a sequence is $(-3)^{(n - 1)}$, write the first 4 elements of the sequence.

Solution:
nth element of a sequence is $(-3)^{(n - 1)}$ ➔
The first element is 1 and the ratio of the consecutive elements is -3 ➔
The first 4 elements of the sequence is 1, -3, 9, -27

5. (Medium)
In a sequence the ratio of the two consecutive elements is 1/2. If the sequence starts with number 7, express the nth element in the sequence.

Solution:
$7(1/2)^{(n - 1)} = 7/2^{(n - 1)}$

6. (Medium)
What are the first 5 elements of the sequence described in the previous question?

Solution:
7, 7/2, 7/4, 7/8, 7/16

7. (Hard)
If the 3rd element of a geometric sequence is 8 and the 7th element is 128, write a general expression for the nth element.

Solution:
Let a and r be the first element and the ratio of the two consecutive elements of the sequence in question.
The 3rd element of a geometric sequence is 8 ➔
$a(r)^{(3 - 1)} = ar^2 = 8$
The 7th element is 128 ➔ $ar^{(7 - 1)} = ar^6 = 128$ ➔
$ar^6/ar^2 = r^4 = 128/8 = 16$ ➔ r = 2 or r = -2 ➔
$ar^2 = 4a = 8$ ➔ a = 2 ➔
The nth element is
$2 \cdot 2^{n-1} = 2^n$ or $2 \cdot (-2)^{n-1} = -(-2)^n$

8. (Medium)
What are the first 5 elements of the sequence described in the previous question?

Solution:
{2, 4, 8, 16, 32, ...} or {2, -4, 8, -16, 32, ...}

Practice Exercises:

1. (Medium)
Express the nth element of the sequence:
-1, -3, -9, -27, ...

2. (Medium)
If the nth element of a sequence is $-1/\left(3^{(n-1)}\right)$, write the first 4 elements of the sequence.

3. (Medium)
If the nth element of a sequence is $(-1/4)^{(n-1)}$, write the first 4 elements of the sequence.

4. (Medium)
In a sequence the ratio of the consecutive elements is -1/5. If the sequence starts with number -8, express the nth element in the sequence.

5. (Medium)
What are the first 5 elements of the sequence described in the previous question?

6. (Hard)
If the 2nd element of a geometric sequence is 54 and the 5th element is 1458, what is the first element of the sequence?

Answers: 1. -3^{n-1}; **2.** {-1, -1/3, -1/9, -1/27, ...};
3. {1, -1/4, 1/16, -1/64, ...}; **4.** $40(-1/5)^n$;
5. {-8, 8/5, -8/25, 8/125, -8/625, ...}; **6.** 18

The Sum and Average of Elements of Geometric Sequences

Some questions in SAT involves the sum of some or all the elements of a geometric sequence.

General formula for the nth element of a geometric sequence is $ar^{(n-1)}$, where “a” is the first element of the sequence and “r” is the ratio of the two consecutive elements, and $n \geq 1$.

The sum of all the elements between n = m (minimum limit) and n = h (maximum (higher) limit) is:

$$S = a(r^{m-1} + r^{m} + r^{m+1} + ... + r^{h-1})$$

To find the sum, S, let’s calculate S - rS:

$$S - rS = S(1 - r) =$$

$$a(r^{m-1} + r^{m} + r^{m+1} + ... + r^{h-1}) -$$

$$a(r^{m} + r^{m+1} + ... + r^{h-1} + r^{h}) = a(r^{m-1} - r^{h}) \Rightarrow$$

$$S = \frac{a(r^{m-1} - r^{h})}{1 - r}$$

Note that in the above derivation, all the terms except the first and the last in the expression in S - rS cancel each other.

The above sum formula is simplified if you want to calculate the sum of the first h elements. For m = 1, it is:

$$S = \frac{a(1 - r^{h})}{1 - r}$$

The above sum formula is also simplified if $|r| < 1$ and you are asked to calculate the sum of all the elements greater than or equal to m. In this case $h = \infty$ and
$r^h = 0$. Hence the sum becomes:

$$S = \frac{ar^{m-1}}{1 - r}$$

This formula can be further simplified when you are asked the sum of all the elements of a geometric series if the ratio of the consecutive terms is less than one. In this case

$$m = 1 \text{ and } S = \frac{ar^{m-1}}{1 - r} = \frac{a}{1 - r}$$

Examples:

1. (Hard)
What is the sum of first 10 elements of the sequence {-1, -3, -9, -27, ... }?

Solution:

In the sequence, the first term, a = -1 and the ratio of the two consecutive elements, r = 3.

Use the simplified formula for:
m = 1, h = 20, r = 3 and a = -1:

$$S = \frac{a(1 - r^{h})}{1 - r} = -\frac{(1 - 3^{10})}{1 - 3} = -29524$$

Alternate Solution:

You don’t need to memorize the formula for the sum. Apply the method used to derive the sum formula. Here is how:
S - rS = S - 3S = S(1 - 3) = -2S =
-1 - 3 - 9 - 27 ... - 19683 -
(- 3 - 9 - 27 ... - 19683 - 59049) =
-1 + 59049 = 59048 ➔ S = -59048/2 = -29524

Note that $19683 = 3^9$ and $59049 = 3^{10}$

Another Alternate Solution:

Just use your calculator to calculate the sum:
S =
-1 - 3 - 9 - 27 - 81 - 243 - 729 - 2187 - 656 - 19683 = -29524

2. (Medium)
What is the average of the first 10 elements of the sequence given in the previous question?

Solution:

The sum, S, of the first 10 elements is -29524. Hence the average is -29524/10 = -2952.4

3. (Hard)
The nth element of the sequence is $(-1/4)^{(n - 1)}$. What is the sum of all the elements, starting with the 3rd element?

Solution:

nth element of the sequence is $(-1/4)^{(n - 1)}$ ➔ first element, a = 1 and the ratio, r, of the consecutive elements is (-1/4).

Since $|r| = |-1/4| < 1$, you can use the simplified formula with a = 1, r = -1/4, m = 3:

$$S = \frac{ar^{m-1}}{1-r} = \frac{(-1/4)^2}{1+\frac{1}{4}} = \frac{4}{5}\cdot\frac{1}{16} = \frac{1}{20} = 0.05$$

Alternate Solution:

To calculate the sum in question, you could first calculate the sum of all the terms and then subtract the first and second terms from it.

Sum of all the elements in the sequence is

$$\frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{1}{1+\frac{1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$$

The sum of the elements starting with the 3rd element is

$$\frac{4}{5} - 1 - \left(-\frac{1}{4}\right) = \frac{1}{20} = 0.05$$

Note that the first and the second elements of the sequence are 1 and -1/4, respectively.

Another Alternate Solution:

You don't need to memorize any formula for the sum. Apply the same method used to derive the sum formula. Here is how:

$S - rS = S + S/4 = S(1 + 1/4) = 5S/4 =$
$[(-1/4)^2 + (-1/4)^3 + (-1/4)^4 + ... + 0] -$
$[(-1/4)^3 + (-1/4)^4 + (-1/4)^5 + ...+ 0] =$

$(-1/4)^2 = 1/16$ ➔ $S = 4/(5 \cdot 16) = 1/20 = 0.05$

Note that the last elements of both S and rS are zero.

Practice Exercise:

1. (Hard)
What is the average of all the elements between 5 and 10 of a geometric sequence if the nth element of the sequence is $-3^{n - 1}$?

2. (Hard)
What is the sum of all the elements of a sequence if the nth element of the sequence is $(1/2)^{n - 1}$?

Answer: 1. -4914; **2.** 2

Counting

Basic Counting

There are two different types of basic counting questions in SAT:

Visual Counting
Numeric Counting.

Both types are usually labeled as "Hard."

For visual types you need to count the number of possibilities from a figure correctly. It sounds easy but it is also easy enough to miss some of the possibilities and count less than the correct number.

If the question is a multiple-choice question, pay special attention to the answers that are multiples of your answer. Make sure they don't represent the correct count.

Examples:

1. (Medium)
How many pages are there between pages 30 and 550, including both pages?

Solution:

It is tempting to say that the answer is 550 - 30 = 520. But the answer is 521 because both page 30 and page 550 are included in the count.

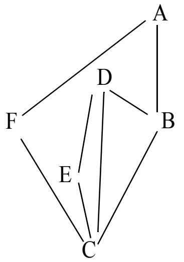
To understand the point better, think about the number of pages between 1 and 3, including both 1 and 3. There are 3 pages (1, 2, 3) not 3 - 1 = 2.

The number of the consecutive integers between two integers is one of the classic questions in SAT. Make sure to remember that number of the integers between "integer low" and "integer high" is:

"integer high" - "integer low" + 1

2. (Hard)
Mary wants to make a round trip between two towns, A and C. Road map is shown in the figure.

In how many different ways can she travel if she is allowed to pass each place only once in each direction?



Solution:

It is tempting to say that the answer is 4: ABC, ABDC, ABDEC, AFC. But the answer is 16, because the question is asking for the different possible round trips. There are 4 different ways to go from A to C, but there are also 4 different ways to come back. For each path that she chooses to go from A to C, there are 4 different ways to return. So the answer is $4 \times 4 = 16$.

Advanced Counting Techniques

Permutations, combinations, mutually exclusive and independent events are advanced counting techniques. These questions are not in each and every SAT. However, it is necessary to learn them if you want to get a high score.

To be able to cover these topics, you need to learn the "Factorial" operator.

Factorial Operator

Factorial operator acts on non-negative integers only. It is represented by exclamation mark (!) symbol. n! is pronounced as "**n factorial**". It is defined as the multiplication of all the integers from 1 to n. For example, $3! = 1 \cdot 2 \cdot 3 = 6$

By definition 0! = 1. Yes, it is one, not zero.

Note that when you divide factorials, you don't need to calculate both the numerator and the denominator separately. It is easier to cancel first as many terms as possible.

Example: (Easy)

$$\frac{5!}{7!} = \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = \frac{1}{6 \times 7} = \frac{1}{42}$$

You don't have to calculate either 5! or 7!.

Practice Exercises:

1. (Easy)
7! =

2. (Easy)
1! =

3. (Easy)
4! =

4. (Easy)
0! =

5. (Easy)
4! - 2! =

6. (Easy)
(4 - 2)! =

7. (Easy)
3!2! =

8. (Easy)
7!/(2!5!) =

Answers:

1. 5040; **2.** 1; **3.** 24; **4.** 1; **5.** 22; **6.** 2; **7.** 12; **8.** 21

Combinations

Combinations are the numbers of possible choices when you pick m different objects from a total of n different objects if the order in which you pick them is NOT important.

The total number of combinations is $\frac{n!}{m!(n-m)!}$.

Examples:

1. (Medium)
Out of 5 roses in different colors, Jay picked 3 of them randomly to prepare a bouquet of 3 flowers. How many different kinds of bouquets can he have?

Solution:

In this case the order is not important. Since he picks three colors randomly out of five colors, the answer is:

$$\frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{4 \times 5}{1 \times 2} = 2 \times 5 = 10$$

Alternate Solution:

Since the number of available colors are relatively small, you can write all the possible color combinations and count them.

Let the five different colors be (R)red, (B)blue, (W)white, (Y)yellow and (G)green.
Here are all the possible color combinations.

{R, B, W}, {R, B, Y}, {R, B, G},
{R, W, Y}, {R, W, G}, {R, Y, G},
{B, W, Y}, {B, W, G}, {B, Y, G},
{W, Y, G}

2. (Hard)
What is the maximum number of lines that can pass through 10 distinct points on a plane?

Solution:

To make a line, you need to pick two points out of 10 possible points. If no three points are on the same line, you will get the maximum number of lines.

You can pick any 2 points to make a line. It doesn't matter in which order you pick them. You will get the same line. For example, picking Point A and Point B will create $\overleftrightarrow{AB}$. It doesn't matter if you first pick Point A and then Point B, or you pick point B first and then Point A. The answer is:

$$\frac{10!}{2!(10-2)!} = \frac{10!}{2!(8)!} = \frac{9 \times 10}{1 \times 2} = 45$$

Practice Exercises:

1. (Medium)
Mary wants to order a pizza with 2 toppings. If there are 6 toppings available, how many different kinds of pizza can she order?

2. (Hard)
What is the maximum number of planes that pass through 10 different points in space?

Hint: To create a plane, you need to pick 3 points out of 10 possible points.

Answer: 1. 15; **2.** 120

Permutations

Permutations are the number of possible choices when you pick m different objects from a total of n different objects if the order in which you pick is important.

The total number of permutations is $\frac{n!}{(n-m)!}$

Examples:

1. (Medium)
Mary wants to prepare her school project in three different colors. One for the background, one for the font color and one for the figures. If she has five colors to choose from, how many different color arrangements can she have for her school project?

Solution:

The order in which she chooses her colors is important. For example, black background, white fonts and yellow figures is different from white background, black fonts and yellow figures, even though the 3 colors she picked for her project are the same. So the answer is:

$$\frac{5!}{(5-3)!} = \frac{5!}{2!} = 3 \times 4 \times 5 = 60$$

Alternate Solution:

Since the number of available colors are relatively small, you can write down all the possible color arrangements and count them.

Let the five different colors be (R)red, (B)black, (W)white, (Y)yellow and (G)green.

Below is the display of all 60 possible color combinations.

{R, B, W},{R, B, Y},{R, B, G},
{R, W, B},{R, W, Y},{R, W, G},
{R, Y, B},{R, Y, W},{R, Y, G},
{R, G, B},{R, G, W},{R, G, Y},

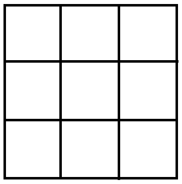
{B, R, W},{B, R, Y},{B, R, G},
{B, W, R},{B, W, Y},{B, W, G},
{B, Y, R},{B, Y, W},{B, Y, G},
{B, G, R},{B, G, W},{B, G, Y},

{W, R, B},{W, R, Y},{W, R, G},
{W, B, R},{W, B, Y},{W, B, G},
{W, Y, R},{W, Y, B},{W, Y, G},
{W, G, R},{W, G, B},{W, G, Y},

{Y, R, B},{Y, R, Y},{Y, R, G},
{Y, B, R},{Y, B, W},{Y, B, G},
{Y, W, R},{Y, W, B},{Y, W, G},
{Y, G, R},{Y, G, B},{Y, G, W},

{G, R, B},{G, R, W},{G, R, Y},
{G, B, R},{G, B, W},{G, B, Y},
{G, W, R},{G, W, B},{G, W, Y},
{G, Y, R},{G, Y, B},{G, Y, W}

2. (Hard)
In how many different ways 4 different objects can be displayed in a bookcase of 9 identical partitions as shown in the figure?



Solution:

To display 4 objects, you need to pick 4 partitions of the bookcase out of 9. Let's label them from 1 to 4. The first partition you picked is Partition 1 and you place Object 1 in this partition. The second partition you pick is Partition 2 and you place Object 2 in this partition. The third partition you pick is Partition 3 and you place Object 3 in this partition. The fourth partition you pick is Partition 4 and you place Object 4 in this partition. Since the objects are different from each other, the order in which you pick the partitions is important. Hence the number of ways you can display these 4 objects is

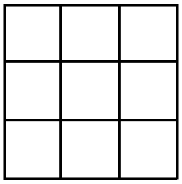
$$\frac{9!}{(9-4)!} = \frac{9!}{5!} = 6 \times 7 \times 8 \times 9 = 3024$$

Practice Exercises:

1. (Medium)
If you are allowed to use each letter only once, how many different 5 letter words (meaningful or meaningless) you can create by using 8 letters?

2. (Hard)
In how many different ways nine friends can arrange themselves in a row to have their photograph taken?

3. (Hard)
In how many different ways 4 identical objects can be displayed in a bookcase of 9 identical partitions as shown in the figure?



Answers:
1. 6720; **2.** 362880; **3.** 126 (Notice the difference between this answer and the answer in Exercise 2 above. The number of choices is dramatically decreased compared to Exercise 2)

Mutually Exclusive Events

Two events are mutually exclusive if the occurrence of one excludes the occurrence of the other.

For example, suppose you are buying two shirts as presents for your twin sisters. There are two different models available, long sleeve and short sleeve. If you want to buy the same model for both, choosing one model excludes the other. Then "picking the long sleeve" and "picking the short sleeve" events are mutually exclusive.

If two or more events are mutually exclusive, the number of ways in which you can select one or the other, is the addition of the number of ways in which each one can be selected separately.

Examples:

1. (Medium)
Suppose you are buying two shirts as presents for your twin sisters. There are two different models available, long sleeve and short sleeve. If you want to buy the same model for both, in how many different ways you can select your presents?

Solution:
You can select either long sleeve or short sleeve shirts for both. Since there is only one way of selecting either, the answer is 1 + 1 = 2

2. (Hard)
In the example above, suppose there are 4 different colors available from the long sleeve model and 5 different colors available from the short sleeve model. If you want to buy the same model for both of your sisters, in how many different ways you can pick your presents? Note that the way you pick your shirts is not important as long as you pick both from the same model.

Solution:
You can pick two long sleeve shirts or two short sleeve shirts.

The number of ways in which you can select two long sleeve shirts from 4 different color is

$$\frac{4!}{2! \times 2!} = 6$$

Similarly the number of ways in which you can select the two short sleeve shirts from the 5 different colors is:

$$\frac{5!}{2! \times 3!} = 10$$

Since these two events are mutually exclusive, the total number of ways of selecting your present:
6 + 10 = 16

Practice Exercises:

1. (Easy)
If you have three hats, black, brown and white, and three shoes, black, brown and white, how many different ways in which you can wear same color hat and shoe?

2. (Hard)
Seven students and five teachers want to go the school game. There is only one car with five seats to carry them. If all the passengers and the driver are either teachers-only or students-only, in how many different ways five of them can be seated in the car?

Answers: 1. 3; **2.** 7!/2! + 5!

Independent Events

Two events are said to be independent if they have no effect on each other. For example, suppose you are buying a sandwich. You can pick from 3 different kinds of bread and 4 different toppings. Picking a particular bread does not effect your choice of topping. The reverse is also true: Picking a particular topping does not effect the kind of bread you can choose. Hence these two events, picking the bread and picking the toppings type, are independent events.

Combined possibilities of two or more independent events are the multiplication of the number of possibilities of each independent event.

Example:

1. (Hard)
Three families have a picnic. One family decided to bring either hamburger or hotdogs for all. The other family will bring either cheesecake or Brownie or ice-cream as desert for all. The third family will bring Coke or Pepsi or Sprite for all. How many different combinations of food will there be at the picnic?

Solution:

All three families' choices are independent of each other. One family's choice does not effect the other.

First family has only 2 choices. Second family has 3 choices. Third family also has 3 choices. So the answer is $2 \times 3 \times 3 = 18$. Let's display them all.

Hamburger, Cheesecake, Coke
Hamburger, Cheesecake, Pepsi
Hamburger, Cheesecake, Sprite
Hamburger, Brownie, Coke
Hamburger, Brownie, Pepsi
Hamburger, Brownie, Sprite
Hamburger, Ice-cream, Coke
Hamburger, Ice-cream, Pepsi
Hamburger, Ice-cream, Sprite
Hotdog, Cheesecake, Coke
Hotdog, Cheesecake, Pepsi
Hotdog, Cheesecake, Sprite
Hotdog, Brownie, Coke
Hotdog, Brownie, Pepsi
Hotdog, Brownie, Sprite
Hotdog, Ice-cream, Coke
Hotdog, Ice-cream, Pepsi
Hotdog, Ice-cream, Sprite

Practice Exercises:

1. (Medium)
If you have three different kinds of bread and 7 different toppings available for pizza, in how many different ways can you order a one-topping pizza?

2. (Hard)
If you have three different kinds of bread and 7 different toppings available for pizza, in how many different ways can you order a 3-topping pizza?

3. (Hard)
Four students and six teachers will have their photograph taken. If the teachers are in the back row and the students are in the front row, how many different arrangements can be done?

Answers: 1. 21; **2.** $3 \cdot \frac{7!}{3!4!}$; **3.** 6!4!

Probability

Definition

Probability is a likelihood of an event. When there are more than one possibilities for the outcome of an event, the likelihood of one of the possibilities occurring is the probability of that event occurring. For example in a heads-or-tails coin flip, there are 2 possibilities, each can happen with equal probability.

The probability is a number between 0 and 1. If the probability of an event is 0 that means the event will not take place for sure. For example, when you roll a single dice, probability of rolling a 7 is 0, because there are no 7s on a dice.

If the probability of an event is 1, it means the event will occur for sure. For example, if you pick a cookie from a cookie jar that has nothing but cookies in it, the probability of selecting a cookie is 1.

The total probability of all the possible outcomes for any given event is always 1.

Examples:

1. (Easy)
In a heads-or-tails coin flip, there are only two possibilities. Getting heads or getting tails.

The probability of getting heads is 1/2, and the probability of getting tails is also 1/2.

The probability of getting heads OR getting tails is $1/2 + 1/2 = 1$

2. (Easy)
Suppose you have four pairs of pants in four different colors, black, brown, blue and white. If you pick a pair of pants randomly,

a. What is the probability of picking a blue pair of pants?

Solution:

Since you are picking one out of four pair, the probability of picking a particular one is 1/4

b. What is the probability of not picking a black pair of pants?

Solution:

Since the probability of picking a particular pair of pants is 1/4, the probability of not picking it is 1 - 1/4 = 3/4.

Addition of Probabilities

The probability of either one specific outcome A, OR, another specific outcome B happening is the addition of the individual probabilities of A and B.

Example:

1. (Medium)
When you roll a dice, the probability of getting 2 is 1/6 and the probability of getting 3 is also 1/6. The probability of getting 2 OR 3 is
1/6 + 1/6 = 1/3

Multiplication of Probabilities

When you calculate the probability of a specific outcome A, AND, another specific outcome B happening is the multiplication of the individual probabilities of A and B.

Examples:

1. (Medium)
When you roll a single dice twice, the probability of getting 2 for the first round is 1/6 and the probability of getting 3 for the second round is also 1/6.

The probability of getting 2 for the first round AND 3 for the second round is

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

2. (Hard)
What is the probability of getting at least one six when you roll 2 dice?

Solution:

You have different possibilities when you roll two dice. They are:

1st Dice	Probability	2nd Dice	Probability	Total Probability
6	1/6	6	1/6	$(1/6) \times (1/6) = 1/36$
6	1/6	not 6	5/6	$(1/6) \times (5/6) = 5/36$
not 6	5/6	6	1/6	$(5/6) \times (1/6) = 5/36$
not 6	5/6	not 6	5/6	$(5/6) \times (5/6) = 25/36$

Note that in the above table, the total probability is the multiplication of the two individual probabilities, because the result of rolling first dice does not effect the second dice. Hence, these two events are independent of each other.

Since the rows in the table are mutually exclusive, i.e., if both dice are 6 (like in the first row), the result can’t be one dice 6 and the other not 6. So the probability of having at least one 6 is the addition of these three probabilities:
1/36 + 5/36 + 5/36 = 11/36

Alternate Solution:

Getting at least one 6 is the “opposite” of getting no 6s. The probability of getting no 6s is 25/36 (The last row in the above table). So the probability of getting at least one 6 is
1 - 25/36 = 11/36

This method is shorter and faster. Try it when you solve exercise 8 below.

Practice Exercises:

1. (Easy)
What is the probability of getting 6 when you roll a dice?

2. (Easy)
What is the probability of not getting 6 when you roll a dice?

3. (Medium)
What is the probability of getting 1 or 2 or 6 when you roll a dice?

4. (Medium)
What is the probability of getting 1 or 2 or 3 or 4 or 5 or 6 when you roll a dice?

5. (Medium)
What is the probability of getting one 3 and one 6 when you roll 2 dice?

Hint: Either the first dice is 3 and the second dice is 6, OR, the first dice is 6 and the second one is 3.

6. (Medium)
What is the probability of getting two 6s when you roll 2 dice?

7. (Medium)
What is the probability of getting two 3s and one 5 when you roll 3 dice?

8. (Hard)
What is the probability of getting at least one 4 when you roll 6 dice?

9. (Hard)
What is the probability of getting at least one 5 and one 6 when you roll 3 dice?

Hint: Make a table of all the possible outcomes of all three dice as it is done in Example 2 above.

Answers:
1. 1/6; **2.** 5/6; **3.** 1/2; **4.** 1; **5.** 1/18; **6.** 1/36; **7.** 1/72; **8.** 31031/46656; **9.** 1 - 64/216 - 96/216 = 30/216

Exercises

Rounding

Below data shows the population of 5 counties in Maryland in the year 2000. The total population of Maryland was 5,296,486 in 2000. Use this data to answer the next 2 questions.

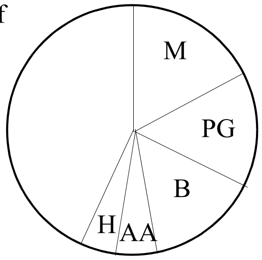
Allegany County	74,930
Anne Arundel County	489,656
Baltimore County	754,292
Calvert County	74,563
Caroline County	29,772

1. (Easy)
Approximate the populations of each of the 5 counties and the total population of Maryland to the nearest thousands digit.

2. (Medium)
Using the results in the previous question, calculate the percentage of the population for each of the 5 counties in the total population. Approximate your results to the nearest tenth digit.

Pie chart shows population of the most populous 5 counties of Maryland in 2000.

M - Montgomery
PG - Prince George's
B - Baltimore
AA - Anne Arundel
H - Howard



Answer the following 3 questions by using this chart.

3. (Medium)
Which of the following best represents the percentage population of the most populous 3 counties combined?

(A) 35%
(B) 45%
(C) 55%
(D) 65%
(E) 75%

4. (Medium)
If the total Maryland population in the year 2000 is 5,296,486, which of the following best represents the Prince George's County's population?

(A) 500,000
(B) 600,000
(C) 700,000
(D) 800,000
(E) 900,000

5. (Medium)
There are 23 counties in Maryland. Which of the following best represents the average population of the 18 smallest counties? Round your answer to the nearest 10,000.

(A) 80,000
(B) 100,000
(C) 120,000
(D) 140,000
(E) 160,000

Data Representation: Tables, Pie Charts & Graphs

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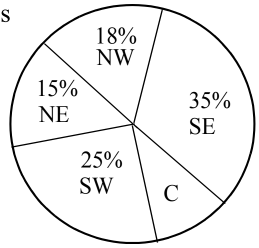
Use the below pie chart for Questions 1 - 3.

Pie chart shows the percentages of the total profit of a corporation from different regions.

NE - North East
NW - North West
SE - South East
SW - South West
C - Central

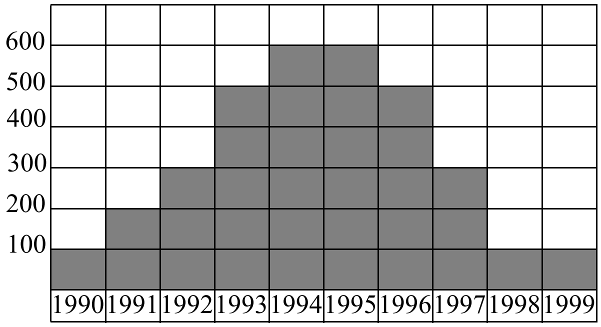


1. (Easy)
What is the percentage of the smallest contribution?

2. (Medium)
If the Central Region brings $5,000,000 yearly profit to the company, how much profit does the NorthWest region bring? Approximate your result to the nearest dollar.

3. (Hard)
The Central Region opened new stores to increase its profit and get some of the North East region's business. If the other three regions remained at the same profit percentage, what percentage of the North East's business must go to the Central for the Central cease to be the smallest contributor of the company?

Use below graph for questions 4 - 7.



The above graph shows change in crime between the years 1990 and 1999.

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4. (Easy)
What is the largest increase in crime and when did it occur?

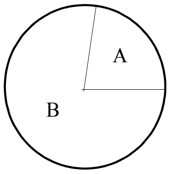
5. (Easy)
What is the largest decrease in crime and when did it occur?

6. (Hard)
What is the largest percentage increase in crime and when did it occur?

Hint: Percentage increase is not the same as absolute increase.

7. (Hard)
What is the largest percentage decrease in crime and when did it occur?

8. (Easy)
In the figure, "A" represents the percentage of population above 65 years old. Approximately what percent of the population is under 65 years old?



(A) 20%
(B) 25%
(C) 70%
(D) 80%
(E) 90%

9. (Easy)
Below is the result of a sales activity regarding 3 kinds of flowers in a nursery. Each Θ represents 50 customers who purchased flowers.

Tulip ΘΘΘЄ
Daisy ΘΘ
Narcissus ΘЄ

How many more customers purchased tulip over narcissus?

Table shows the home sales activity in 4 different counties in the USA.

Use this table to answer questions 10 - 13.

County	Homes Sold	Average Price
M	1050	200K
P	802	150K
F	358	178K
H	400	195K

---

10. (Easy)
How many homes are sold in all 4 counties combined?

11. (Medium)
How much money the purchasers paid to buy all the homes in 4 counties combined?

12. (Hard)
What is the average home price for all 4 counties?

(A) \$180750

(B) \$180,852

(C) \$160,312

(D) \$193,750

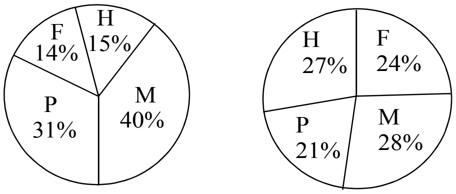
(E) There isn't enough information to answer the question?

13. (Hard)
Which of the below figures represents the home sale data the best?

I. □ = 50 Homes, \$ = \$25,000

County	**Homes Sold**	**Average Price**
M	□□□□□□□□□□□□□□□□□□□□□	\$\$\$\$\$\$\$\$
P	□□□□□□□□□□□□□□□□	\$\$\$\$\$\$
F	□□□□□□□	\$\$\$\$\$\$\$
H	□□□□□□□□	\$\$\$\$\$\$\$\$

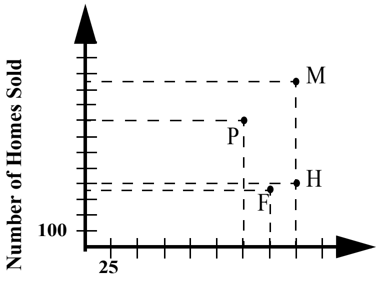
II.



Number of Homes Sold

Average Prices

III.



(A) I only

(B) II only

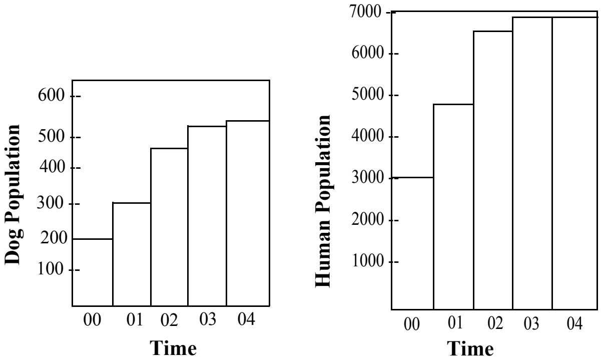
(C) III only

(D) I and III

(E) I, II and III

Below graph shows the changes in the dog and human populations in a certain town, between the years 2000 (00) and 2004 (04).

Use these graphs to answer questions 14 - 16.



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14. (Medium)
Which of the following statements is <u>not</u> true?

(A) Largest human population increase occurred in 2001 and 2002.

(B) As human population increases, dog population also increases.

(C) Dog population increased approximately 50% in between 2001 and 2002.

(D) Yearly percentage increase in the human population is approximately 60%, two years in a row, from 2000 to 2001 and from 2001 to 2002.

(E) Total increase in the dog population between 2000 and 2004 is approximately 350.

15. (Medium)
In the year 2003, approximately how many dogs each person has owned?

16. (Hard)
If you are trying to display visually that more humans means more dogs, which of the following graphs is the most suitable?

(A) Correlation Graph - dog population versus human population.

(B) Histogram - Showing both populations on the same graph.

(C) Two pie charts - each showing dog and human populations respectively.

(D) A table - Showing both populations.

(E) All of the above.

17. (Hard)
In the year 2002, 60% of the population did not have a dog in their homes. Among the remaining population, if each household had only one dog, what was the average size of a household that had a dog? Approximate your answer to the nearest whole number.

- **(A)** 3
- **(B)** 4
- **(C)** 6
- **(D)** 7
- **(E)** 8

18. (Hard)
Below table shows the distance of a car from its destination as time passes.

Time (minutes)	2	11	15	20	30
Distance from Destination (miles)	60	54	52	40	30

a. Graph Time-versus-Distance in a scattered graph.

b. Approximately, how far the destination is in the beginning?

- **(A)** 30 miles
- **(B)** 60 miles
- **(C)** 65 miles
- **(D)** 70 miles
- **(E)** 75 miles

c. Approximately, how fast the car is moving?

- **(A)** 40 miles/hour
- **(B)** 50 miles/hour
- **(C)** 60 miles/hour
- **(D)** 70 miles/hour
- **(E)** 80 miles/hour

d. Approximately, what is the distance from the destination after 5 minutes?

- **(A)** 60 miles
- **(B)** 50 miles
- **(C)** 40 miles
- **(D)** 30 miles
- **(E)** There is not enough information to answer the question.

e. Approximately, how long the whole trip will take?

- **(A)** 75 minutes
- **(B)** 65 minutes
- **(C)** 55 minutes
- **(D)** 45 minutes
- **(E)** There is not enough information to answer the question.

f. Approximately, at what time is the distance from the destination equals to 25 miles?

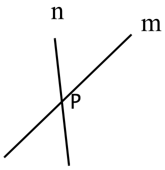
g. Write an equation that gives the relationship between the time in minutes and the distance in miles.

Sets

1. (Easy)

Consider the two lines in the figure. Set A is the points on line m and Set B is the points on line n.

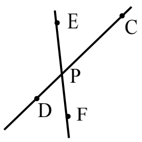
What is the intersection of Set A and Set B?



2. (Easy)

Consider the two line segments in the figure. Set A is the points on $\overline{CD}$ and Set B is the points on $\overline{EF}$.

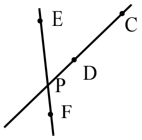
What is the intersection of Set A and Set B?



3. (Medium)

Consider the two line segments in the figure. Set A is the points on $\overline{CD}$ and Set B is the points on $\overline{EF}$.

What is the intersection of Set A and Set B?



4. (Medium)

20 seniors are in both the school play and the chorus. 80 seniors are either in the school play only or the chorus only or neither. How many seniors are there in the school?

5. (Hard)

In Richmond High, there are 50 seniors who are in either the school play or the chorus, but not both. 65 seniors are in at least one of the two. 80 seniors are either in the school play only, or the chorus only, or neither. How many seniors are there in the school?

6. (Hard)

In a large fish tank, there are a total of 500 fish, in three colors: blue, yellow and red. 80 of them have at least red and yellow on them. 70 of them have at least blue and red on them. 40 of them have at least blue and yellow on them. If 20 fish are in all three colors, how many single-color fish are there in the tank?

Defined Operators

1. (Medium)

^a is the sum of integers from 1 to a. What is ^^5?

2. (Medium)

The number of "@"s and "O"s in a diagram represents the units digit and the 10s digit of a number respectively. For example: $\boxed{\text{@@@OO}}$ = 23

Which of the diagrams below is NOT defined in this representation?

(A) $\boxed{\qquad\qquad}$

(B) $\boxed{\quad\text{O}\quad}$

(C) $\boxed{\text{O@O@O@O@O@O@}}$

(D) $\boxed{\text{@@@@@@@@@@}}$

(E) $\boxed{\text{OOOOOOOOO}}$

3. (Medium)
How many integers can be represented by the representation in question 2?

- **(A)** 99
- **(B)** 999
- **(C)** 100
- **(D)** 1000
- **(E)** All the integers

4. (Medium)
If a& is 1/a, what is 5&&&&&&&&&&&& ?

- **(A)** 0.0016
- **(B)** 0.04
- **(C)** 0.2
- **(D)** 2
- **(E)** 5

5. (Medium)
$a = 10a, !b = b/100. What is ($!$!$!30)/!30?

- **(A)** 3/100
- **(B)** 30/100
- **(C)** 0.1
- **(D)** 1/30
- **(E)** 3/10

6. (Medium)
Consider the configuration of numbers shown in the figure.

8 operators are defined as follows.

a	b	c
d	e	f
g	h	i

$\overrightarrow{e}$ gives the number to the right of e. In the above figure, it is f.

$\overleftarrow{e}$ gives the number to the left of e. In the above figure, it is d.

$\uparrow$e gives the number above e. In the above figure, it is b.

$\downarrow$e gives the number below e. In the above figure, it is h.

$\nearrow$e gives the number to the upper right diagonal of e. In the above figure, it is c.

$\swarrow$e gives the number to the lower left diagonal of e. In the above figure, it is g.

$\nwarrow$e gives the number to the upper left diagonal of e. In the above figure, it is a.

$\searrow$e gives the number to the lower right diagonal of e. In the above figure, it is i.

What is

$$(\searrow(\nearrow(\overrightarrow{33}))) \cdot \overleftarrow{(\searrow(\overrightarrow{57}))}$$

in the below figure?

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69

7. (Medium)
$a\Delta b = 1/(b - a)$, $a\nabla b = b - a$

If $-(3\nabla5)\Delta(c\nabla8) = c\nabla10$, then what is the value of c?

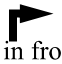
8. (Medium)
— is a mirror operator that reflects the image in front of it. For example:

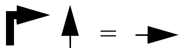
$\underline{\;\Delta\;} = \nabla$, and $\triangleright\Big| = \triangleleft$

What is $\Big|\Big(\overline{\Big(\nearrow\Big|\Big)}\Big)$?

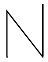
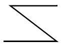
- **(A)** $\nearrow$
- **(B)** $\searrow$
- **(C)** $\swarrow$
- **(D)** $\nwarrow$
- **(E)** None of the above.

9. (Hard)

 is a rotation operator that rotates the figure in front of it by 90^o clockwise.

For example: 

By using the rotation operator and the mirror operator, defined in the previous question, what combination of operators must be used to get an

 from  ?

Hint: Rotate the paper if you need to.

10. (Hard)
b^a is the sum of integers from b to a. If x^y = 147, what are the two possible values of x and y?

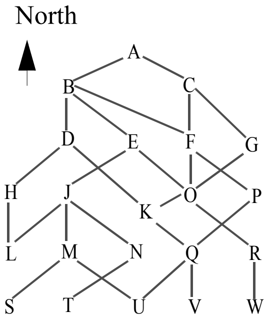
Logic

1. (Easy)
Peter and his friend Liz each have only one car. Peter's car is blue. Liz's car is red. They went to a party together in a red car. Which of the following is true.

- **I.** They went to the party in Liz's car.
- **II.** Liz was the driver for the trip.
- **III.** They did not use Peter's car to go to the party.

- **(A)** I only
- **(B)** III only
- **(C)** I and II
- **(D)** I and II and III
- **(E)** None

2. (Medium)
Instructions to travel from north to south in the network of roads shown in the figure are as follows.



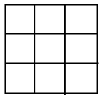
Start from A and each time you come to an intersection, take the middle path of the available paths if the number of available paths in the intersection is odd.

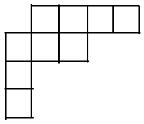
If there are even number of available paths, take the first available road on the right in the first occasion. In the next occasion, take the first available left road and alternate between right and left for each occasion.

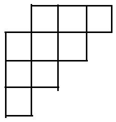
You are not allowed to go through the same location more than once. In addition, you are only allowed to go south from where you are. So the available paths are only the paths leading to towns that you did not visit and south of where you are.

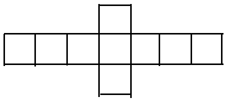
At the end of your journey, which location are you at?

3. (Hard)
The following 5 shapes are composed of squares and each square is to be painted in red, blue, yellow or green. No two squares that share the same side can be painted the same color. If this condition is not met, the square is left unpainted. Which of these shapes can be totally painted?

(A) 

(B) 

(C) 

(D) 

(E) None of the above.

Hint: There are only 4 colors. So any square that has more than 3 neighbors should not be painted.

Use the information in the following paragraph to answer the next 3 questions. For one week, Joy ate cereal for breakfast on odd days, and eggs on even days. For example, if she started her week on the 10th of March, she ate eggs on 10th, 12th, 14th and 16th; cereal on the 11th, 13th and 15th.

---

4. (Medium)
Which of the following may be correct?

I. On Wednesday she ate eggs for breakfast.
II. She ate cereal two days in a row.
III. If she ate cereal on Monday, she ate eggs on Tuesday.

(A) I only
(B) I and III only
(C) III only
(D) I and II and III
(E) None

Hint: In the end of the month, there can be two consecutive odd days. For example, 31st of January and 1st of February.

5. (Hard)
Which of the following is always wrong?

(A) On Wednesday she ate eggs for breakfast.
(B) She ate cereal two days in a row.
(C) She ate eggs on Saturday and Sunday.
(D) If she ate cereal on Monday, she ate eggs on Tuesday.
(E) She did not eat cereal on Tuesday.

6. (Hard)
Which of the following is always correct?

(A) On Wednesday she ate eggs for breakfast.
(B) She ate cereal two days in a row.
(C) She ate eggs on Saturday and Sunday
(D) If she ate cereal on Monday, she ate eggs on Tuesday.
(E) None of the above.

Statistics

1. (Medium)
Class A's and class B's math grade averages are the same, 8 out of 10. Which of the following statements must always be correct?

(A) Class A's math grade median is greater than the Class B's math grade median.
(B) Class A's math grade median is less than the Class B's math grade median.
(C) Class A's math grade median is equal to the Class B's math grade median.
(D) Class A's math grade median is not 8.
(E) None of the above.

2. (Medium)
The median age of employees in ABC Company is 41. There are 5 employees in ABC. No two employee are at the same age. Which of the following must be true?

(A) There is at least one employee who is 41 years old.
(B) Sum total of the ages of all employees is $41 \cdot 5 = 205$
(C) There are only two employees whose ages are 41 or less.
(D) The average age in ABC is 41.
(E) None of the above.

3. (Medium)
In both the first and second math exams the class average was 80. Maximum possible grades are 100 and 90 in the first and second exams respectively. Which of the following statements is correct?

(A) Average performance in the first exam is better than the average performance in the second exam.
(B) Average performance in the first exam is worse than the average performance in the second exam.
(C) Average performance in the first exam is the same as the average performance in the second exam.
(D) Median grade in the first exam is the same as the median grade in the second exam.
(E) Median grade in the first exam is less than the median grade in the second exam.

4. (Medium)
Out of 3 sets of 10 cards, each labeled from 1 to 10, if you pick 10 cards, what is the highest possible average?

5. (Medium)
Out of 3 sets of 10 cards, each labeled from 1 to 10, if you pick 10 cards, what is the highest possible median?

6. (Medium)
Out of 3 sets of 10 cards, each labeled from 1 to 10, if you pick 10 cards, what is the highest possible mode?

7. (Medium)
For a parade, 2 students are chosen from each age, among students ranging from ages 5 to 15. Which of the following is true?

(A) Average age in the parade is less than the median age of the student in the parade.

(B) Average age in the parade is greater than the median age of the student in the parade.

(C) Average age in the parade is equal to the median age of the student in the parade.

(D) Average age can not be 10.

(E) Median age can not be 10.

8. (Medium)
In January, there are 31 days, ranging from day 1 to day 31. What is the average and the median day in January?

9. (Hard)
Consider a set of distinct integers. One less and one more than any element of the set is also an element of the set - except for the smallest and the largest integers in the set. Which of the following statements is correct?

(A) The average of the numbers in the set is smaller than the median of the set.

(B) The average of the numbers in the set is greater than the median of the set.

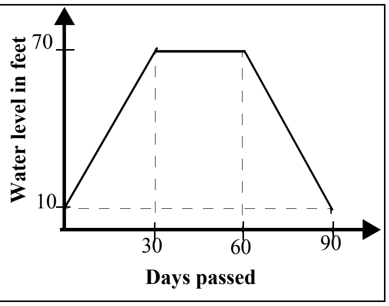
(C) The average of the numbers in the set is equal to the median of the set.

(D) If you add another integer which is greater than all the elements of the set, the median of the set will remain the same.

(E) None of the above.

The graph shows the water level in a dam.

Use the graph for the questions 10 - 12



10. (Medium)
What is the mode of daily water level during the 90 day period?

11. (Hard)
What is the daily average water level during the 90 day period?

12. (Hard)
What is the daily median water level during the 90 day period?

13. (Hard)
Mary walks at a speed of 2 miles/hr for ½ hour. Then she sits and rests for 10 minutes. Finally she walks with a constant speed for 40 more minutes. If her average speed is 2 miles/hr, what is her speed for the final part of her walk?

Hint: Average speed =
(total distance traveled) / (total travel time)

14. (Very Hard)
Certain species reproduce themselves by giving birth to 2 offsprings. Starting from the next day of their birth, each offspring starts to reproduce itself in the same way until it dies on the 6th day, at the age of 5 days. The offsprings don't reproduce on the day that they die. Starting with a new born on day one, what is the average age of all the offsprings on day 6?

Sums

1. (Medium)
John decided to save some money for the vacation he wants to take in September. He puts $4 aside each day from 10th of each month to the 20th, starting on January 10th, for 8 months. How much does he have for his vacation on September 1st?

2. (Hard)
What is the sum of the even integers between 200 and 929?

3. (Hard)
What is the sum of the odd integers between 200 and 929?

4. (Hard)
Which of the following is equal to the sum of the even numbers between two odd integers a and b?

(A) The sum of the odd integers between a and b.

(B) (The sum of the integers between a and b)/2

(C) (The sum of the odd integers between a and b) - a

(D) (The sum of the odd integers between a and b) - b

(E) (The sum of the odd integers between a and b) - (a + b)/2

5. (Hard)
What is the average of the even integers between the two odd integers a and b?

Sequences

1. (Medium)
A sequence starts with 2 and each element is the square of the previous element minus 1. What is the median of the first 5 elements?

2. A sequence starts with -5 and has 24 elements. Each element is 4 less than the previous element

a. (Easy)
Write the first 4 elements of the sequence.

b. (Medium)
Write an expression for the nth element in the sequence.

c. (Medium)
What is the 17th element of the sequence?

d. (Hard)
What is the sum of all the elements?

e. (Hard)
What is the average of the last 10 elements?

3. (Medium)
The first element of an arithmetic sequence is -5. If the 7th element is 19, what are the first 5 elements of the sequence?

4. (Medium)
What is the next 3 elements of the sequence:
{0, 1, 1, 2, 3, 5, 8, 13, ... }

5. The first element of a sequence is 2. If each element in the sequence is 5 times greater than the previous element,

a. (Easy)
Write the first 5 elements of the sequence.

b. (Medium)
Write an expression for the nth element in the sequence.

c. (Medium)
What is the 7th element of the sequence?

6. In a sequence if the nth element is $3 \cdot 7^{n-1}$, then

a. (Easy)
What is the first element?

b. (Medium)
What is the ratio of the consecutive elements?

c. (Medium)
What is the ratio of the 5th to the 2nd elements?

7. (Medium)
The ratio of two consecutive terms in a geometric sequence is 1/5. If the sum of the first 3 terms is 62, what is the first term?

8. (Medium)
The number of bacteria in a petri dish doubles every 5 minutes. If the number is 17600 after half an hour, how many were there in the beginning of the experiment?

9. (Medium)
If the average of the 8th, 10th and 12th elements of an arithmetic sequence of distinct integers is 102, which of the following can NOT be the 11th element?

(A) 102
(B) 103
(C) 104
(D) 105
(E) There is not enough information to answer the question.

10. (Medium)
If the addition of the 8th and 9th elements of an arithmetic sequence is 102, and the first element is 3, which of the following is the 11th element?

(A) 1
(B) 16
(C) 25
(D) 67
(E) There is not enough information to answer the question.

11. (Medium)
How many elements are there in the sequence:
{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, ..., 99}

12. (Hard)
What is the average of the first 50 elements in the sequence given in the previous question?

13. (Hard)
If the 4th element of a geometric sequence is -162 and the 7th element is 6, write a general expression for the nth element.

14. (Medium)
What are the first 10 elements of the sequence described in the previous question?

15. (Hard)
If the addition of the 8th and 9th elements of an arithmetic sequence is 102, and the 11th element is an integer, which of the following can be the first element?

(A) 8
(B) 9
(C) 10
(D) 11
(E) There is not enough information to answer the question.

16. (Hard)
If the 8th and 10th terms of a geometric series are 2 and 4 respectively, what is the ratio of the 11th to the 5th element?

Counting

Basic Counting

1. (Easy)
A, B, C are 3 points on the same line and D is a point outside of this line. How many lines pass through at least 2 of these 4 points?

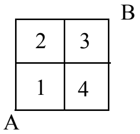
2. (Medium)
What is the minimum number of regions you can separate by four lines on a plane?

Hint: Minimum number of lines are obtained when all 4 lines are parallel.

3. (Medium)
How many even pages are there between pages 30 and 550, including both pages?

Hint: To understand the point of the question, think of the number of pages between 2 and 6.

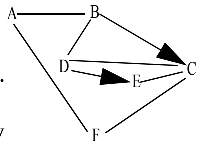
4. (Medium)
In how many different ways can you go from A to B without visiting the same cell more than once?



5. (Medium)
Starting with red, yellow and blue, how many different colors can you get if every mixture of two or more original colors is another color?

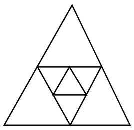
Hint: Don't forget mixing all 3 together.

6. (Medium)
Mary wants to make a round trip between A and C. The road map is shown in the figure. One-way roads are indicated by one-sided arrows. In how many different ways can Mary travel if she is allowed to pass through each place only once in each direction?



Hint: There are 4 different ways of going from A to C, and only 2 ways for the return trip.

7. (Hard)
In the figure, each inscribed triangle's corners are bisecting the outer triangle's sides. All three triangles are equilateral. How many trapezoids are there in the figure?



Hint: A parallelogram is a special type of trapezoid.

Advanced Counting

Combinations

1. (Medium)
Two crossing lines define a point. What is the maximum number of points defined by 4 lines?

2. (Medium)
Two crossing lines define 4 angles. What is the maximum number of angles defined by 4 lines?

3. (Medium)
In a colored pencil box, there are 10 pencils in 10 different colors. How many different color combinations are available if you randomly pick :

a. 1 color

b. 2 colors

- **c.** 3 colors
- **d.** 4 colors
- **e.** 5 colors
- **f.** 6 colors
- **g.** 7 colors
- **h.** 8 colors
- **i.** 9 colors
- **j.** 10 colors

Assume that the order in which you pick the colors is not important. Draw a graph of number of colors picked vs. number of choices you have. What are the minimum and maximum number of choices?

4. (Hard)
How many diagonals does a hexagon have?

5. (Hard)
How many diagonals does an n-sided polygon have?

Permutations

1. (Medium)
In a classroom of 18 students, 2 students will be picked as class president and vice president. In how many different ways can this be done?

2. (Medium)
In a classroom of 11 girls and 7 boys, 2 students will be picked as class president and vice president. If both the president and the vice president must be a girl, in how many different ways can this be done?

3. (Medium)
In a classroom of 11 girls and 7 boys, 3 students will be picked as the class president, the vice president and the treasurer. If all three positions must be filled by boys, in how many different ways can this be done?

4. (Medium)
How many different 2-letter words, meaningful or not, can you form with the letters S, H, O, P, if you are allowed to use each letter only once?

5. (Medium)
How many different 3-letter words, meaningful or not, can you form with the letters S, H, O, P, if you are allowed to use each letter only once?

6. (Medium)
How many different 4-letter words, meaningful or not, can you form with the letters S, H, O, P, if you are allowed to use each letter only once?

7. (Medium)
Using all 5 integers from 1 to 5 as the digits of a whole number, how many different whole numbers can you create starting with 1 and ending with 5? You are allowed to use each of the 5 integers only once.

8. (Hard)
In a classroom, there are 3 girls, Mary, Sue and Kim and 3 boys, Joe, Rick and Tim. You are trying to form 3 teams from these boys and girls. You want each team to have a boy and a girl. How many different sets of 2-team groups can you arrange?

Mutually Exclusive Events

1. (Hard)
In a classroom of 11 girls and 7 boys, 2 students will be picked as class president and vice president. If both the president and the vice president must be of the same gender, in how many different ways can this be done?

2. (Hard)
Helen has 3 white, 4 black and 2 purple bracelets. In how many different ways can she wear two bracelets of the same color?

3. (Hard)
If you want to create integers by arranging the integers from 1 to 5, how many <u>even</u> integers can you create? You are allowed to use each number from 1 to 5 only once.

4. (Hard)
If you want to create integers by arranging the integers from 1 to 5, how many <u>odd</u> integers can you create? You are allowed to use each number from 1 to 5 only once.

Independent Events

1. (Medium)
If you have 5 T-shirts and 4 shorts, in how many different ways can you wear a T-shirt-short pair?

2. (Medium)
Suppose you are having a soup and salad lunch. If you have 3 different salads and 3 different soups to choose from, in how many different ways can you have your lunch?

3. (Medium)
In a classroom of 11 girls and 7 boys, 2 students will be picked as the class president and the vice president. If the president must be a girl and the vice president must be a boy, in how many different ways can it be done?

4. (Hard)
In a classroom of 11 girls and 7 boys, 2 students will be picked as the class president and the vice president. If the president and the vice president must be of different gender, in how many different ways can it be done?

5. (Hard)
You have 10 different color pencils and 7 different color papers. In an art project, you decide to use three different color pencils and two different color papers. How many different choices do you have?

6. (Hard)
Kim has invited 4 of her friends to dinner. There are 6 chairs around the dinner table. Kim wants to sit on one end of her rectangular table and she wants the other end stay empty. In how many different ways can she and her friends be seated?

Hint: First find the number of possible ways in which the friends can sit. Then multiply it by 2, because Kim can sit in either chair on the two ends of the table.

7. (Hard)
In a classroom of 24 students, the teacher forms groups of 6 student each. In how many different ways can these groups be formed?

8. (Hard)
In a classroom of 12 female and 12 male students, the teacher forms a group of 6 student to represent the class in a school parade. If she wants to have equal number of male and female students in the group, in how many different ways that she can form it?

(A) $4!\left(\frac{12!}{3!9!}\right)$

(B) $\frac{12!}{3!9!}$

(C) $\left(\frac{12!}{3!9!}\right)^2$

(D) $\frac{24!}{6!18!}$

(E) $\left(\frac{12!}{9!}\right)^2$

Probability

1. (Medium)
What is the probability of getting 6 sixes when you roll 6 dice?

2. (Medium)
What is the probability of not getting any sixes when you roll 5 dice?

3. (Medium)
Calculate the probabilities of both events presented across a row and determine which column is bigger.

	Column A	Column B
a.	Probability of getting 2 sixes when you roll 2 dice.	Probability of getting one six and one five when you roll 2 dice.
b.	Probability of getting 2 sixes when you roll 2 dice.	Probability of getting 2 sixes when you roll the same dice two times.

There are 12 red, 7 green and 10 yellow candies in a box. Use this information for the questions 4 - 8.

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4. (Medium)
What is the probability of getting a yellow candy if you randomly pick one candy?

5. (Hard)
What is the probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and keep it?

(A) $\frac{10}{29} \cdot \frac{12}{29}$

(B) $\frac{10}{29} + \frac{12}{29}$

(C) $\frac{10}{29} \cdot \frac{12}{28}$

(D) $\frac{10}{29} + \frac{12}{28}$

(E) $1 - \frac{7}{29}$

6. (Hard)
What is the probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and put it back in the box after picking?

(A) $\frac{10}{29} \cdot \frac{12}{29}$

(B) $\frac{10}{29} + \frac{12}{29}$

(C) $\frac{10}{29} \cdot \frac{12}{28}$

(D) $\frac{10}{29} + \frac{12}{28}$

(E) $1 - \frac{7}{29}$

7. (Hard)
Calculate the probabilities of both events presented across a row, and determine which column is bigger:

	Column A	Column B
a.	Probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and keep it.	Probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and put it back in the box after picking.
b.	Probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and keep it.	Probability of getting one yellow and one red candy if you pick one candy at a time and keep it.
c.	Probability of getting a yellow candy first and a red candy the next if you pick one candy at a time and keep it.	Probability of getting 2 yellow candies if you pick one candy at a time and keep it.
d.	Probability of getting 2 yellow candies if you pick one candy at a time and keep it.	Probability of getting 2 yellow candies if you pick two candies together at one time.

8. (Hard)
If you are allowed to eat only one candy from each color, and you are allowed to have 3 tries to pick a candy, what are the probabilities of eating:

a. At least one candy?

b. No candy,?

c. Only one candy?

d. 3 candies?

Note that if you are not allowed to eat the candy that you pick, you need to put it back in the box.

9. (Hard)
A class has 3 females and 6 males. To form a debate team, 3 students are randomly picked to defend the idea that college education is important for a person's happiness. From the remaining students, 3 of them are randomly picked to defend that college education is not important for a person's happiness. The remaining 1/3 is the jury.

a. What is the probability of all the three members of the jury being female?

b. If the first group is the jury, what is the probability of all the three members of the jury being female?

c. If the second group is the jury, what is the probability of all three members of the jury being female?

d. If the groups are formed first and the jury group is randomly picked out of the 3 groups, what is the probability of all the three members of the jury being female?

10. (Hard)
There are 7 different courses thought in Richmond High: Math, English, Biology, History, Geography, Music and Spanish. Each morning there are 3 class periods.

Wei has randomly picked three different classes on Tuesday morning for the 1st, the 2nd and the 3rd periods. First one turned out to be Math and the last one turned out to be English.

Mary also has randomly picked three different classes on Tuesday morning. First one turned out to be English and the last one turned out to be Math.

a. What is the probability of both taking the Biology class on Tuesday morning at the same time?

b. What is the probability of both students taking the same class on Tuesday morning at the same time?

c. What is the probability of both students not taking the same class on Tuesday morning at the same time?

Answers

Rounding

1. See the solutions
2. See the solutions
3. (B)
4. (D)
5. (C)

Tables, Pie Charts & Graphs

1. 7%
2. $12,857,143
3. More than 26.7%
4. 200, from 1992 to 1993
5. 200, from 1996 to 1997 and 1997 to 1998
6. 100%, from 1990 to 1991
7. 67%, from 1997 to 1998
8. (D)
9. 100
10. 2610
11. $472,024,000
12. (B)
13. (A)
14. (D)
15. 0.08
16. (A)
17. (C)
18.
  a. See the solution
  b. (C)
  c. (D)
  d. (A)
  e. (C)
  f. Around 34 minutes.
  g. $d = -\frac{6}{5}t + 65$

Sets

1. Point P
2. Point P
3. ∅
4. 100
5. 95
6. 350

Defined Operators

1. 120
2. (D)
3. (C)
4. (E)
5. (C)
6. 2448
7. 9 or 11
8. (B)
9. See the solution
10. {73, 74} or {48, 50}

Logic

1. (B)
2. V
3. (B)
4. (D)
5. (C)
6. (E)

Statistics

1. (E)
2. (A)
3. (B)
4. 8.8
5. 9
6. 10
7. (C)
8. Ave. = Median = 16
9. (C)
10. 70 feet
11. 50 feet
12. 55 feet
13. 2.5 miles/hour
14. 11.5 hours

Sums

1. $352
2. 205,860
3. 206,225
4. (E)
5. (a + b)/2

Sequences

1. 8
2.
  a. -5, -9, -13, -17
  b. -4n - 1
  c. -69
  d. -1224
  e. -79
3. -5, -1, 3, 7, 11
4. 21, 34, 55
5.
  a. 2, 10, 50, 250, 1250
  b. $2 \cdot 5^{n-1}$
  c. 31,250
6.
  a. 3
  b. 7
  c. 343
7. 50
8. 275
9. (A)
10. (D)
11. 90
12. 30.26
13. $-(-1)^n 2 \cdot 3^{8-n}$
14. {4374, -1458, 486, -162, 54, -18, 6, -2, 2/3, -2/9, ...}
15. (B)
16. 8

Basic Counting

1. 4
2. 5
3. 261
4. 5
5. 7
6. 8
7. 12

Combinations

1. 6
2. 24
3.
  a. 10
  b. 45
  c. 120
  d. 210
  e. 252
  f. 210
  g. 120
  h. 45
  i. 10
  j. 1
4. 9
5. $\frac{n!}{2!(n-2)!} - n$

Permutations

1. 306
2. 110
3. 210
4. 12
5. 24
6. 24
7. 6
8. 6

Mutually Exclusive Events

1. 152
2. 10
3. 48
4. 72

Independent Events

1. 20
2. 9
3. 77
4. 154
5. 2520
6. 48
7. 2,308,743, 493,056
8. (C)

Probability

1. 1/46656
2. 3125 / 7776
3.
  a. Column B
  b. They are equal
4. 10/29
5. (C)
6. (A)
7.
  a. Column A
  b. Column B
  c. Column A
  d. They are equal
8.
  a. 1
  b. 0
  c. $\frac{2514}{22736}$
  d. $\frac{60}{261}$
9.
  a. 1/84
  b. 1/84
  c. 1/84
  d. 1/84
10.
  a. 1/25
  b. 1/5
  c. 4/5

Solutions

Rounding

1.

Allegany County	75,000
Anne Arundel County	490,000
Baltimore County	754,000
Calvert County	75,000
Caroline County	30,000

The total population of Maryland 5,296,000

2.

Allegany County	$\frac{75000}{5296000}100 = 1.4\%$
Anne Arundel County	$\frac{490000}{5296000}100 = 9.3\%$
Baltimore County	$\frac{754000}{5296000}100 = 14.2\%$
Calvert County	$\frac{75000}{5296000}100 = 1.4\%$
Caroline County	$\frac{30000}{5296000}100 = 0.6\%$

3. Answer: (B)
Since the addition of the percentage populations of the three largest counties (Montgomery, Prince George's and Baltimore) is a little less than half (50%) of the total population, the answer is 45%, (B).

4. Answer: (D)
From the chart, Montgomery, Prince George's and Baltimore have approximately the same population. Since the combined percentage population of the Montgomery, Prince George's and Baltimore counties are around 45%, Prince George's county's population is about 45/3 = 15% of the total Maryland population. Hence PG county's population is

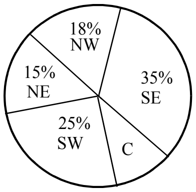
$$5296000 \cdot \frac{15}{100} = 794400$$

Among the answer choices, the closest answer is 800,000. So the answer is (D).

5. Answer: (C)
Since the combined percentage population of the 5 largest counties is about 60% of the total, the remaining 18 counties' percentage population is 100 - 60 = 40% of the total population. The average population of these 18 counties is

$$\left(\frac{5296000 \cdot 40}{100}\right)/18 = 117689 \cong 120000$$

Data Representation: Tables, Pie Charts & Graphs



1. Answer: 7%
Central Region with
100 - 15 - 18 - 35 - 25 = 7%.

2. Answer: $1,285,714
North West brings 18% of the profit. It is

$$\frac{5000000}{7} \times 18 = \$12,857,143$$

3. Answer: More than 26.7%
Combined contribution of North East and Central regions is 15 + 7 = 22%. For the Central region not to be the lowest contributor, the Central region must have more than 11% of the total profit. North East will drop to less than 11%.

The difference in the North East region's profit percentage is 15 - 11 = 4% of the total profit.

Let the total profit of the company be P. ➔
The North East region's original profit is $\frac{15P}{100}$ and
The change in the North East region's profit is $\frac{4P}{100}$
The percentage change in the North East region's profit is

$$\left(\left(\frac{4P}{100}\right) \cdot 100\right)/\left(\frac{15P}{100}\right) = 4 \times \frac{100}{15} \cong 26.7\,\%$$

4. Answer: 200, from 1992 to 1993

5. Answer: 200, from 1996 to 1997 and 1997 to 1998

6. Answer: 100%, from 1990 to 1991.
The crime has increased from 100 in 1990 to 200 in 1991. This represents a

$$\left(\frac{200-100}{100}\right)100 = 100\,\% \text{ increase.}$$

Note that 200 increase from 1992 to 1993 is

$$\left(\frac{200}{300}\right)100 \cong 67\,\%,$$

which is less than the percentage increase in 1991.

7. Answer: 67%, from 1997 to 1998

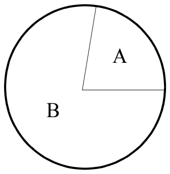
$$\left(\frac{200}{300}\right)100 \cong 67\,\%, \text{ from 1997 to 1998.}$$

Note that 200 decrease from 1996 to 1997 is

$$\left(\frac{200}{500}\right)100 = 40\,\%,$$

which is less than the decrease in 1998.

8. Answer: (D)
Since region A is close to but less than one quarter of the pie, the region B must be a little over 75%. Hence the best answer is 80%.



9. Answer: 100
The answer is $(3.5 - 1.5) \times 50 = 100$

10. Answer: 2610
1050 + 802 + 358 + 400 = 2610

11. Answer: $472,024,000
$1050 \cdot 200 + 802 \cdot 150 + 358 \cdot 178 + 400 \cdot 195 =$
210,000 + 120,300 + 63,724 + 78,000 =
472,024K = $472,024,000

12. Answer: (B)
Average price = 472,024,000/2610 = $180,852 (approximated to the nearest dollar amount).
The answer is (B).

Note that the average price is <u>not</u> the average of 200K, 150K, 178K and 195K. The average of these 4 numbers is 180,750 which is case (A). However it is one of the wrong answers.

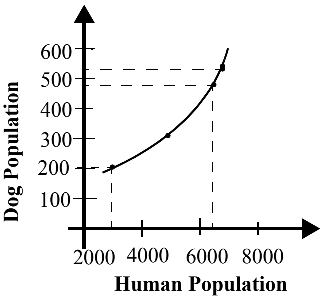
13. Answer: (A)
Case II is wrong because four average prices are not parts of a whole. So the 2nd pie chart in this case is meaningless.

Case III is wrong because it is trying to show a correlation between home prices and the number of homes sold in four different locations. The number of homes sold may depend on several other factors such as population, income levels, etc. Since none of these factors can be assumed to be the same, a coloration graph is not the best way to display the data.

14. Answer: (D)
Percent increase in 2001 is about 60%, but percent increase is only about 37% in 2002.

15. Answer: 0.08
In 2003, the dog and human populations were about 520 and 6800, respectively. Hence the average dog population per house is:
520/6800 = 0.08 dogs.

16. Answer: (A)
<u>Correlation Graph</u>
Dog population versus human population will best show the increase in dog population as the human population increases. This graph is shown in the figure.



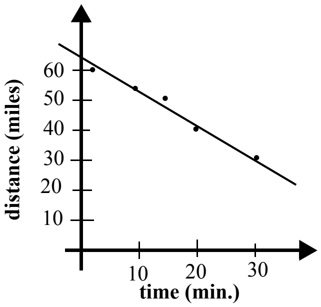
17. Answer: (C)
Human population in 2002 is about 6500.
40% of 6500 = $\frac{40 \times 6500}{100}$ = 2600
people have a dog in their household.

There are about 460 dogs in 2002 and each one is owned by a household. That means 2600 people, corresponding to 460 households.

Hence the average household has 2600/460 = $5.65 \cong 6$ people. The answer is (C).

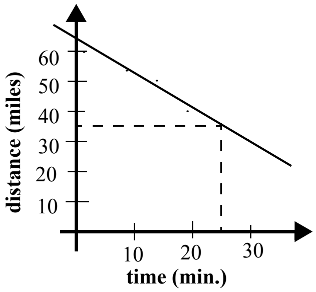
18.

a.



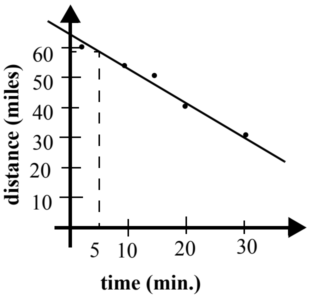
b. Answer: (C)
From the above graph: The distance at t = 0 is about 65 miles. The answer is (C).

c. Answer: (D)
The speed is the distance traveled per unit time. You can use any two points on the line to calculate the speed.

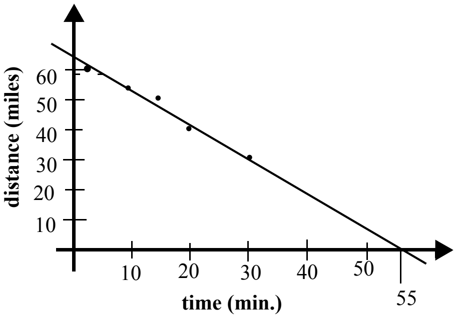


In the figure, we choose t = 0, d = 65 miles and t = 25 min., d = 35 miles to calculate the speed, which is also the absolute value of the slope of the line in the figure. In the first 25 minutes the car traveled 65 - 35 = 30 miles. So the speed is: 30/25 = 6/5 miles/min. = 72 miles/hour. The nearest approximation is 70 miles/hour. So the answer is (D).

d. Answer: (A)
As shown in the figure, the distance from the destination is about 58 miles at t = 5 minutes. The closest answer is 60 miles. The answer is (A).



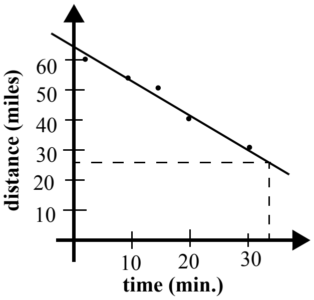
e. Answer: (C)



As shown in the figure, you can extrapolate the original graph to determine the total time. It is clear that the car reaches its destination in about 55 minutes. The answer is (C).

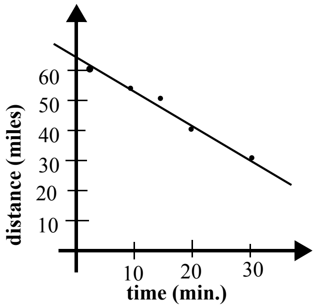
f. Answer: Around 34 minutes.

You can read the time as approximately 34 minutes as shown in the figure.



g. Answer: $d = -\frac{6}{5}t + 65$

The equation for distance vs. time is the equation for the line in the figure. -6/5 is the slope of this line and 65 is its y-intercept.



So the equation of the line is $d = -\frac{6}{5}t + 65$, where d is the distance in miles and t is time in minutes.

Sets

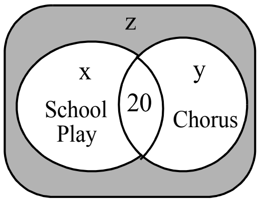
1. Answer: Point P
Since line m and line n cross at point P, the intersection of these two sets is point P.

2. Answer: Point P
Since $\overline{CD}$ and $\overline{EF}$ cross at point P, the intersection of these two sets is point P.

3. Answer: $\varnothing$
Since $\overline{CD}$ and $\overline{EF}$ don't cross, the intersection of these two sets is null, $\varnothing$.

4. Answer: 100
x - is the number of seniors in the school play only.

y - in the number of seniors in the chorus only.



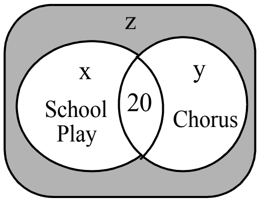
z -is the number of seniors who are not in the school play or in the chorus.

80 seniors are either in the school play only or in the chorus only or neither ➔ $x + y + z = 80$ ➔

Total number of seniors =
$x + y + z + 20 = 80 + 20 = 100$

5. Answer: 95
x - is the number of seniors in the school play only.

y - is the number of seniors in the chorus only.



z -is the number of seniors who are not in the school play or in the chorus.

w - is the number of seniors both in the school play and the chorus.

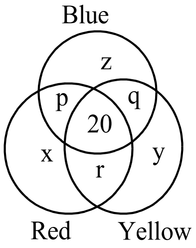
50 seniors are either in the school play or in the chorus, but not both. ➔ $x + y = 50$

65 seniors are in at least one of the two activities.
➔ $x + y + w = 65$ ➔ $50 + w = 65$ ➔ $w = 15$

80 seniors are either in the school play only or in the chorus only or neither. ➔ $x + y + z = 80$

Total number of seniors =
$x + y + z + w = 80 + 15 = 95$

6. Answer: 350
x - number of red fish.
y - number of yellow fish.
z -number of blue fish.
p - number of red and blue fish.
q - number of blue and yellow fish.
r - number of red and yellow fish.



80 of them are at least red and yellow ➔
$r + 20 = 80$ ➔ $r = 60$

70 of them are at least blue and red ➔
$p + 20 = 70$ ➔ $p = 50$

40 of them are at least blue and yellow ➔
$q + 20 = 40$ ➔ $q = 20$

Total number of fish =
x + y + z + p + q + r + 20 =
x + y + z + 50 + 20 + 60 + 20 = 500 ➔

Number of single-color fish =
x + y + z = 500 - 150 = 350

Defined Operators

1. Answer: 120
^^5 = ^(^5) = ^(1 + 2 + 3 + 4 + 5) =
^15 = (1 + 2 +…+ 15) = 120

2. Answer: (D)
A is 0, B is 10, C is 66 and E is 90. D has 10 @'s. 10 can not be unit's digit.

3. Answer: (C)
From 0 to 99, there are 100 numbers.

4. Answer: (E)
Each & switches 5 to 1/5 and 1/5 to 5. Even number of operator "&" operating on a number will result in the number itself. Since there are 12 "&" operators in the question, the result is 5.

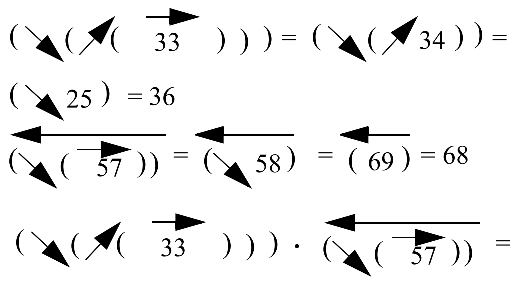
5. Answer: (C)
Each $!x pair first divides x by 100 and multiplies the result by 10. In effect $! divides the number in front of it by 10.

Since there are 3 $! pairs in $!$!$!30,
$!$!$!30 = 30/1000 = 3/100
!30 = 30/100
Hence the answer is

$$\frac{3}{100} \div \frac{30}{100} = \frac{3}{100} \cdot \frac{100}{30} = \frac{1}{10} = 0.1$$

6. Answer: 2448

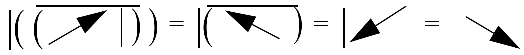
0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69



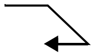
36 × 68 = 2448

7. Answer: 9 or 11
$-(3\nabla5)\Delta(c\nabla8) = c\nabla10$ ➔
$-(5-3)\Delta(8-c) = 10-c$ ➔
$-2\Delta(8-c) = 10-c$
$\frac{1}{8-c+2}$ = 10 - c ➔ $\frac{1}{10-c}$ = 10 - c ➔
$(10 - c)^2 = 1$ ➔
10 - c = 1 ➔ c = 9 or
10 - c = -1 ➔ c = 11

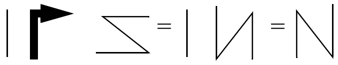
8. Answer: (B)



The answer is (B).



9. Answer:



10. Answer: {73, 74} and {48, 50}
x^y = x + x + 1 + x + 2 + ... + y =
(x + y)(y - x + 1)/2 = 147 ➔
(x + y)(y - x + 1) = 294

Note that in the second step above, we employed the method explained in the section Sums, subsection Sum of Consecutive Integers, both in this chapter. In this step, x + y is the sum of the first and the last integers, respectively. y - x + 1 is the number of integers between x and y.

To find the possible values, let's consider two cases:

Case 1:
Let y - x = 1 ➔ y = x + 1 ➔
(x + y)(y - x + 1) = 2(2x + 1) = 294 ➔
x = 146/2 = 73 and y = 74

Case 2:
Let y - x = 2 ➔ y = x + 2 ➔
(x + y)(y - x + 1) = 3(2x + 2) = 294 ➔
x = 48 and y = 50

Quicker Solution:
Assume that x and y are consecutive integers.
Then, x = (147 - 1)/2 = 73 and y = 73 + 1 = 74

Assume that x and y are two consecutive even integers, separated by an odd integer to make the sum odd.
Then, x + x + 1 + x + 2 = 147 ➔ 3x + 3 = 147 ➔
x = (147 - 3)/3 = 48 and y = 48 + 2 = 50

Logic

1. Answer: (B)
Case I is wrong because “Liz has a red car” and “They went to a party in a red car” does not mean that the red car they have used to go to the party is Liz’s car.

Case II is wrong. Nothing is said about who is driving.

III is correct. Peter has only one car and it’s blue. Since they went to the party in a red car, it can not be Peter’s.

So the answer is (B)

2. Answer: V
Path is A, B, E, O, K, Q, V

3. Answer: (B)
Since there are 4 available colors, if a square has more than 3 neighbors, it has to stay unpainted.
Only in (B) none of the squares have more than 3 neighbors. The answer is (B).

4. Answer is (D)
All these cases may be correct, depending on which week of the year Joy chooses to follow her diet.

Case (I) can be correct if Wednesday happens to be an even day.

Case (II) can be correct because it is possible to have two consecutive odd days as indicated in the hint.

Case (III) can be correct, because Monday and Tuesday are two consecutive days. If Monday is an odd day, Tuesday will be even most of the time.

The answer is (D), all three cases.

5. Answer: (C)
Case (A) can be correct if Wednesday happens to be an odd day.

Case (B) can be correct because it is possible to have two consecutive odd days as indicated in the hint.

Case (C) can never be correct, because Saturday and Sunday are consecutive days and for Joy to eat eggs on both days, both of them must be even days. But there are no consecutive even days in a year.

6. Answer: (E)
Case (A) can not be correct all the time. If Wednesday happens to be an even day, it is wrong.

Case (B) can not always be correct. In fact, most of the consecutive days are not both odd.

Case (C) is always wrong, because Saturday and Sunday are consecutive days and for Joy to eat eggs on both days, both of them must be even days. But there are no consecutive even days in a year.

Case (D) is correct only if Monday and Tuesday are not both odd.

The answer is (E).

Statistics

1. Answer: (E)
Unless the elements in the group has some order, the information about the average is not enough to determine the median. With the information given in the question, you can not draw any conclusions. The answer is (E).

2. Answer. (A)
Since there are 5 (odd number) employees, median age corresponds to the actual age. There are two employees younger then 41 and there are two employees older then 41 and one employee at 41 years old. The answer is (A).

3. Answer: (B)
Students’ score in the second exam is greater than the students’ score in the first exam because in the first exam, the students answered
$\frac{8}{10}100 = 80\%$ of the questions correctly.
In the second exam, the students answered
$\frac{8}{9}100 = 88.9\%$ of the questions correctly.

4. Answer: 8.8
Ten cards with maximum values are three 10s, three 9s, three 8s and one 7. The average of these 10 cards is
(10 + 10 + 10 + 9 + 9 + 9 + 8 + 8 + 8 + 7)/10 =
(30 + 27 + 24 + 7)/10 = 88/10 = 8.8

5. Answer: 9
Ten cards with maximum values are three 10s, three 9s, three 8s and one 7. The median of these cards is 9.

6. Answer: 10
Ten cards with maximum values are three 10s, three 9s, three 8s and one 7. The highest mode of these cards is 10.

7. Answer: (C)
Since there are 2 students in each age group, the ages of the students in the group are increasing regularly. Hence the average is equal to the median. So the answer is (C).

8. Answer: Average: 16, Median: 16
The median day in January is 16. There are 15 days before 16th of January and 15 days after the 16th of January. Since the days of the month are increasing regularly, the average day of each month is the same

as the median day of each month. Hence the average day in January is also 16.

9. Answer: (C)
"One less and one more of each element of the set is also an element of the set" means the elements of the set is increasing regularly. Hence the median and the average of the set are the same. The answer is (C).

10. Answer: 70 feet
Mode: 70 feet, since the water level remained at 70 feet the longest time, 30 days.

11. Answer: 50 feet
During the first 30 days, the water level increased steadily and reached 70 feet. The average water level is (10 + 70)/2 = 40 feet.

In the next 30 days the water level remained the same at 70 feet.

During the last 30 days the water level decreased steadily and dropped down to 10 feet. The average water level for this period is (70 + 10)/2 = 40 feet.

Since the number of the days in each time interval are equal (30 days), you can take the average of the averages to calculate the average water level for the whole 90 days. The average water level is (40 + 40 + 70) / 3 = 50 feet.

12. Answer: 55 feet
Total period is 90 days. Water level must be less than the median for 45 days and more than the median for 45 days. Half of the lower water levels, 45/2 = 22.5 days, is in the beginning of the 90-day period and the other half is at the end of the 90-day period. So starting from 10 feet high, the water level will reach its median in 22.5 days. Since it is increasing at a rate of (70 - 10)/30 = 2 feet/day, the median is
$10 + 22.5 \cdot 2 = 55$ feet.

13. Answer: 2.5 miles/hour.
The whole walk took 80 minutes. Since the average speed is 2 miles/hr. = 2/60 miles/min., the distance she traveled is $80 \cdot 2/60 = 8/3$ miles.

Let x be the number of miles that she traveled during the final part of her walk. Then
8/3 =
distance for the first part +
distance for the final part =
$30 \cdot 2/60 + x = 1 + x$ ➔
x = 8/3 – 1 = 5/3 miles.

Her speed = distance/time =
(5/3)/(40/60) = 15/6 = 2.5 miles/hour.
Note that 40 minutes = 40/60 hours.

14. Answer: 11.5 hours
Below is the chart of population and their ages, tabulated by days.

Day Number	Age in Days					
	0	1	2	3	4	5
1	1					
2	2	1				
3	6	2	1			
4	18	6	2	1		
5	54	18	6	2	1	
6	162	54	18	6	2	1^{st} one died

On day 1, there is only one specimen at age 0.
On day 2, there are 2 offsprings (of the first one) at age 0 and the original one reaches age 1.
On the 3rd day, the two offsprings of the first specimen produces 2 offsprings (grandchildren of the first one) each and the original one will have its own two offsprings, a total of 6 new born, 2 one-day old and 1 (the original one) 2-days old.

In the same way, the population is calculated for each age, for the rest of the days, and displayed in the following table.

Average Age on the 6th day =

$$\frac{162 \cdot 0 + 54 \cdot 1 + 18 \cdot 2 + 6 \cdot 3 + 2 \cdot 4}{162 + 54 + 18 + 6 + 2} \cong 0.479 \text{ day}$$

$\cong 11.5$ hours

Sums

1. Answer: $352
There are (20 - 10 + 1) = 11 days between the 10th and the 20th of the month. If John puts $4 aside for each day for 11 days each month for 8 months, he will have $11 \times 4 \times 8$ = $352 by September 1st.

2. Answer: 205,860
The even integers between 200 and 929 can be expressed by 2n, where n is an integer and
$100 \le n \le 464$

The sum, S, in question is
$S = 2S_n$,where S_n is the addition of consecutive integers between 100 and 464.

S = 2(464 - 100 + 1)(464 + 100)/2 =
$365 \times 564 = 205860$

3. Answer: 206225
The odd integers between 200 and 929 can be expressed by 2n + 1, where n is an integer and
$100 \le n \le 464$

The sum, S, in question is
$S = 2S_n + S_1$, where S_n is the sum of consecutive integers between 100 and 464 =
(464 - 100 + 1)(464 + 100)/2 = 102930
and S_1 = 464 - 100 + 1 = 365

$S = 2 \times 102930 + 365 = 206225$

4. Answer: (E)
This question is answered best by using two of the guessing techniques presented in Chapter 4. Here is how:

If case (A) is correct, sum of the odd integers between the two odd numbers is equal to the sum of the even integers between the two odd numbers. If this were the case, case (B) would also be correct. Since both case (A) and case (B) can not be correct, you can eliminate both.

Since both boundaries, a and b, are odd, neither a nor b is counted toward the sum of even numbers. So you can argue correctly that the sum of the even integers between a and b is less than the sum of the odd integers between a and b. However, all the remaining choices are less than the sum of even numbers.At this point you can use "the solution by example" method to choose between (C), (D) and (E). Here is how:
Let a = 1 and b = 5. Then, the addition of the odd numbers is 1 + 3 + 5 = 9 and the addition of the even numbers is 2 + 4 = 6. Since 6 = 9 - (5 + 1)/2, you can correctly guess the answer as (E).

Proper Solution:

The sum of the even integers between the two odd integers, a and b, is:
The average of these even integers multiplied by their quantity.
Since the even integers are regularly increasing, their average equals their median =
(a + b)/2
The number of the even integers between the two odd integers, a and b, is (a - b)/2
Hence the sum of even integers between the two odd integers, a and b, is:

$$\frac{a+b}{2} \cdot \frac{b-a}{2} = \frac{b^2-a^2}{4}$$

The sum of the odd integers between the two odd integers, a and b, is:
Average of these odd integers multiplied by their quantity.
Since the odd integers are regularly increasing, their average equals their median =
(a + b)/2
The number of odd integers between the two odd integers, a and b, is (a - b)/2 + 1
Hence the sum of the even integers between the two odd integers, a and b, is:

$$\frac{a+b}{2} \cdot \left(\frac{b-a}{2} + 1\right) = \frac{b^2-a^2}{4} + \frac{a+b}{2} =$$

(Sum of the even integers between the two odd integers a and b) + (a + b)/2 ➔

(Sum of the even integers between the two odd integers, a and b) =

(Sum of the odd integers between the two odd integers a and b) - (a + b)/2

The answer is (E).

5. Answer: (a + b)/2
Since the even integers are regularly increasing numbers, the average and the medium is the same and it is the number in the middle of a and b: (a + b)/2

Sequences

1. Answer: 8
The 3rd element is the median. The first three elements of the sequence are 2, 3, 8. So the median is 8.

2.

a. Answer: {-5, -9, -13, -17, ... }

b. Answer: -4n - 1
-5 - 4(n - 1) = -5 - 4n + 4 = -4n - 1

c. Answer: -69
$a_{17} = -4 \cdot 17 - 1 = -69$

d. Answer: -1224
The sum of all the elements is
$-4S_n - S_1$, where
S_n is the sum of integers from 1 to 24 and
S_1 is the sum of 24 "one"s.
$S_n = (24 - 1 + 1)(1 + 24)/2 = 300$
$S_1 = 24$

So the sum of all the elements in the sequence is $-4 \cdot 300 - 24 = -1224$
-1200 - 24 = -1224

e. Answer: -79
The sum of the last 10 elements is
$-4S_n - S_1$, where
S_n is the sum of integers from 15 to 24 and
S_1 is the sum of 10 "one"s.
$S_n = (24 - 15 + 1)(15 + 24)/2 = 195$
$S_1 = 10$

So the sum of the last 10 elements in the sequence is $-4 \cdot 195 - 10 = -790$

The average of the last 10 elements:
-790/10 = -79

3. Answer: {-5, -1, 3, 7, 11, ... }
7th element is d(7 - 1) - 5 = 19 ➔ d = 4 ➔
The sequence starts with -5 and increases by 4 ➔
The first 5 elements are -5, -1, 3, 7, 11, ...

4. Answer: 21, 34, 55
This sequence is neither arithmetic nor geometric. After the first 2 terms, which are 0 and 1, each term is

the addition of the previous two terms. So the next three terms are
8 + 13 = 21, 13 + 21 = 34 and 21 + 34 = 55

5.

a. Answer: {2, 10, 50, 250, 1250, ... }

b. Answer: $2 \cdot 5^{n-1}$

c. Answer: 31,250
$2 \cdot 5^{7-1} = 2 \cdot 5^6 = 31250$

6.

a. Answer: 3

b. Answer: 7

c. Answer: 343
$\frac{3 \cdot 7^{5-1}}{3 \cdot 7^{2-1}} = 7^{4-1} = 7^3 = 343$

7. Answer: 50
Let a be the first element of the sequence.

The sum of the first 3 elements =
$a + a/5 + a/25 = 31a/25 = 62$ ➔
$a = 50$

8. Answer: 275
Let x be the original number of bacteria.
In half an hour the bacteria doubles 30/5 = 6 times. At the end of half an hour, the number of bacteria =

$x \cdot 2^6 = 17600$ ➔ $x = 17600/64 = 275$

9. Answer: (A)
If the median of 8th, 10th and 12th elements of an arithmetic sequence is 102, then the 10th element must be 102. Since the 11th element can not be the same as the 10th element, the answer is (A).

10. Answer: (D)
Let d be the difference between two consecutive elements.

Since the first element of the sequence is 3, the addition of the 8th and the 9th elements is:
$3 + d(8 - 1) + 3 + d(9 - 1) = 6 + 15d = 102$ ➔
$d = 6.4$

The 11th element is:
$3 + 6.4(11 - 1) = 67$

11. Answer: 90
This sequence contains all the positive integers, less than 100, except the squares of integers. So the number of elements is 99 - 9 = 90.

12. Answer: 30.26
The squares of 1, 2, 3, 4, 5, 6 and 7 are missing from the first 50 integers. Hence the first 50 elements are the positive integers from 1 to 57, except 1, 4, 9, 16, 25, 36 and 49.

The addition of the first 50 terms is
$(57 + 1)(57/2) - 1 - 4 - 9 - 16 - 25 - 36 - 49 = 1513$

The average of the first 50 terms is
$1513/50 = 30.26$

13. Answer: $-(-1)^n 2 \cdot 3^{8-n}$
Let a and r be the first element and the ratio of the consecutive terms of the sequence.
The 4th element of a geometric sequence is -162 ➔
$a(r)^{(4-1)} = ar^3 = -162$

The 7th element is 6 ➔ $ar^{(7-1)} = ar^6 = 6$ ➔

$ar^6/ar^3 = r^3 = 6/(-162) = -1/27$ ➔ $r = -1/3$ ➔

$ar^3 = -a/27 = -162$ ➔ $a = 4374 = 2 \cdot 3^7$ ➔

The nth element is

$4374 \cdot (-1/3)^{n-1} = -(-1)^n 2 \cdot 3^{8-n}$

14. Answer: {4374, -1458, 486, -162, 54, -18, 6, -2, 2/3, -2/9, ...}

15. Answer: (B)
Let a and d be the first element and the difference between the two consecutive elements.

The addition of 8th and 9th elements =
$a + d(8 - 1) + a + d(9 - 1) = 2a + 15d = 102$ ➔
$d = (102 - 2a)/15$

The 11th element is $a + d(11 - 1) = a + 10d =$
$a + 10(102 - 2a)/15 = 68 - a/3$

If the 11th element is an integer, a, the first element must be divisible by 3. Hence the answer is (B).

16. Answer: 8
If the 8th and the 10th terms of a geometric series are 2 and 4 respectively, then $ar^9/ar^7 = r^2 = 4/2 = 2$ ➔

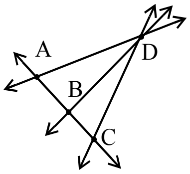
$r = \sqrt{2}$ or $r = -\sqrt{2}$

The ratio of 11th to 5th term: $ar^{10}/ar^4 = r^6 = 8$ for both values of r.

Counting

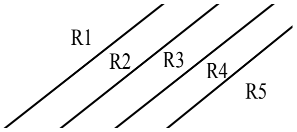
Basic Counting

1. Answer: 4
Let's draw these 4 points and the 4 lines passing through them.



2. Answer: 5

Minimum number of lines are obtained when all 4 lines are parallel as shown in the figure.



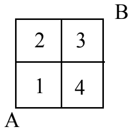
5 different regions, R1, R2, R3, R4 and R5, are also shown.

3. Answer: 261
(550 - 30)/2 + 1 = 261

4. Answer: 5

There are 5 different paths from A to B. They are:

123, 143, 13, 1243, 1423

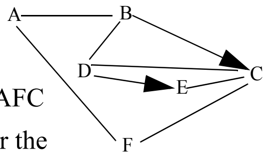


5. Answer: 7
Red, blue, yellow, red-blue, red-yellow, yellow-blue, red-yellow-blue.
The answer is 7.

6. Answer: 8

There are 4 different ways of going from A to C:
ABC, ABDC, ABDEC, AFC

There are only 2 ways for the return trip: CDBA, CFA.

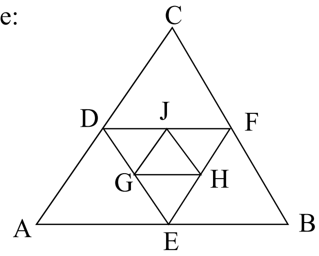


Since going from A to C and returning from C to A are independent events, the total number of possible ways is $2 \times 4 = 8$

7. Answer: 12

The 12 trapezoids are:

AEFC, EDCB,
ADFB, ADFE,
EDFB, DEFC,
GJFE, JHED,
GHFD, GHFJ,
GJHE, JHGD



Note that a parallelogram is a special trapezoid. See Chapter 6, Geometry, Quadrangles.

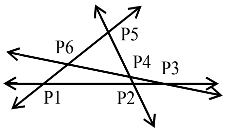
Advanced Counting

Combinations

1. Answer: 6

To form a point, you pick 2 lines out of four. So the number of points is

$$\frac{4!}{2!2!} = 6$$



You can also see four lines and all 6 points from P1 to P6 in the figure. Note that the order in which you pick the two lines is not important. The same point is created by the same two lines.

2. Answer: 24
4 Lines cross at a maximum of 6 points. (See the solution for the previous question.) Four angles are formed at each of these 6 intersections. Hence the maximum number of angles is $4 \times 6 = 24$

3.

a. Answer: 10
$$\frac{10!}{1!9!} = 10$$

b. Answer: 45
$$\frac{10!}{2!8!} = 45$$

c. Answer: 120
$$\frac{10!}{3!7!} = 120$$

d. Answer: 210
$$\frac{10!}{4!6!} = 210$$

e. Answer: 252
$$\frac{10!}{5!5!} = 252$$

f. Answer: 210
$$\frac{10!}{6!4!} = 210$$

g. Answer: 120
$$\frac{10!}{7!3!} = 120$$

h. Answer: 45
$$\frac{10!}{8!2!} = 45$$

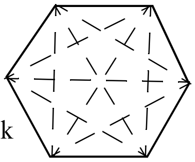
i. Answer: 10
$$\frac{10!}{9!1!} = 10$$

j. Answer: 1
$$\frac{10!}{10!0!} = 1$$. Note that 0! = 1

Prepare the graph yourself and notice the symmetry around 5 pens. The maximum number of choices, 252, is at 5 pens and the minimum number of choices, 1, is at 10 pens.

4. Answer: 9

A hexagon has 6 corners. A diagonal of the hexagon is a line segment that joins the two different corners. So the problem is "in how many ways can we pick pairs of 6 corners?"



However, 6 of these pairs are the 6 sides of the hexagon. Hence the answer is

$$\frac{6}{2! \times 4!} - 6 = 15 - 6 = 9$$

They are shown in the above figure.

5. Answer: $\frac{n!}{2!(n-2)!} - n$

Each diagonal of an n-sided polygon is formed by picking 2 corners out of n corners and combining them to form a line segment (diagonal). The total number of line segments formed this way is

$$\frac{n!}{2!(n-2)!}$$

However, n of these line segments are the sides of the polygon, so the total number of diagonals is

$$\frac{n!}{2!(n-2)!} - n$$

Permutations

1. Answer: 306

The order is important here: "President Joe and Vice President Mary" is different than "President Mary and Vice President Joe."

So the number of ways =

$$\frac{18!}{(18-2)!} = \frac{18!}{16!} = 17 \times 18 = 306$$

2. Answer: 110

The order is important here: "President Susan and Vice President Mary" is different than President Mary and Vice President Susan."

Since both president and the vice president will be girls, the answer is

$$\frac{11!}{(11-2)!} = 10 \times 11 = 110$$

3. Answer: 210

$$\frac{7!}{(7-3)!} = 5 \times 6 \times 7 = 210 .$$

If you have difficulty understanding this answer, read the explanations given to the previous two questions.

4. Answer: 12

There are 4 letters and we need to pick 2. The order is important. So the answer is

$$\frac{4!}{(4-2)!} = 3 \times 4 = 12$$

5. Answer: 24

There are 4 letters and we need to pick 3. The order is important. So the answer is

$$\frac{4!}{(4-3)!} = 2 \times 3 \times 4 = 24$$

6. Answer: 24

There are 4 letters and we need to pick 4. The order is important. So the answer is

$$\frac{4!}{(4-4)!} = \frac{4!}{0!} = 1 \times 2 \times 3 \times 4 = 24$$

Note that 0! = 1

7. Answer: 6

Since 1 and 5 are already reserved for the first and the last place, we have only 3 integers to choose from. Since the order is important in this choice, the answer is

$$\frac{3!}{(3-3)!} = \frac{3!}{0!} = 1 \times 2 \times 3 = 6$$

These six numbers are:
12345, 12435, 13245, 13425, 14325, 14235

8. Answer: 6

Below are the 6 possible sets of teams.
M,J – S,R – K,T
M,J – S,T – K,R
M,R – S,T – K,J
M,R – S,J – K,T
M,T – S,J – K,R
M,T – S,R – K,J

In the above notation, M for Mary, S for Sue, K for Kim, J for Joe, R for Rick and T for Tim are used.

Alternate Solution:

Let each girl sit in a chair and let the boys arrange themselves in different permutations in front of the girls to form pairs. The pairs formed this way depend on the order in which the boys arrange themselves. Since there are 3 boys, there are $\frac{3!}{0!} = 3! = 6$ different arrangements of the boys.
So the answer is 6.

Mutually Exclusive Events

1. Answer: 152

The number of ways of picking an all-girls team =

$$\frac{11!}{9!} = 110$$

The number of ways of picking an all-boys team =

$$\frac{7!}{5!} = 42$$

Since choosing an all-girls team excludes choosing an all-boys team (and vice versa), these two events are mutually exclusive and the total number of ways of choosing either an all-girls OR all-boys team is
110 + 42 = 152

2. Answer: 10
She can have two white OR two black OR two purple bracelets.

The number of ways she can have 2 white bracelets out of 3 is
$\frac{3!}{2!1!} = 3$
The number of ways she can have 2 black bracelets out of 4 is
$\frac{4!}{2!2!} = 6$
The number of ways she can have 2 purple bracelets out of 2 is 1.

These three events are mutually exclusive: Helen can only have two white or two black or two purple bracelets. Choosing one color excludes the other two. Thus, the total number of ways she can have two same-color bracelets is 3 + 6 + 1 = 10

3. Answer: 48
To have an even integer, you need to have an even number as the units digit. So the integer you create must end with 2 or 4.

If the units digit is 2, you are left with four integers, 1, 3, 4 and 5, to create the rest of the integer. Since the order is important, the number of integers you can create by using four integers is
$\frac{4!}{0!} = 24$

If the units digit is 4, you are left with four integers, 1, 2, 3 and 5, to create the rest of the integer. Since the order is important, the number of integers you can create by using four integers is
$\frac{4!}{0!} = 24$

Since the integer may have either 2 or 4 as its units digit, the total number of integers is 24 + 24 = 48

4. Answer: 72
The units digit of the integer may be 1 or 3 or 5. In each case, the number of integers that can be created is 24. (See the previous question). So the total number of integers that can be created is 24 + 24 + 24 = 72

Independent Events

1. Answer: 20
There are 5 different ways to pick a T-shirt from 5 different T-shirts. There are 4 different ways to pick a short from 5 different shorts. Since picking a short and picking a T-shirt are two independent events (one does not effect the other), the total possible outfits is $5 \times 4 = 20$

Note that we did not use the general formula to find the unique possible ways of picking a short or a T-shirt, because the answer is quite obvious. Nevertheless, it fits to the general formula:
The number of ways to pick one short out of 4 is 4!/(1!3!) = 4

2. Answer: 9
$3 \times 3 = 9$

3. Answer: 77
$11 \times 7 = 77$

4. Answer: 154
There are 11 ways of choosing a girl for president. There are 7 ways of choosing a boy for vice president.

The number of ways choosing a girl president and a boy vice president is $11 \cdot 7 = 77$

The number of ways choosing a girl vice president and a boy president is $11 \cdot 7 = 77$

Since these two events are mutually exclusive, the total number of ways choosing a president and a vice president of different gender is
77 + 77 = 154

5. Answer: 2520
There are $\frac{10!}{3!7!} = 120$ ways of picking three different color pencils out of 10.
There are $\frac{7!}{2!5!} = 21$ ways of picking two different color papers out of 7.

Since picking the pencils and picking the papers are two independent events, the number of possible combinations is $120 \cdot 21 = 2520$

6. Answer: 48
Because the table has two ends, Kim has two different ways of picking her seat.

The remaining four seats will all be picked by four of Kim's friends. The order (who is sitting in which chair) is important in this case. So the total number of permutations is 4!/(4 - 4)! = 4! = 24

Since Kim's picking her seat does not effect the way her friends pick their seats, the final number of permutations is $2 \cdot 24 = 48$

7. Answer: 2,308,743,493,056
The number of ways the first 6 can be chosen to form the first group is
$\frac{24!}{6!18!} = 134596$

Once the first group is chosen, only 24 - 6 = 18 students remain. The number of ways in which 6 students out of 18 can be chosen to form the second group is

$\frac{18!}{6!12!} = 18564$

After the second group is chosen, only 18 - 6 = 12 students remain. The number of ways in which 6 students out of 12 can be chosen to form the third group is

$\frac{12!}{6!6!} = 924$

After the third group is chosen, only 12 - 6 = 6 students remain who form the last group.

The total number of ways the groups can be formed is

$134596 \times 18564 \times 924 = 2{,}308{,}743{,}493{,}056$

8. Answer: (C)
Since the group will have an equal number of girls and boys, we can first select 3 girls and 3 boys separately. The number of ways 3 girls (boys) can be selected from 12 is

$\frac{12!}{3!9!} = 220$

It means that for each of 220 possible girls group, there will be 220 possible boys groups. Hence the total number of ways the grouping can be done is

$\left(\frac{12!}{3!9!}\right)\left(\frac{12!}{3!9!}\right) = \left(\frac{12!}{3!9!}\right)^2$

So the answer is (C).

Probability

1. Answer: 1/46656
The probability of getting 6 for one dice is 1/6. Since rolling one dice does not effect the result of the other, the probability of getting all 6s for all 6 dice is equal to the multiplication of all six probabilities.

$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{46656}$

2. Answer: 3125/7776
The probability of not getting 6 for one dice is 5/6. Since rolling one dice does not effect the result of the other, the probability of not getting any 6s is the multiplication of all five probabilities.

$\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = \frac{3125}{7776}$

3.

a. Answer: Column B
Probability of getting 2 sixes when you roll 2 dice =

$\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$

Probability of getting the first dice six and the second one five when you roll 2 dice =

$\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$

You can also get the first dice five and the second one six. Hence the total probability of getting one dice 6 and another one 5 is

$\frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$

Column B is bigger than Column A.

b. Answer: They are equal.
As far as the probabilities are concerned, rolling two dice at once and rolling the same dice twice are the same. Column A and Column B are equal.

4. Answer: 10/29
Total number of candies = 12 + 7 + 10 = 29
Since there are 10 yellow candies, the probability of picking a yellow candy is 10/29.

5. Answer: (C)
The probability of getting a yellow candy first is 10/29.

Once you pick a yellow candy, the total number of candies becomes 29 - 1 = 28. So the probability of getting a red candy the next time is 12/28

The probability of getting a yellow candy first and a red candy the next time =

$\frac{10}{29}\cdot\frac{12}{28}$

So the answer is (C).

6. Answer: (A)
The probability of getting a yellow candy first is 10/29.

Once you pick a yellow candy, the total number of candies is still 29 because you put the first candy back into the box. So the probability of getting a red candy the next time is 12/29

The probability of getting a yellow candy first and a red candy the next time =

$\frac{10}{29}\cdot\frac{12}{29}$

The answer is (A).

7.

a. Answer: Column A
As you calculated in the previous two questions, Column A, $\frac{10}{29}\cdot\frac{12}{28}$, is bigger than Column B, $\frac{10}{29}\cdot\frac{12}{29}$.

b. Answer: Column B
The probability of getting a yellow candy first and a red candy the next time =

$\frac{10}{29}\cdot\frac{12}{28}$

Getting one yellow and one red candy can occur in two ways: The first pick is yellow and the second pick is red, or, the first pick is red and the second pick is yellow.

The probability of getting a red candy first and a yellow candy the next time = $\frac{12}{29} \cdot \frac{10}{28}$

Hence, the probability of getting one yellow and one red candy is $\frac{10}{29} \cdot \frac{12}{28} + \frac{12}{29} \cdot \frac{10}{28} = 2\left(\frac{10}{29} \cdot \frac{12}{28}\right)$

The Column B is bigger.

c. Answer: Column A
Probability of getting a yellow candy first and a red candy the next if you pick one candy at a time is
$\frac{10}{29} \cdot \frac{12}{29} = 0.148$ (See Question 5)

After picking a yellow candy once, the number of the yellow candies is reduced to 9 and the total number of candies is reduced to 28. So the probability of picking a yellow candy for a second time is 9/28.

The probability of getting two yellow candies in a row, if you pick one candy at a time is
$\frac{10}{29} \cdot \frac{9}{28} = \frac{90}{812} = 0.111$

Column A is greater than the Column B.

d. Answer: They are equal.
The probability of getting 2 yellow candies if you pick one candy at a time without replacing them is the same as picking the two yellow candies simultaneously. In either case, you can’t pick the same yellow candy twice.

8.

a. Answer: 1
Since the first candy you pick is always the first of the three different colors, you are allowed to eat it. So the probability of eating at least one candy is 1 (100%).

b. Answer: 0
Since the first candy you pick is always the first of the three different colors, you are allowed to eat it. So the probability of eating no candy is zero. In other words, you will eat at least one candy.

c. Answer: $\frac{2514}{22736}$
Eating only one candy means, you pick the same color candy three times in a row. Below table shows 3 different ways this can happen.

Picking Order	Probability
{Y, Y, Y}	$\frac{10}{29} \cdot \frac{9}{28} \cdot \frac{9}{28} = \frac{810}{22736}$
{R, R, R}	$\frac{12}{29} \cdot \frac{11}{28} \cdot \frac{11}{28} = \frac{1452}{22736}$
{G, G, G}	$\frac{7}{29} \cdot \frac{6}{28} \cdot \frac{6}{28} = \frac{252}{22736}$

The probability of eating only one candy is
$\frac{810}{22736} + \frac{1452}{22736} + \frac{252}{22736} = \frac{2514}{22736} \cong 0.11$

d. Answer: $\frac{60}{261}$
To be able to eat 3 candies, you need to pick different color candies each time you pick a candy.

There are six different ways to pick a different color candy. Below table shows each combination and its probability:

Picking Order	Probability
{Y, R, G}	$\frac{10}{29} \cdot \frac{12}{28} \cdot \frac{7}{27} = \frac{10}{261}$
{Y, G, R}	$\frac{10}{29} \cdot \frac{7}{28} \cdot \frac{12}{27} = \frac{10}{261}$
{R, Y, G}	$\frac{12}{29} \cdot \frac{10}{28} \cdot \frac{7}{27} = \frac{10}{261}$
{R, G, Y}	$\frac{12}{29} \cdot \frac{7}{28} \cdot \frac{10}{27} = \frac{10}{261}$
{G, Y, R}	$\frac{7}{29} \cdot \frac{10}{28} \cdot \frac{12}{27} = \frac{10}{261}$
{G, R, Y}	$\frac{7}{29} \cdot \frac{12}{28} \cdot \frac{10}{27} = \frac{10}{261}$

Since all six possible picking orders will result in three different candies, the probability of eating three candies is
$6 \cdot \frac{10}{261} = \frac{60}{261} \cong 0.23$, slightly less then 25%

9.

a. Answer: 1/84
The first 6 students picked must be male in order to have an all-female jury.

Since 6 out of 9 students are male, the probability of picking a male student in the beginning is 6/9.

After the first student is picked, 5 males remain out of 8 students. So the probability of picking another male student on the second turn is 5/8.

In the same way, the probabilities of picking a male student on the 3rd, 4th, 5th and 6th turns are 4/7, 3/6, 2/5, and 1/4 respectively.

Hence the probability of picking male students, 6 in a row is:

$$\frac{6}{9}\times\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4} = \frac{1}{84}$$

b. Answer: 1/84
Since 3 out of 9 students are female, the probability of picking a female student in the beginning is 3/9.

After the first student is picked, 2 females remain out of 8 students. So the probability of picking another female student on the second turn is 2/8 = 1/4.

Similarly, the probability of picking another female student on the third turn is 1/7.

So the probability of the first group being all female is:

$$\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7} = \frac{1}{84}$$

c. Answer: 1/84
In this case the first 3 students must be males and the next 3 students must be females. The probability of the first 3 students being male is

$$\frac{6}{9}\times\frac{5}{8}\times\frac{4}{7}\text{ and}$$

the probability of the next 3 students being female is

$$\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}$$

Hence the probably of having an all-female jury is:

$$\left(\frac{6}{9}\times\frac{5}{8}\times\frac{4}{7}\right)\cdot\left(\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\right) = \frac{1}{84}$$

d. Answer: 1/84
In this case one of the 3 groups must be all females.

It does not matter if it is the 1st, 2nd or 3rd group.

The probability of the 1st group being all female is 1/84. The probability of 2nd group being all-female is 1/84. The probability of 3rd group being all-female is also 1/84.

So the probability of one of the groups being all-female is

$$\frac{1}{84}+\frac{1}{84}+\frac{1}{84} = \frac{3}{84}$$

If you pick randomly from these three groups, the probability of getting the all-female group is 1/3.

So getting a group of all females and picking that group as jurors is

$$\frac{3}{84}\cdot\frac{1}{3} = \frac{1}{84}$$

You can see in this question that the probability of getting a particular outcome doesn't change no matter how you approach the problem.

10.

a. Answer: 1/25
Both Wei and Mary have five courses left for the second period: Biology, History, Geography, Music and Spanish. Each has 1/5 chance of picking biology for the second period. So the probability of both picking biology is

$$\frac{1}{5}\cdot\frac{1}{5} = \frac{1}{25}$$

b. Answer: 1/5
Both Wei and Mary have five courses left for the second period: Biology, History, Geography, Music and Spanish. The probability of their both picking the same particular course (like biology) is

$$\frac{1}{5}\cdot\frac{1}{5} = \frac{1}{25}$$

The probability of their both picking the same course from one of the five remaining courses is:

$$\frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25} = \frac{1}{5}$$

c. Answer: 4/5
Since the probability of their taking the same class is 1/5, the probability of their not taking the same class is

$$1-\frac{1}{5} = \frac{4}{5}$$

9

WORD QUESTIONS

There are many word questions in SAT. They come in at every level of difficulty and on every subject. In the previous chapters, you have already seen several examples of them. However, some students don't like the word questions and simply skip them. Hence we include this chapter to emphasize the importance of word questions, to present you methods to tackle them and provide you with extra exercises.

The word questions in SAT are in three general categories:

1. Regular Word Questions.
2. Formulation Only.
3. Describing Figures

In this chapter, you will find examples and exercises in each category.

Regular Word Questions

In a typical word question, the situation is described instead of formulated. You need to convert the statements into formulas and/or equations first and then solve them.When answering “word questions”, write a formula for each statement as you read. Especially long word questions may look confusing if you don’t break them down into smaller parts and solve them one step at a time. The most difficult part of these long word questions is to read them without losing your concentration.

Examples:

1. (Easy)
Gary is two years younger that his brother and four years older than his sister. If his brother is 12 years old, how old is his sister?

Solution:

Gary is two years younger that his brother and his brother is 12 years old. ➔
Gary is 12 - 2 = 10 years old.

Gary is four years older than his sister. ➔
His sister is 10 - 4 = 6 years old.

2. (Medium)
Karen collected donations from seven people for the school band. Minimum donation was \$7 and the maximum was \$13. Which of the following can be the total amount of money that Karen collected?

(A) \$20

(B) \$40

(C) \$50

(D) \$65

(E) 87

Solution:

If 6 people donate only the minimum amount, \$7, and one person donates the maximum amount, \$13, the total amount will be $(6 \times 7) + 13 = \$55$

If 6 people donate the maximum amount of \$13 and only one person donates the minimum amount of \$7, the total amount will be $(6 \times 13) + 7 = \$85$

These are the lower and upper limits of the money collected. Among the answer choices, only \$65 is within these limits. Hence the answer is (C).

3. (Medium)
The first two terms of a sequence is 2 and 3. Each term after the first two terms is the addition of the previous two terms if the addition is 10 or less. If the addition of the previous two terms is greater than 10, then the term is 0. What is the 67th term?

Solution:

Let’s write the first few terms, until the pattern becomes obvious.The first 13 terms of the sequence is:
{2, 3, 5, 8, 0, 8, 8, 0, 8, 8, 0, 8, 8, ...}

Note that 2 + 3 = 5 (third term)
3 + 5 = 8 (fourth term)
Since 5 + 8 = 13 > 10, the fifth term is 0.
8 + 0 = 8 (sixth term)
0 + 8 = 8 (seventh term)
Since 8 + 8 = 16 > 10 the eighth term is 0.

As you can see, after the first three terms, the rest are the repetition of the pattern 8, 0, 8. So after the first three terms, if the remainder of n/3 is 0 or 1 then the nth term is 8; if it is 2, the nth term is 0.

For example:
The remainder of 4/3 is 1. So the 4th term is 8.
The remainder of 5/3 is 2. Thus the 5th term is 0.
The remainder of 6/3 is 0. Hence the 6th term is 8.
The remainder of 67/3 is 1. So the 67th term is 8.

4. (Hard)
Joe had 79 on the first math test. On the 2nd test he had 5/4 of what he had in the 1st test. On the 3rd and final test he did well but he could not learn his grade. However, he had an “A” (average of 90 or above up to 100) for the quarter. What is the range of his grade for the 3rd test?

Solution:

“Joe had 79 on the first math test.” ➔
1st test = 79

“On the 2nd test he had 5/4 of what he had in the 1st test.” ➔
$2^{nd}\text{ test} = \frac{5}{4} \times 79 = 98.75$

“On the 3rd and final test he did well but he could not learn his grade.” ➔
3rd test = x

“However he had an “A” (average of 90 or above up to 100) for the quarter.)” ➔
$\frac{79 + 98.75 + x}{3} \geq 90$ ➔ $79 + 98.75 + x \geq 270$ ➔

$x \geq 92.25$

The answer is $100 \geq x \geq 92.25$

5. (Medium)
The area of a circle is A. What would be the area if you triple the radius of the circle?

(A) A/3

(B) 3A

(C) 9A

(D) 27A

(E) None of the above.

Solution:

The area of a circle is A. ➔ $\pi r^2 = A$ where r is the radius of the circle.

If you triple the radius of the circle, the area of the new circle is

$\pi(3r)^2 = 9(\pi r^2) = 9A$

The answer is (C).

6. (Hard)
Joe and his friend Sue combined their money to spend it together. Originally, Joe had ½ of what Sue had. They gave 6% of what Sue had to John and spend one dollar more than ¼ th of the rest of the money. They lost 10% what Joe originally had. At the end they are left with \$12. How much money did Sue have originally? Approximate your answer to the nearest cent.

Solution:

Let J and S be Joe's and Sue's money before they combine their money.

"Joe has ½ of what Sue has." ➔ $J = S/2 = 0.5S$

Let T be the total amount of money that Joe and Sue have together at the beginning.

"Joe and Sue combined their money." ➔
$T = J + S = 0.5S + S = 1.5S$

Let H be John's money.
"They gave 6% of what Sue had to John." ➔
$H = 0.06S$

Let R be the rest of the money after they gave H amount to John. ➔
$R = T – H = 1.5S – 0.06S = 1.44S$

Let P be the money that they spend.

"…and spend one dollar more then the ¼ th of the rest of the money." ➔
$P = 1+ (1.44S)/4 = 1 + 0.36S$

Let L be the money lost.
"They lost 10% what Joe originally had." ➔
$L = \frac{10J}{100} = 0.1 \times 0.5S = 0.05S$

Let F be the remaining money.
"At the end they have left with \$12" ➔
$F = R – P - L = 12$

Substitute R, P, and L to the last equation.

$1.44S –1 – 0.36S – 0.05S = 12$ ➔
$1.03S – 1 = 12$ ➔ $S = (12 + 1)/1.03 = 12.62$

Sue had \$12.62 originally.

7. (Hard)
Jack's age is the addition of his four sons' ages. If his sons were born three years apart, and if Jack was 41 years old last year, how old was he when his oldest son was born?

Solution:

Let x and y be Jack's age and his youngest son's age, respectively.
"...his sons were born three years apart..." ➔
The ages of his sons are y, y + 3, y + 6 and y + 9.

"Jack's age is the addition of his four son's age."
➔ $y + y + 3 + y + 6 + y + 9 = 4y + 18 = x$

"...Jack was 41 years old last year..." ➔
$x = 41 + 1 = 42$ ➔ $4y + 18 = 42$ ➔ $y = \frac{42 - 18}{4} = 6$
Since his youngest son is 6 years old, his oldest son is 6 + 9 = 15 years old. ➔
When his oldest son was born 15 years ago, Jack was 42 - 15 = 27 years old.

Formulation Only

Some of the SAT word questions state the facts and ask you to express them in a formula. These questions require abstract thinking. They are almost always in the "Hard" category. Here are some examples.

1. (Medium)
Ben runs h hours a day. If Jim runs two hours less than Ben each week, how long Ben runs every day on the average?

Solution:

Ben runs h hours a day. ➔ He runs 7h each week.

Jim runs two hours less than Ben each week. ➔
Ben runs 7h - 2 hours each week. ➔
He runs $\frac{7h - 2}{7}$ hours a day.

2. (Medium)
In a lab experiment, students measure the density, d, of a fluid. The results vary between 0.86 gr./cm^3 and 0.91gr/cm^3. Which of the following describes the result of this experiment most accurately and completely?

(A) $d > 0.86$

(B) $d < 0.91$

(C) $d - 0.885 = 0$

(D) $|d - 0.885| < 0.025$

(E) $|d - 0.885| > 0.025$

Solution:

The density of the fluid is more than 0.86 and less than 0.91.
Case (A) and (B) give lower or upper limits of the density, not both. So neither one is the complete answer.

Case (C) gives the mid value between the two limits but not the range. So it is not the complete answer.
Case (D) gives the range and the limits accurately. So it is the correct answer.

3. (Medium)
Sue is s years old. John's age is two-third of Sue's age. 10 years from now, what will be the ratio of Sue's age to John's age?

(A) 3/2

(B) 2/3

(C) $\frac{3(s+10)}{2s}$

(D) $\frac{3s+10}{2s+10}$

(E) $\frac{3s+30}{2s+30}$

Solution:

Let j be John's age.

"John's age is two-third of Sue's age." ➔
j = 2s/3

In ten years, Sue will be s + 10 years old and John will be 2s/3 + 10 years old.

The ratio of Sue's age to John's age in 10 years is

$$\frac{s+10}{\frac{2s}{3}+10} = \frac{3s+30}{2s+30}$$

The answer is (E).

4. (Hard)
A newly open business neither lost nor gained any money for the first three years. After three years, the business's profit, P, increased proportionally with the time, t, passed. Which of the following functions could be the company's profit on and after the 3rd year?

(A) $P(t) = -3 + 3t$

(B) $P(t) = -12 + 4t$

(C) $P(t) = 3t$

(D) $P(t) = 3 - t$

(E) $P(t) = t^2 - 9$

Solution:

Since P increased proportionally with time, P has to be a linear function of t with a positive slope. Since P = 0 at t = 3 years, P(3) = 0.

The only answer choice that satisfies both conditions is (B). It has a positive slope, 4, and $P(3) = -12 + 4 \times 3 = 0$

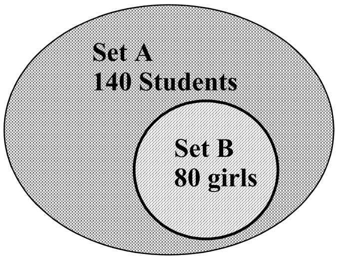
Describing Figures

In some other word questions, the figures, charts or geometrical shapes are described. To answer these questions, you need to draw the figure, construct the table etc. first. To do that, all you need to do is follow the instructions given in the question. Once the figure is drawn, it is easier to solve the problem. Here are some examples.

1. (Easy)
Set A is a set of students in Wilson Elementary whose ages are 8 or less. Set B is a set of girls in the same school whose ages are 8 or less. If the number of elements in Set A and set B are 140 and 80 respectively, how many boys are there in Wilson Elementary who are 8 years old or younger?

Solution:

Let's draw both sets.



As you see in the figure, Set B is a subset of Set A.

So the number of boys who are 8 years old or younger is
140 - 80 = 60

2. (Medium)
Peter is writing a paper. He estimated that it will take 40 hours to finish the job. He starts on the 5th of March to write his paper. On odd days he will work four hours a day. On the 10th of March he will take the whole day off. If the job needs to finish in 10 days, how many hours he needs to work on even number of days?

Solution:

Let's draw a chart showing Peter's work hours. In the below table, first row is the days of the month and the second row is the number of hours that Peter worked. In this table, x is the number of hours Peter worked on even days.

5	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
4	x	4	x	4	0	4	x	4	x

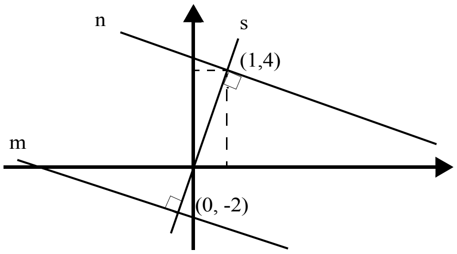
The addition of all the hours has to be 40. ➔
$4 \cdot 5 + x \cdot 4 = 40$ ➔ $x = 5$ hours.

3. (Hard)
Two lines n and m are parallel. A third line, s, passing through (0,0), is perpendicular to line n and intersects n at point (1, 4). If line m intersects the y axis at point -2, what is the intersection of s and m?

(A) (1, -4)
(B) (1, 2)
(C) (-1, -2)
(D) (-8/17, 32/17)
(E) (-8/17, -32/17)

Solution:

Let's draw the figure and write down the data on the figure.



Guess:

At this point, you can guess the answer easily.

The intersection of s and m has negative x- and y- coordinates. Only (C) and (E) have both coordinates negative.

But (C) can not be the answer because y- coordinate of the intersection must be larger (less negative) than -2. Therefore the answer is (E).

Proper Solution:

The equation for line s is the equation of a line passing through (1, 4) and (0, 0). It is $y = 4x$

The equation for line m is the equation of a line passing through (0, -2) and perpendicular to line s. So the slope of line m is -1/4 and y-intercept of it is -2. The equation of line m is

$$y = -\frac{1}{4}x - 2$$

The intersection of s and m is when

$4x = -\frac{1}{4}x - 2$ ➔ $x = -8/17$ ➔ $y = -32/17$

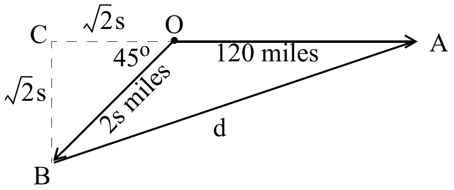
The answer is (E).

4. (Hard)
Two trains leave the same station at the same time. First one goes to East at 60 miles/hour.
The second train goes to South West with a constant speed, s. If after 2 hours the distance between them is d, which of the following relationships between d and s is correct?

(A) $d = 120 + 2s$
(B) $d^2 = 120^2 + 4s^2$
(C) $d^2 = 120^2 + 240\sqrt{3}s + 6s^2$
(D) $d^2 = 120^2 + 240\sqrt{2}s + 4s^2$
(E) $d^2 = 120^2 - 240\sqrt{2}s + 4s^2$

Solution:

Let's draw the locations of the trains after 2 hours as shown in the figure below.



After 2 hours, the first train is at location A, $2 \cdot 60 = 120$ miles away from the station O.

The second train is at location B, 2s miles away from the station O.

The distance between them = AB = d

In the figure, $\overline{OC}$ is the extension of $\overline{OA}$ and $\overline{CB} \perp \overline{OC}$.

Since the second train travels to South West, $\angle BOC = 45^o$. ➔ ΔOBC is an isosceles right triangle. ➔

$OC = CB = \frac{\sqrt{2}}{2}2s = \sqrt{2}s$ as shown in the figure.

ΔABC is a right triangle. Hence you can use the Pythagorean Theorem to find the relationship between d and s.

$d^2 = (120 + \sqrt{2}s)^2 + (\sqrt{2}s)^2 =$
$d^2 = 120^2 + 240\sqrt{2}s + 4s^2$

The answer is (D).

Exercises

1. (Easy)
Mr. Brown's yearly income was $30,000 last year. He gets 3% increase each year. What is his income this year?

2. (Easy)
ΔABC is a right triangle with $\angle A = 90°$.
If AB = 3 and AC = 4, what is BC?

3. (Easy)
Brown family spends one quarter of their monthly income on food, one-third of it on mortgage and utilities, two-sevenths on clothing and entertainment and the rest on child care. If their monthly income is $4200, how much they spend for child care?

4. (Medium)
The volume of a swimming pool is "v" cubic feet. How many hours are required to fill it, if it is filled at the speed of "s" cubic feet per hour?

(A) v/s
(B) s/v
(C) sv
(D) s^3v
(E) sv^3

5. (Medium)
The average of 8, and two other numbers is 10. What is the average of these two numbers?

6. (Medium)
In a basketball game, Mary scored four points in the first quarter. In the second quarter, she scored two less than twice as much as she scored in the first quarter. In the third quarter, she scored 5 less than what she scored in the second quarter. In the last quarter, she did not score. What is her average score per quarter?

7. (Medium)
In an arithmetic sequence, n^{th} element is x less than the m^{th} element. If the first element is y and n < m, what is the k^{th} element?

(A) y + xk/(n - m)
(B) y + xk/(m - n)
(C) y + x(k - 1)/(m - n)
(D) y + x(k - 1)/(n - m)
(E) y + x(n - m)/k

8. (Medium)
Peter is writing a paper. He works 3 hours on odd days and 5 hours on even days. After working 3 days, he takes every fourth day off. If he finished the paper by working a total of 38 hours on 2nd of April, when did he start writing his paper?

9. (Medium)
Line m passes through (2, 1) and has a slope of -1. Which of the following is WRONG?

(A) y-intercept of line m is positive.
(B) x-intercept of line m is positive.
(C) x and y intercepts of line m are equal.
(D) Line m goes through point (1, -1).
(E) Line y = x is perpendicular to line m.

10. (Medium)
The speed, s, of an accelerating object moving on a strait line is given by the formula: $s = s_o + at$, where s_o is the initial speed, "a" is the acceleration and "t" is the time. If the object starts off with a speed of 10 m/sec. and reaches 16 m/sec. in 2 minutes, what is its acceleration? Assume that the acceleration is constant.

11. (Medium)
The highest math grade in an exam is 300% higher than the lowest grade. 20% of the highest grade's 40% equals to what percent of the lowest grade's 30%?

12. (Medium)
Mary's mother's age is 6 times the difference between the ages of her two younger brothers' ages. If Mary is 12 years old and one of her brothers is 6 years old, which of the following can be her mother's age?

- **(A)** 34
- **(B)** 35
- **(C)** 36
- **(D)** 40
- **(E)** 42

13. (Medium)
The volume of a swimming pool is "v" cubic feet. How many hours are required to fill it, if is filled at the speed of "s" cubic feet per hour until the half of the pool is filled and then the speed at which the pool is filled is dropped to half?

- **(A)** 2v/s
- **(B)** v/2s
- **(C)** 3v/s
- **(D)** 3v/2s
- **(E)** v/3s

14. (Hard)
The addition of Jack's and Jill's ages is 72. When Jack was at Jill's current age, Jill's age was three-fourth of Jack's age. What is Jack's current age?

15. (Hard)
Sue is s years old. John's age is one-third of Sue's age. How many years from now John will be half of Sue's age? Give your answer in terms of s.

16. (Hard)
In a dam, the water level rises first at the speed of 1,000 gallons/day in the first time interval. Then it drops 1,500 gallons/day in the second time interval. If it took 5 days for the water level to reach its original value, how long it took to reach its maximum value?

17. (Hard)
The distance traveled by an accelerated object is proportional to its acceleration, a, and the square of the time passed, t. Which of the following best describes the distance, d, traveled by the object?

- **(A)** $d = a + t$
- **(B)** $d = a + t^2$
- **(C)** $d = c + a t^2$, where c is a constant.
- **(D)** $d = c(a + t^2)$, where c is a constant.
- **(E)** $d = c + at$, where c is a constant.

18. (Hard)
Set A contains four -legged animals in a zoo. Set B contains animals that can swim. Set C contains vegetarian animals.

Set D contains all the four legged animals and all the animals that can swim. Set E contains all the elements in Set A and Set C combined. Set F contains all the vegetarian animals that can swim.

Set G is the union of three sets:
1. The intersection of Set A and Set B.
2. The intersection of Set A and Set C.
3. Set F.

If a, b, c, d, e, f and g are the number of elements in Sets A, B, C, D, E, F and G, respectively, which of the following represents the number of elements of the intersection of Set A, Set B and Set C?

(A) $a + b - d - g$

(B) $a + b + c - d - e - g$

(C) $(a + b + c - d - e - g + f)/2$

(D) $2a + b + c - d - e + f - g$

(E) $(2a + b + c - d - e + f - g)/2$

19. (Hard)
An electrician charges “a” dollars for the first hour and “p” percent less for the additional hours. If he worked for $h > 1$ hours for a project, which of the following represents his average charge per hour?

(A) $\left(a + \left(\frac{p}{100}\right)a\right) \div 2$

(B) $\left(a + (h-1)\left(\frac{p}{100}\right)a\right) \div h$

(C) $\left(a + h\left(\frac{100-p}{100}\right)a\right) \div h$

(D) $\left(a + (h-1)\left(\frac{100-p}{100}\right)a\right) \div h$

(E) $\left(a + \left(\frac{100-p}{100}\right)a\right) \div 2$

20. (Hard)
A swimming pool is filled by three different water sources. The first one supplies water at the speed of “a” m^3/hr.; the second one supplies water with a speed of “b” m^3/hr.; and the third one supplies water at a speed of “c” m^3/hr. If all together they filled the pool in “h” hours, how many hours will take for only the first one to fill it?

(A) $\frac{h(a+b+c)}{a}$ hours.

(B) $\frac{h(a+b+c)}{3}$ hours.

(C) $\frac{a+b+c}{a}$ hours.

(D) $\frac{a+b+c}{3}$ hours.

(E) $\frac{h}{3}$ hours.

Answers

1.	\$30, 900	**6.**	2.75	**11.**	80%	**16.**	3 days
2.	5	**7.**	(C)	**12.**	(E)	**17.**	(C)
3.	\$550	**8.**	21st of March	**13.**	(D)	**18.**	(E)
4.	(A)	**9.**	Answer: (D)	**14.**	40	**19.**	(D)
5.	11	**10.**	0.05 m/sec.^{2}	**15.**	s/3	**20.**	(A)

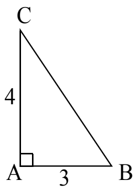
Solutions

1. Answer: \$30, 900
"Mr. Brown's income was \$30,000 last year. He gets 3% increase each year." ➔
This year's income is 3% more than \$30,000.

$$3\%\text{ of }30{,}000 = \frac{3 \times 30000}{100} = 900$$

His income this year is 30000 + 900 = \$30,900

2. Answer: 5
Once we draw the triangle, it is clear that $\overline{BC}$ is the hypotenuse of ΔABC.



Using the Pythagorean Theorem:

$$AB = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

3. Answer: \$550
They spend $\frac{1}{4} + \frac{1}{3} + \frac{2}{7} = \frac{73}{84}$ of their income for non-child care expenses. ➔

They spend $1 - \frac{73}{84} = \frac{11}{84}$ for child care. ➔

They spend $\frac{11}{84} \times 4200 = \$550$ for child care.

4. Answer: (A)
If in one hour, s cubic feet of the pool is filled, all of it is filled in v/s hours. The answer is (A).

5. Answer: 11
Let x and y be the two numbers in question.

"The average of 8 and two other numbers is 10" ➔
$(x + y + 8)/3 = 10$ ➔
$x + y = 3 \times 10 - 8 = 30 - 8 = 22$ ➔

The average of x and y = $(x + y)/2 = 22/2 = 11$

6. Answer: 2.75
"In the second quarter, she scored two less than twice as much as she scored in the first quarter." ➔
In the second quarter, she scored $2 \cdot 4 - 2 = 6$

"In the third quarter, she scored 5 less than what she scored in the second quarter." ➔
In the third quarter, she scored 6 - 5 = 1 points.

The average score of all four quarters is:

$$\frac{4 + 6 + 1}{4} = 2.75$$

Note that even though she did not score in the last quarter, we divide the total score by four, to find the average.

7. Answer: (C)
Let d be the difference between the two consecutive elements.

"nth element is x less than the mth element" ➔
$d = x/(m - n)$

Since the sequence is an arithmetic sequence, the kth element is
$y + d(k - 1) = y + x(k - 1)/(m - n)$

The answer is (C).

8. Answer: 21st of March
Let's draw a chart showing Peter's work hours each day. In preparing the below table, you need to start from 2nd of April and work your way back. First you go back enough to cover 38 hours without any break as shown below:

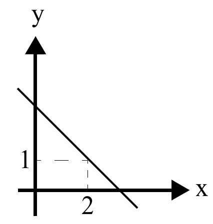
March								April	
24	25	26	27	28	29	30	31	1	2
5	3	5	3	5	3	5	3	3	5

In the above table, the first row displays the days in March and April and the second row displays the number of hours Peter works on each day. It is clear that it takes 10 days to finish the project if he takes no breaks. It means, throughout the entire project, Peter will have 3 full days off. (10/3 = 3.33). To be able to take 3 days off, he must start on 24 - 3 = 21st of March, and not on 24th of March. As shown in the table below, if he starts working on March 21st, he will finish the paper on target. The answer is March 21st.

March											April	
21	22	23	24	25	26	27	28	29	30	31	1	2
3	5	3	0	3	5	3	0	3	5	3	0	5

The total number of hours that Peter works is $6 \times 3 + 4 \times 5 = 38$ hours.

9. Answer: (D)
Let's draw the line m.



In the figure, it is clear that line m does not go through (1, -1). So the answer is (D).

In the figure, you can also easily see that all the other cases are correct.

10. Answer: 0.05 m/sec.^{2}
"... the object starts of with a speed of 10 m/sec...." ➔ $s_0 = 10$

"... reaches 16 m/sec. in 2 minutes..." ➔
$s = 16$ and $t = 2 \times 60 = 120$ sec. ➔
$16 = 10 + 120a$ ➔ $a = 6/120 = 0.05$ m/sec.^{2}

11. Answer: 80%
Let H, and L be the highest and the lowest grades respectively and let x be the percentage in question.

"The highest math grade in an exam is 300% higher than the lowest grade." ➔ H= 3L

"20% of 40% of the highest grade equals to the 30% of x percent of the lowest grade." ➔

$$\frac{20}{100} \cdot \frac{40}{100} \cdot H = \frac{30}{100} \cdot \frac{x}{100} L \text{ ➔}$$

$$\frac{20}{100} \cdot \frac{40}{100} \cdot 3L = \frac{30}{100} \cdot \frac{x}{100} L \text{ ➔}$$

$$20 \cdot 40 \cdot 3L = 30xL \text{ ➔ } x = \frac{2400}{30} = 80\%$$

12. Answer: (C)
Since the mother's age is 6 times the age difference between the two younger brothers, mother's age has to be a multiple of 6. Answer choices (C) and (E) satisfy this condition.

If the 6 years old brother is the older of the two, the youngest brother can be a new born and thus 0 years old. In this case the age difference is 6 - 0 = 6 years and the mother's age is $6 \times 6 = 36$

On the other hand, let's assume that the 6 years old brother is the youngest of the two. Since both brothers are younger than Mary who is 12 years old, the oldest brother can be at most 11 years old. In this case the age difference is 11 - 6 = 5 years and the mother's age is $6 \times 5 = 30$, which is not one of the answer choices.

The answer is (C).

13. Answer: (D)
The first half of the pool is filled in $\frac{\left(\frac{v}{2}\right)}{s} = \frac{v}{2s}$ hours.

The second half of the pool is filled in $\frac{\left(\frac{v}{2}\right)}{\left(\frac{s}{2}\right)} = \frac{v}{s}$ hours.

The whole pool is filled in $\frac{v}{2s} + \frac{v}{s} = \frac{3v}{2s}$ hours. The answer is (D).

14. Answer: 40
Let x and y be Jack's and Jill's current ages, respectively.

"The addition of Jack's age and Jill's age is 72." ➔
x + y = 72 ➔ y = 72 - x

"d" years ago Jack was at Jill's current age, "y" ➔
x - d = y ➔ d = x - y

"When Jack was at Jill's current age, Jill's age was three-fourth of Jack's age." ➔
d years ago, Jill's age = y - d = $(x - d)\frac{3}{4}$
Substitute d = x - y into the above equation yields
$y - (x - y) = (x - (x - y))\frac{3}{4}$ ➔ $2y - x = \frac{3y}{4}$

Substituting y = 72 - x into the above equation yields

$$2(72 - x) - x = \frac{3(72 - x)}{4} \text{ ➔ } 144 - 3x = 54 - \frac{3x}{4} \text{ ➔}$$

$$144 - 54 = 3x - \frac{3x}{4} \text{ ➔ } \frac{9x}{4} = 90 \text{ ➔ } x = 40$$

Jack is 40 years old.

15. Answer: s/3
Let j be John 's age.

"Sue is s years old. John's age is one-third of Sue's age." ➔ j = s/3

Let y be the number of years that has to pass so that John's age will be half of Sue's age. ➔

$$2 = \frac{s + y}{j + y} = \frac{s + y}{\frac{s}{3} + y} = \frac{3s + 3y}{s + 3y} \text{ ➔}$$

2s + 6y = 3s + 3y ➔ y = s/3

16. Answer: 3 days
Let t_1 and t_2 be the first and the second time intervals, respectively.

The amount of increase in the first time interval is $1000t_1$.
The amount of decrease in the second time interval is $1500t_2$.

If the water level reaches its original value at the end, then $1000t_1 = 1500t_2$

"...it took 5 days for the water level to reach to its original value..." ➔ $t_1 + t_2 = 5$ ➔ $t_2 = 5 - t_1$ ➔

$1000t_1 = 1500(5 - t_1)$ ➔ $7500 = 2500t_1$ ➔
$t_1 = 3$ days

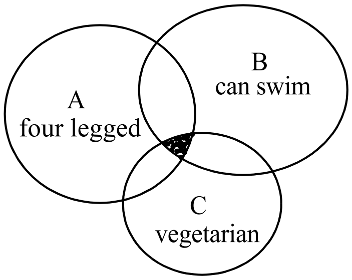
Since the water level is at its maximum at the end of the first time period, the answer is 3 days.

17. Answer: (C)
Since d is proportional to both a and t^2, d has to be proportional to at^2 as well. The only answer choice that gives this proportionality is (C).

In this case, c is the original distance in the beginning, when t = 0.

18. Answer: (E)

Let's draw the three primary sets: Set A, Set B and Set C.



Let the intersection of A and B be X and the intersection of A and C be Y. Let x and y are the number of elements in Set X and Set Y, respectively.

Since Set D is the union of Set A and Set B, $d = a + b - x$. In this equation $a + b$ counts the number of elements in the intersection of A and B twice. To find the correct number of elements in D, you need to subtract x from $a + b$.
$d = a + b - x$ ➔ $x = a + b - d$

Similarly $y = a + c - e$

Set F is the intersection of Set B and Set C. Let W be the intersection of sets X, Y and F and w be the number of elements in W. Note that W is the small shaded region in the above figure.

Set G is the union of three sets:
1. The intersection of Set A and Set B (Set X).
2. The intersection of Set A and Set C (Set Y).
3. Set F.

Since Set G is the union of the sets X, Y and F, $g = x + y + f - 2w$. Note that 2w is subtracted from the addition of $x + y + f$, because w is counted three times, one for each set. Hence two of them must be subtracted from the total.

$g = x + y + f - 2w$ ➔ $w = (x + y + f - g)/2$ ➔
$w = [(a + b - d) + (a + c - e) + f - g]/2 =$
$(2a + b + c - d - e + f - g)/2$

The answer is (E).

19. Answer: (D)

The total charge is the addition of the first hour charge, a, the charge for the remaining h - 1 hours. Since he charges p percent less for the additional hours, his charge for these hours is

$$\left(\frac{100 - p}{100}\right)a$$

His charge for h - 1 hours is

$$(h - 1)\left(\frac{100 - p}{100}\right)a$$

His total charge is

$$a + (h - 1)\left(\frac{100 - p}{100}\right)a$$

His average charge per hour is

$$\left(a + (h - 1)\left(\frac{100 - p}{100}\right)a\right) \div h$$

The answer is (D)

20. Answer: (A)

All three of them together will supply $(a + b + c)$ m^3 of water each hour. So the pool will be filled in

$h = \frac{v}{a + b + c}$ hours, where "v" is the volume of the pool. ➔ $v = h(a + b + c)$

Hence the first water supply alone will fill the pool in

$\frac{v}{a} = \frac{h(a + b + c)}{a}$ hours.

The answer is (A).

10

SAMPLE DIAGNOSTIC & PRACTICE TESTS

Test 1

Date:

General Directions:

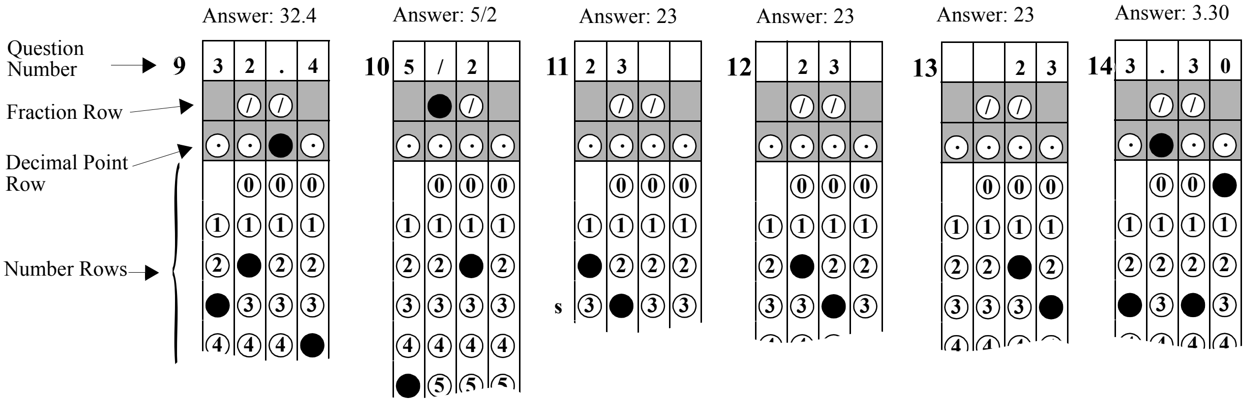
- The test is 70 minutes long.
- It has 3 sections.
  Section 1: 20 multiple choice questions, 25 minutes.
  Section 2: 8 multiple choice and 10 grid-in questions, 25 minutes.
  Section 3: 10 multiple choice questions, 20 minutes.
- You can only work on one section at a time.
- You are not allowed to transfer your answers from the test to the Answer Sheet after the time is up for each section. Therefore mark your answer on the Answer Sheet as soon as you finish answering a question.
- Make sure that the question number in the test matches the question number on the answer sheet.
- Mark only one answer for each question.
- If you want to change your answer, erase the old answer completely.
- You receive one point for each correct answer.
- You lose one point for 4 incorrect multiple choice answers.
- You don't lose any points for incorrect grid-in questions.
- You neither gain nor loose points for missing answers.

Directions for the Grid-In Questions

- You can only have zero or positive numbers as answers for the grid-in questions. Therefore, if your answer is negative it is wrong.
- The upper limit of your answer is 9999. If your answer is greater than 9999, it is wrong.
- You can express your answers as integers, fractions or decimal numbers. Do not spend any time to convert fractions to decimals or decimals to fractions.
- Mixed numbers have to be converted to fractions or decimals. For instance $3\frac{1}{2}$ has to be converted to $\frac{7}{2}$ or 3.5 before it is marked on the answer sheet.
- Write your answer in the first row. Note that this is optional and in real SAT it will not count as an answer. The only answer that counts is the answer that you mark on Fraction, Decimal Point and Number rows.
- If your answer has less than 4 characters, you can start from any column you wish. Leave the rest of the columns empty.
- If your answer has more than 4 characters, you can either mark the first 4 characters or you can round it to an appropriate 4-character number.

  For example, 10/3 = 3.33333333 can be marked in as 10/3, 3.33 or 3.30, but not as 3, 3.0, 3.3 or 3.4.
  200/3 can be marked in as 66.6 or 66.7. But 66 or 67 are not the correct answers.

Here are several examples to clarify the above statements.



Answer Sheet - Test 1

Section 1

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Section 3

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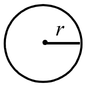
Test 1 - Section 1

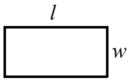
25 Minutes, 20 Questions

Reference Information

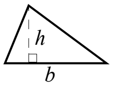


$A = \pi r^2$

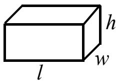
$C = 2\pi r$



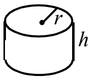
$A = lw$



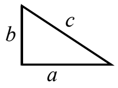
$A = \frac{1}{2}bh$



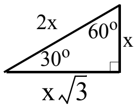
$V = lwh$

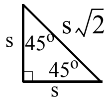


$V = \pi r^2 h$



$c^2 = a^2 + b^2$





Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

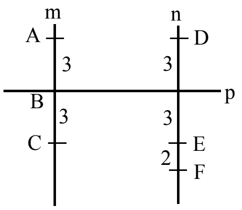
1. If x + 7 = 5, then x - 1 = ?

(A) -3
(B) -2
(C) 1
(D) 2
(E) 3

2. Set A = {a, c, e, k, m} and Set B are the vowels of the English alphabet. If Set C is all the members of Set A that are NOT in Set B, which of the following is Set C?

(A) {c, k, m}
(B) {a, c, e, i, j, k, m, o, u}
(C) {a, e}
(D) {a, c, e, k, m}
(E) {e, a, i, j, o, u}

3. In the figure, line m and line n are both perpendicular to line p. Which of the distances below is the longest?



(A) AC
(B) AE
(C) DE
(D) BD
(E) BF

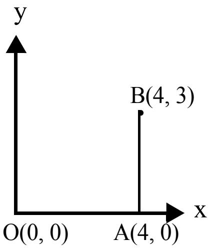
4. The median of three consecutive integers is 6. What is the sum of these numbers?

(A) 3
(B) 6
(C) 12
(D) 15
(E) 18

5. Sue has p/2 pounds of sugar to make cupcakes. If each cupcake uses c pounds of sugar, and k pounds of sugar is left after baking the cupcakes, how many cupcakes did Sue bake?

(A) (p/2 - k)/c
(B) (k - p/2)/c
(C) (p - 2k)/c
(D) pc - 2kc
(E) (p/2 - k)c

6. In the figure, what is the area of the triangle OAB (not drawn)?



- **(A)** 3
- **(B)** 4
- **(C)** 6
- **(D)** 7
- **(E)** 12

7. In a parking lot there are 36 cars. 1/3 of them are SUVs. 12 cars are white. If 3/4 of SUVs are not white, how many cars that are <u>not</u> SUV are white?

- **(A)** 3
- **(B)** 4
- **(C)** 8
- **(D)** 9
- **(E)** 12

1/3

8. a and b are two positive numbers.
If $a^2 + 4b + 1 = (2b + 1)^2$, then b = ?

- **(A)** b = a/2
- **(B)** b = -a/2
- **(C)** b = -a/4
- **(D)** b = a/4
- **(E)** b = 2a

9. a and b are unit digits of two 2-digit numbers, 2a and 1b, respectively. If

$$\begin{array}{r} 2a \\ 1b \\ + \\ \hline 44 \end{array}$$

which of the following values can the (a, b) pair have?

- **(A)** (10, 1)
- **(B)** (9, 2)
- **(C)** (9, 5)
- **(D)** (8, 5)
- **(E)** (10, 2)

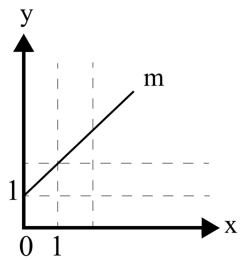
10. The height of a certain tree is given by the function, $h(t) = a + b\sqrt{t}$, where a and b are constants, h is the height in feet, t is time in years. When the tree was first planted, it was 2′ high and after 9 years it became 20′ tall. What is the value of b?

- **(A)** 2
- **(B)** 2.44
- **(C)** 6
- **(D)** 7.5
- **(E)** -6

1/2

11. Which of the following points is <u>not</u> on the line m shown in the figure?

- **(A)** (-1, 0)
- **(B)** (-1/2, 1/2)
- **(C)** (-1/2, 3/2)
- **(D)** (1, 2)
- **(E)** (2, 3)



12. The sum of 6 consecutive odd integers, a, b, c, d, e and f is 36. If a < b < c < d < e < f, what is the median of {a, b, c, d, e, f, 50}?

(A) 43
(B) 6
(C) 36
(D) 5
(E) 7

13. How many 4×2^6 is in 2^{12}?

(A) 4×2^6
(B) 2^4
(C) $2^4/2$
(D) 32
(E) 8

14. When an integer i is divided by 9 the remainder is 7. What is the remainder if 3i + 5 is divided by 9?

(A) 8
(B) 7
(C) 6
(D) 5
(E) 3

2/3

15. The median of three numbers, a, b and c, is 1.5. The arithmetic average of these numbers is 2. If a < b < c, which of the following statements must be true?

(A) (a + c)/3 = 1.5
(B) (a + c)/2 = 2
(C) a > 0
(D) 1.5 > a + c > 0
(E) (a + b)/3 = 1.5

16. A circle with the radius r is divided into four quarter circles. What is the perimeter of each quarter circle?

(A) $\pi\frac{r}{4}$
(B) $\pi\frac{r}{2}$
(C) $r\left(\frac{\pi}{2} + 2\right)$
(D) $r\left(\frac{\pi}{r} + 1\right)$
(E) $r\left(\frac{\pi}{4} + 2\right)$

17. In the figure, if AC > BC, which of the following CAN NEVER be true?

(A) y > x
(B) x < 30
(C) x > 30
(D) AB > AC
(E) y - x > 0

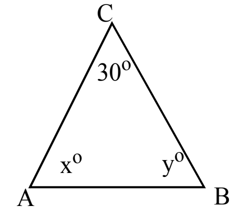


Figure is not drawn to scale.

18. ABCD is an equilateral quadrangle inscribed inside the rectangle EFGH.
If GD = AG/2, what is the area of EFGH?

- **(A)** $2\sqrt{2}$
- **(B)** 8
- **(C)** 4.8
- **(D)** 6
- **(E)** 12

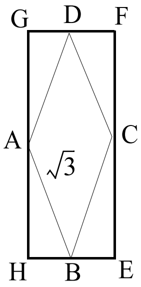


Figure is not drawn to scale.

19. If | 5 - | a | | > 3, which of the following may be correct?

- **I.** $a < -8$
- **II.** $-2 < a < 2$
- **III.** $a > 8$

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I or III
- **(E)** I or II or III

20. Out of 8 students, 2 of them will be randomly chosen to form a debate team. How many different teams are possible?

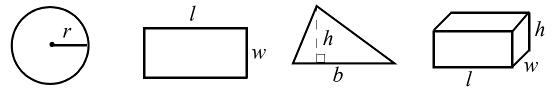
- **(A)** 56
- **(B)** 20160
- **(C)** 28
- **(D)** 8
- **(E)** 1

L

Test 1 - Section 2

25 Minutes, 18 Questions

Reference Information



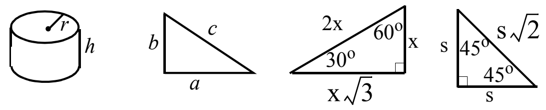
$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

Part 1

1. Which of the following numbers is not between 0.015 and 0.045?

- **(A)** 1/50
- **(B)** 0.025
- **(C)** 1/40
- **(D)** 1/30
- **(E)** 1/20

2. Two cylinders are placed side by side on a flat surface as shown in the figure. A and B are the centers of the circles on the top surfaces of the smaller and larger cylinders, respectively.

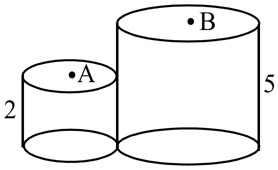


Figure is not drawn to scale.

The diameters of the circles A and B are 3 and 5, respectively. The heights of the cylinders are shown in the figure. What is the distance between A and B?

- **(A)** 2
- **(B)** 3
- **(C)** 4
- **(D)** 5
- **(E)** 6

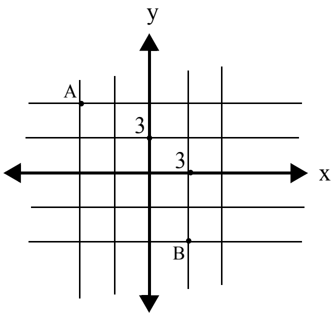
1/4

3. If $3^a = 27$ and $5^b = 125$, what is a - b = ?

- **(A)** 2
- **(B)** 1
- **(C)** 0
- **(D)** -1
- **(E)** -2

1/3

4. In the figure, what is the distance between the points A and B?



- **(A)** 5
- **(B)** 7
- **(C)** 9
- **(D)** 15
- **(E)** 21

1/2

5. If $0 < 2x < 1$, then $x^2 - x/2$ must be

- **(A)** Positive.
- **(B)** Zero.
- **(C)** Negative.
- **(D)** Greater than one.
- **(E)** Between zero and one.

6. In a deck of cards, each card is labeled by a positive integer. Ten of them have an integer greater than 18. If the probability of picking a card with number 18 or less is 2/3, how many cards are there in total?

- **(A)** 10
- **(B)** 15
- **(C)** 20
- **(D)** 25
- **(E)** 30

2/3

7. John has some money. Mary has \$6.00 more than John. Sue has two-thirds of the money that Mary has. Which of the following must be true?

- **(A)** Sue has \$0.01
- **(B)** Sue has \$1.00
- **(C)** Sue has \$4.00
- **(D)** Sue has more than \$4.00
- **(E)** Sue has less than \$6.00

8. $f(x) = x^2$ and $g(x) = 5x^2 - 9$
f(x) and g(x) cross each other at two points, A and B. What is the area defined by $\overline{AB}$, x-axis, line perpendicular to $\overline{AB}$ at point A and a line perpendicular to $\overline{AB}$ at point B?

- **(A)** 3
- **(B)** 9/2
- **(C)** 9/4
- **(D)** 27/4
- **(E)** 27/8

L

Part 2

9. 3 less than 1/2 of a number is 12. What is the number?

10. At the end of the year, an employer gives \$400.00 bonus to each employee who worked for the company 5 years or more. If the employer's bonus is 20% less for the employees who worked less than five years for the company, how much bonus do they get?

1/4

11. ΔABC is a triangle with AB = 5, AC = 3 and BC = 4.5. If ΔXYZ is another triangle with XY = 10, XZ = 6 and $\angle X = \angle A$, what is the value of YZ?

1/3

12. At the end of the year, an employer gives \$400.00 bonus to each employee. If the employer reduces the bonus by 20% and saves a total of \$6000.00 at the end of the following year, how many employees are working for the company? Assume that the number of employees remains the same during these two years.

13. Each of the 100 consecutive integers starting with 7 are divided by 3. What is the addition of the remainders?

1/2

14. John's phone schedule from 5/25/2004, Tuesday to 6/04/2004 is shown in the table below:

Date	Time	Number Called	Duration
5/25/2004	8:30 AM	222-222-2222	1 min., 23 sec.
5/27/2004	11:44 AM	222-222-2222	2 min., 01 sec.
5/27/2004	1:03 AM	333-333-3333	10 min., 59 sec.
5/29/2004	7:48 PM	222-222-2222	0 min., 27 sec.
5/30/2004	9:30 AM	222-222-2222	55 min., 23 sec.
6/03/2004	1:23 PM	222-222-2222	0 min., 23 sec.
6/04/2004	2:30 PM	333-333-3333	4 min., 03 sec.
6/04/2004	10:30 AM	222-222-2222	3 min., 23 sec.

Telephone company charges $0.20 per minute, Monday through Friday from 8:00 AM to 6:00 PM. In all other times, the charge is $0.10 per minute. If the minutes are approximated UP to the nearest higher minute, what is the total amount in dollars John has to pay for the calls he made to 333-333-3333 during the days shown in the table?

15. In the figure, lines m and n are parallel. If $u + v + y = 200$, what is the value of $w + z$ in degrees?

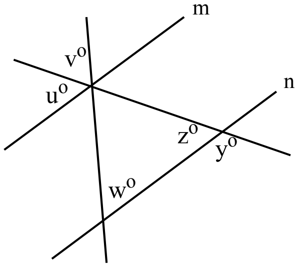


Figure is not drawn to scale.

2/3

16. Figure below shows some of the 11 distinct integers in increasing order. If you remove the smallest two numbers, arithmetic mean of the remaining numbers becomes equal to their median. What is the arithmetic mean of the original 11 numbers? Approximate your result to the nearest hundredth digit.

1	2					8				

17. A sport team has a circular emblem with concentric black and white rings around a black circle. The figure shows the central black circle and the first few rings starting from the center. The central black circle has a diameter of 2″ and each black ring is 2″ thick and 3″ apart. If the last ring on the circumference of the emblem is black and if there are 9 white rings on the emblem, what is the diameter of the emblem in inches?

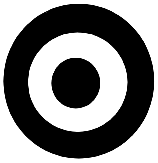


Figure is not drawn to scale.

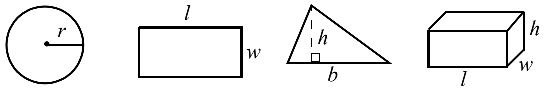
18. "s" is a positive even integer. "r" is a positive integer, divisible by 3. What is the greatest value of "sr" such that s + r is even, less than 100 and divisible by 3?

L

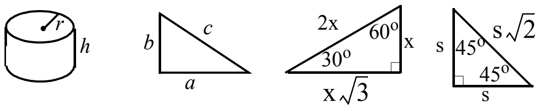
Test 1 - Section 3

20 Minutes, 16 Questions

Reference Information



$A = \pi r^2$ $C = 2\pi r$ $A = lw$ $A = \frac{1}{2}bh$ $V = lwh$



$V = \pi r^2 h$ $c^2 = a^2 + b^2$ Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

1. $12.85/10^2 =$

- **(A)** 1285
- **(B)** 128.5
- **(C)** 1.285
- **(D)** 0.1285
- **(E)** 0.01285

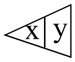
2. If $m > 2n - 8$ and $n = 7$, which of the following must be true?

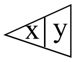
- **(A)** $m - 10 = -4$
- **(B)** $m - 10 > -4$
- **(C)** $m - 10 > 0$
- **(D)** $m - 10 > 4$
- **(E)** $m - 10 > 6$

3. For which of the following bread types you pay the same price for the equal amount?

Bread Type	Amount (lbs.)	Price
Wheat	2	$1.50
Oat	3	$2.50
Rye	4.5	$3.75
Multi-Grain	4.5	$3.75

- **(A)** Rye and Multi-Grain only.
- **(B)** Wheat and Oat only.
- **(C)** Wheat, Rye and Multi-Grain only.
- **(D)** Oat and Rye only.
- **(E)** Oat, Rye and Multi-Grain only.

4. Let  $= y^x/x$

For which of the (x, y) pairs  = 9?

- **(A)** (3, 2)
- **(B)** (2, 3)
- **(C)** (3, 3)
- **(D)** (2, 4)
- **(E)** (4, 2)

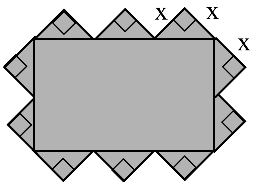
1/4

5. Mary uses a cookie recipe to bake boxes of cookies for a school event. Each box has 10 cookies. The recipe uses 3 eggs to make x cookies. If she uses 2 dozen eggs how many boxes of cookies she can make?

- **(A)** 8x
- **(B)** 4x/5
- **(C)** 3x/5
- **(D)** 2x/5
- **(E)** 5x/4

1/3

6. What is the area of the shaded region if all the legs of the 10 right triangles shown in the figure are equal to x?

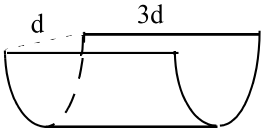


- **(A)** $34x^2$
- **(B)** $(6\sqrt{2} + 5)x^2$
- **(C)** $17x^2$
- **(D)** $22x^2$
- **(E)** $11\sqrt{2}x^2$

7. A 10'x10' room is covered by 1'x1' white tiles in the center. Two rows of 1'x1' black tiles are used as a trim around the edges. What is the white-tile, black-tile ratio?

- **(A)** 16/9
- **(B)** 8/5
- **(C)** 2/3
- **(D)** 16/19
- **(E)** 9/16

8. A company charges \$2.00 per square foot to clean a brass surface. How much it is going to cost, in dollars, to clean both sides of a half-cylinder sheet of brass as shown in the figure?



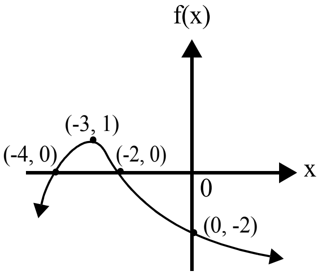
(A) $\pi d^2/2$
(B) πd^2
(C) $3\pi d^2$
(D) $6\pi d^2$
(E) $3\pi d^3/8$

1/2

9. $y = x^2$ and $y - 2x = 8$. How many different (x, y) pairs satisfy both of these equations?

(A) None
(B) 1
(C) 2
(D) 3
(E) 4

10. For which negative values of x, f(x) is negative?



(A) $x < 4$
(B) $x < -4$
(C) $x < 0$
(D) $-4 < x < -2$
(E) $x < -4$ or $-2 < x < 0$

2/3

11. a and b are two positive numbers.

If $a@b = \sqrt{a+b}$, then $(1@2)(3@3) =$

(A) 7@11
(B) $9@3^2$
(C) 1@17
(D) 14@4
(E) All of the above

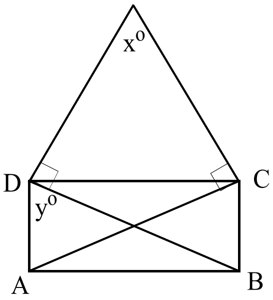
12. The sale volumes of a company with 4 different departments are represented with a pie chart of radius 3. The sale volumes from these 4 departments have relative ratios of 4:3:2:1. What is the area covered by the sales from the 2nd and 3rd largest departments combined, on the pie chart?

(A) 4.5π
(B) 3.6π
(C) 2.7π
(D) 1.8π
(E) 0.9π

13. The speed, s, of a car decreases when it comes to a hill. The decrease in speed is proportional to time, t, with a proportionality constant of a > 0. Originally, a car has 30 miles/hour speed on a flat road, then it approaches a hill. Which of the following formulas gives the speed of the car on the hill?

(A) $s = at$
(B) $s = -at$
(C) $s = 30 + at$
(D) $s = 30 - at$
(E) $s = 30 - at^2$

14. In the figure, ABCD is a rectangle. What is the value of y in terms of x?



(A) x
(B) 90 - x
(C) 90 - x/2
(D) 45 + x
(E) 45 + x/2

15. If $2^a = 8^{\frac{(b-1)^2}{b(b+1)} \cdot b^2 \cdot \frac{(b+1)^2}{b(b-1)}}$ for positive values of a and b, what is the value of b in terms of a?

(A) a/3
(B) $\sqrt{\frac{a}{3}+1}$
(C) a/3 + 1
(D) $\sqrt{\frac{a}{3}-1}$
(E) a/3 - 1

16. In a bakery, x percent of the rye bread is sold with x percent profit. The remaining y percent of the same rye bread is sold with y percent loss. What is the total percentage profit (or loss) in terms of x?

(A) x - 100
(B) 100 - x
(C) 50 - x
(D) 2x + 100
(E) 2x - 100

L

Answer Key - Test 1

Section 1

1. (A)
2. (A)
3. (B)
4. (E)
5. (A)
6. (C)
7. (D)
8. (A)
9. (C)
10. (C)
11. (C)
12. (E)
13. (B)
14. (A)
15. (A)
16. (C)
17. (D)
18. (C)
19. (E)
20. (C)

Section 2

Part 1

1. (E)
2. (D)
3. (C)
4. (D)
5. (C)
6. (E)
7. (D)
8. (D)

Part 2

9. 30
10. 320
11. 9
12. 75
13. 100
14. 2.10
15. 160
16. 6.82
17. 92
18. 2304

Section 3

1. (D)
2. (B)
3. (E)
4. (C)
5. (B)
6. (C)
7. (A)
8. (D)
9. (C)
10. (E)
11. (E)
12. (A)
13. (D)
14. (C)
15. (B)
16. (E)

Calculate Your Score

Calculate Your Raw Score

1. Count the number of correct answers and write it in Line 1 of the following work sheet.

2. Count the number of incorrect (<u>not</u> missing) answers in Section 1, Section 2, Part 1 and Section 3. Divide this count by 4 and write it in the line 2.

Note that Part 2 of Section 2 has grid-in questions. No penalty is given for the incorrect answers in this part. That is why you don't need to count incorrect answers here.

Also note that there is no penalty for missing any answers. That is why you don't need to count the missing answers.

3. Subtract Line 2 from Line 1 and write the result in Line 3.

4. Round the score on line 3 to the nearest whole number and write the result on Line 4. This is your raw score.

Work Sheet to Calculate the Raw Score

1.	Correct Answers	
2.	(Incorrect Answers)/4	
3.	Not rounded Raw Score	
4.	Raw Score	

Calculate Your SAT Score

Find your raw score in the below table and read the corresponding SAT score. Note that the actual SAT scores are expressed in ranges, not definite numbers. For our purposes, we assign only one SAT score for each raw score.

Raw Score	**SAT Score**	**Raw Score**	**SAT Score**	**Raw Score**	**SAT Score**
-6 or less	200	15	430	36	590
-5	205	16	435	37	595
-4	215	17	440	38	600
-3	225	18	450	39	610
-2	235	19	460	40	620
-1	245	20	465	41	630
0	260	21	470	42	640
1	280	22	480	43	650
2	290	23	490	44	655
3	310	24	495	45	660
4	320	25	500	46	670
5	330	26	510	47	690
6	340	27	520	48	700
7	360	28	525	49	710
8	370	29	530	50	730
9	380	30	540	51	740
10	385	31	550	52	760
11	390	32	560	53	775
12	400	33	565	54	800
13	410	34	570		
14	420	35	580		

Name:

Subject Table - Test 1

Date:

Question Number	Categories - Subjects				Difficulty Level
1.1	S. A - One Variable Equations				Easy
1.2	Others - Sets				Easy
1.3	Geo. - Points and Lines	Geo. - Triangles			Easy
1.4	Others - Statistics				Easy
1.5	WQ -Formulation Only				Medium
1.6	Geo. - Triangles	Geo. - Coordinate Geo.			Medium
1.7	Arith. - Basic Arithmetic	Others - Logic	WQ - Regular		Medium
1.8	S. A - Multiple Unknowns				Medium
1.9	S. A - Multiple Unknowns				Medium
1.10	Arith. - Square Root	A. A - Functions			Medium
1.11	Geo. - Coordinate Geo.	A. A - Linear Functions			Medium
1.12	Others - Statistics	Others - Sequences	Others - Sums		Medium
1.13	Arith. - Powers				Medium
1.14	Arith. - Divisibility				Medium
1.15	Others - Statistics	WQ - Formulation Only			Hard
1.16	Geo. - Circles	WQ - Describing Figures			Hard
1.17	Geo. - Triangles				Hard
1.18	Geo. - Triangles	Geo. - Quadrangles	Geo. - Rectangles		Hard
1.19	S. A - Inequalities	S. A - Exprs. with Abs. Value			Hard
1.20	Others - Perm., Comb.	WQ - Regular			Hard
2.1	Arith. - Decimals	Arith. - Fractions, Ratios			Easy
2.2	Geo. - Triangles	Geo. - 3-D Objects			Easy
2.3	S. A - Exprs. with Powers				Easy
2.4	Geo. - Triangles	Geo. - Coordinate Geo.			Easy
2.5	Arith. - Nums. Between –1 & 1	A. A - Linear Functions	A. A - Quad. Functions		Medium
2.6	Arith. - Fractions, Ratios	Others - Probability			Medium
2.7	Arith. - Fractions, Ratios	Others - Logic	WQ - Regular		Hard
2.8	A. A - Quadratic Functions	WQ - Describing Figures			Hard
2.9	Arith. - Fractions, Ratios	WQ - Regular			Easy
2.10	Arith. - Percentages	WQ - Regular			Easy
2.11	Geo. - Triangles	WQ - Describing Figures			Medium
2.12	Arith. - Percentages	WQ - Regular			Medium
2.13	Arith. - Divisibility	Others - Basic Counting			Medium
2.14	Others - Rounding	Others - Tbls, Chrts, Grphs			Medium
2.15	Geo. - Points and Lines	Geo. - Angles	Geo. - Triangles		Medium
2.16	Others - Rounding	Others - Statistics			Hard
2.17	Geo. - Circles	Others - Basic Counting	WQ - Desc. Figures		Hard
2.18	Arith. - Divisibility	Arith. - Even Odd Numbers			Hard
3.1	Arith. - Basic Arithmetic	Arith. - Powers			Easy
3.2	S. A - Multiple Unknowns	S. A - Inequalities			Easy
3.3	Arith. - Basic Arithmetic	Others - Tbls, Chrts, Grphs	WQ - Regular		Easy
3.4	Arihmetic - Powers	Others - Defined Operators			Easy
3.5	Arith. - Basic Arithmetic	S. A - Proportionality	WQ - Formulation Only		Medium
3.6	Geo. - Triangles	Geo. - Rectangles	WQ - Formulation Only		Medium
3.7	Arith. - Fractions, Ratios	Geo. - Squares	WQ - Desc. Figures		Medium
3.8	Geo. - Rectangles	Geo. - Circles	Geo. - 3D Objects	WQ - Form. Only	Medium
3.9	S. A - Multiple Unknowns	A. A - Linear Functions	A. A - Quad. Functions		Medium
3.10	Geo. - Coordinate Geo.	S. A - Inequalities	A. A - Functions		Medium
3.11	Arith. - Square Root	Others - Defined Operators			Medium
3.12	Arith. - Fractions, Ratios	Geo. - Circles	Others - Tbls, Chrts, Grphs	WQ - Desc. Figures	Medium
3.13	S. A - Proportionality	WQ - Formulation Only			Hard
3.14	Geo. - Quadrangles	Geo. - Rectangles	WQ - Formulation Only		Hard
3.15	S. A - One Variable Equations	S.A - Exprs. with Powers			Hard
3.16	Arith. - Percentages	WQ - Formulation Only			Hard

Skipped:

Wrong:

Analysis Chart - Test 1

Name:

Date:

Category	Subject	Easy	Medium	Hard
Arithmetic	Basic Arithmetic	3.1, 3.3	1.7, 3.5	
	Decimals	2.1		
	Fractions, Ratios	2.1, 2.9	2.6, 3.7, 3.12	2.7
	Percentages	2.10	2.12	3.16
	Powers	3.1, 3.4	1.13	
	Square Root		1.10, 3.11	
	Radicals			
	Negative Numbers			
	Numbers Between –1 and 1		2.5	
	Divisibility		1.14, 2.13	2.18
	Even & Odd Numbers			2.18
	Absolute Value			
Geometry	Points and Lines	1.3	2.15	
	Angles		2.15	
	Polygons			
	Triangles	1.3, 2.2, 2.4	1.6, 2.11, 2.15, 3.6	1.17, 1.18
	Quadrangles			1.18, 3.14
	Rectangles		3.6, 3.8	1.18, 3.14
	Squares		3.7	
	Circles		3.8, 3.12	1.16, 2.17
	Trigonometry			
	Coordinate Geometry	2.4	1.6, 1.11, 3.10	
	Symmetry			
	3-D Objects	2.2	3.8	
Simple Algebra (S. A.)	One Variable Equations	1.1		3.15
	Multiple Unknowns	3.2	1.8, 1.9, 3.9	
	Equations with Powers	2.3		3.15
	Radical Equations			
	Inequalities	3.2	3.10	1.19
	Expressions with Absolute Value			1.19
	Proportionality		3.5	3.13
Advanced Algebra (A. A.)	Functions		1.10, 3.10	
	Linear functions		1.11, 2.5, 3.9	
	Quadratic Functions		2.5, 3.9	2.8
Others	Rounding		2.14	2.16
	Tables, Charts, Graphs	3.3	2.14, 3.12	
	Sets	1.2		
	Defined Operators	3.4	3.11	
	Logic		1.7	2.7
	Statistics	1.4	1.12	1.15, 2.16
	Sequences		1.12	
	Sums		1.12	
	Basic Counting		2.13	2.17
	Permutations, Combinations			1.20
	Mutually Exclusive Events			
	Independent Events			
	Probability		2.6	
Word Questions (WQ)	Regular	2.9, 2.10, 3.3	1.7, 2.12	1.20, 2.7
	Formulation Only		1.5, 3.5, 3.6, 3.8	1.15, 3.13, 3.14, 3.16
	Describing Figures		2.11, 3.7, 3.12	1.16, 2.8, 2.17

Skipped:

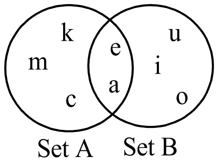
Wrong:

Solutions - Test 1

Section 1

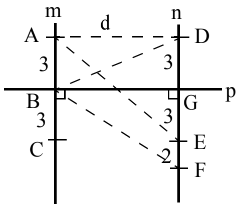
1. Answer: (A)
$x + 7 = 5$ ➔ $x = 5 - 7 = -2$ ➔ $x - 1 = -2 - 1 = -3$
The answer is (A).

2. Answer: (A)



In the figure, e and a are members of both sets. Hence the members of set A, which are not in Set B, are c, k and m. The answer is (A).

3. Answer: (B)



In the figure, AE looks the longest distance among the answer choices. Since the figure is to scale, i.e., you don't see the phrase "Figure is not drawn to scale", you can conclude that AE is really the longest distance as it appears to be. The answer is (B).

Alternate Solution:
In the figure, AC = DE = 3 + 3 = 6

BA = DG and $m \parallel n$ ➔ $\overline{AD} \perp \overline{DG}$ ➔

$\angle ADG = \angle BGF = 90°$ ➔

AE is the hypotenuse of ΔADE. ➔
AE > DE, AE > AC and AE > BD

BG = AD = d ➔
$BF = \sqrt{d^2 + (3+2)^2} = \sqrt{d^2 + 5^2}$

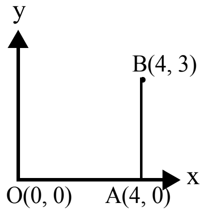
On the other hand, $AE = \sqrt{d^2 + 6^2}$ ➔ AE > BF
Since AE is longer than all the other distances, the answer is (B).

4. Answer: (E)
Median of consecutive numbers is equal to their average. So if the median of 3 consecutive integers is 6, their average is also 6. Therefore, their sum is $3 \times 6 = 18$. The answer is (E).

Alternate Solution:
If the median of 3 consecutive integers is 6, these integers are 5, 6 and 7. ➔
The sum is 5 + 6 + 7 = 18
The answer is (E).

5. Answer: (A)
k pounds of sugar is left after baking the cupcakes ➔
Sue used p/2 - k pounds of sugar to bake the cupcakes.
Each cupcake is c pounds. ➔
Sue baked (p/2 - k)/c cupcakes.
The answer is (A).

6. Answer: (C)



OA = 4 - 0 = 4
BA = 3 - 0 = 3
The area of triangle OAB =
$\frac{1}{2}(OA \times BA) = \frac{1}{2}(4 \times 3) = 6$
The answer is (C)

7. Answer: (D)
1/3 of 36 cars are SUVs ➔ There are 36/3 = 12 SUVs.

3/4 of SUVs are not white ➔
$\frac{3}{4} \cdot 12 = 9$ of the SUVs are not white. ➔
12 - 9 = 3 SUVs are white.

3 out of 12 white cars are SUVs ➔
12 - 3 = 9 white cars are not SUVs.

The answer is (D).

8. Answer: (A)
$a^2 + 4b + 1 = (2b + 1)^2 = 4b^2 + 4b + 1$ ➔
$a^2 = 4b^2$ ➔ $a = 2b$ ➔ $b = a/2$

Note that $a \neq -2b$ because both a and b are positive. They can not have opposite signs.

The answer is (A).

9. Answer: (C)
If

$$\begin{array}{r} 2a \\ +\ 1b \\ \hline 44 \end{array}$$

then a + b = 14. Only in case (C) the addition of the two numbers, 9 and 5, is 14. The answer is (C).

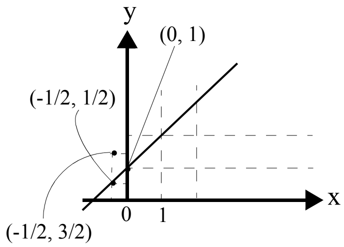
10. Answer: (C)
When the tree was first planted, it was 2′ high ➔
h(0) = a = 2

After 9 years, it became 20' tall. ➔

$h(9) = a + b\sqrt{9} = 2 + 3b = 20$ ➔ b = (20 - 2)/3 = 6

The answer is (C).

11. Answer: (C)



The line has slope 1 and y-intercept 1. So the equation of this line is $y = x + 1$. The only (x, y) pair which doesn't satisfy this equation is (-1/2, 3/2).

The answer is (C).

Alternate Solution:
The points in Cases (B) and (C) can not be on the same line, because their x-coordinates are the same but y-coordinates are different. Hence one of them is wrong. You can see in the above figure that (-1/2, 1/2) is on line m and (-1/2, 3/2) isn't. The answer is (C).

12. Answer: (E)
The sum of 6 consecutive integers is 36 →
Their average and median is 36/6 = 6
The median of {a, b, c, d e, f} is (c + d)/2 = 6 →
c and d are 5 and 7, respectively.

The median of {a, b, c, d, e, f, 50} is d, which is 7. The answer is (E).

Alternate Solution:
The sum of 6 consecutive odd integers from a to f is 3(a + f) = 36 → a + f = 12

Since the odd integers are consecutive,
$f = a + 2 \cdot 5 = a + 10$

Hence a + f = a + a + 10 = 2a + 10 = 12 → a = 1 →
6 odd integers are 1, 3, 5, 7, 9, 11 →
The median of {1, 3, 5, 7, 9, 11, 50} is 7.

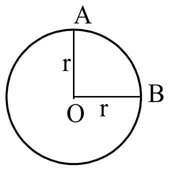
13. Answer: (B)
$4 \times 2^6 = 2^2 \times 2^6 = 2^8$
To find the answer you need to divide 2^{12} by 2^8.
$2^{12} / 2^8 = 2^{12 - 8} = 2^4$
The answer is (B).

14. Answer: (A)
If an integer i is divided by 9 the remainder is 7. →
i/9 = n + 7/9, where n is an integer. → i = 9n + 7
(3i + 5)/9 = 3i/9 + 5/9
Substituting i = 9n + 7 into the above equation yields
3i/9 + 5/9 = 3(9n + 7)/9 + 5/9 = 3n + 2 + 8/9 =
m + 8/9, where m = 3n + 2 is an integer. →
The remainder is 8. The answer is (A).

15. Answer: (A)
The median of three numbers, a, b and c is 1.5.
a < b < c → b = 1.5

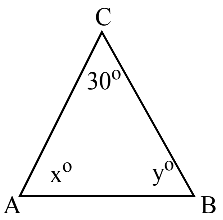
The arithmetic mean of these numbers is 2. →
(a + b + c)/3 = (a + 1.5 + c)/3 = 2 →
(a + c)/3 = 2 - 1.5/3 = 2 - 0.5 = 1.5
The answer is (A).

16. Answer: (C)
The circumference of the circle with radius r is $2\pi r$. →
The length of the arc AB in the figure is



$\widehat{AB} = (2\pi r)/4 = \pi r/2$ →
The perimeter of the quarter circle OAB is
$\pi\frac{r}{2} + r + r = r\left(\frac{\pi}{2} + 2\right)$
The answer is (C).

17. Answer: (D)
The length of the side of a triangle is longer if the angle across it is larger. →
AC > BC → y > x.
So (A) and (E) are always correct.



y can not be smaller than 30, because if y < 30, then x < 30. Then the addition of the three inner angles of ΔABC =
x + y + 30 < 30 + 30 + 30 → x + y + 30 < 90
Since the addition of all the angles of any triangle is 180, y, the value of the largest of all three angles can not be less than 30. → AB can not be larger than AC. The answer is (D).

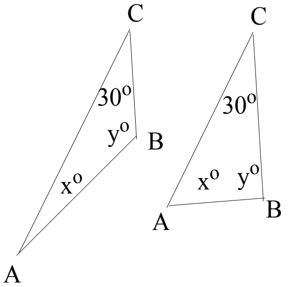
Note that x can take any value as long as it is less than y. Hence (B) or (C) can be correct.

Alternate Solution:
Let's draw the figure to scale in two different ways.
In each figure,

$\angle C = 30^\circ$ and

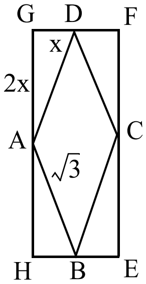
AC > BC.

You can see that all the cases except Case (D) may be correct.



18. Answer: (C)
Let GD = x. Then AG = 2x.
Since ABCD is an equilateral quadrangle, AD = AB = $\sqrt{3}$.
ΔAGD is a right triangle. →
$x^2 + (2x)^2 = 3$ → $5x^2 = 3$ →
$x^2 = 3/5$

GF = 2GD = 2x
GH = 2GA = 4x



The area of EFGH is $GF \cdot GH = 2x \cdot 4x = 8x^2 = 8 \cdot (3/5) = 24/5 = 4.8$
The answer is (C).

19. Answer: (E)
There are 2 cases:

a. $a \geq 0$ → $| 5 - |a| | = | 5 - a | > 3$
There are again 2 cases

I. $5 - a \geq 0$ → $| 5 - a | = 5 - a > 3$ → $a < 2$
Since $a \geq 0$, then $0 \leq a < 2$

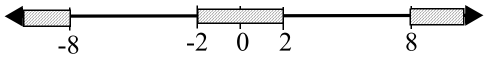
II. $5 - a < 0$ → $5 < a$
$5 - a < 0$ → $| 5 - a | = a - 5 > 3$ → $a > 8$

b. $a < 0$ → $| 5 - |a| | = | 5 + a | > 3$
There are again 2 cases

I. $5 + a \geq 0 \rightarrow |5 + a| = 5 + a > 3 \rightarrow a > -2$
Since $a < 0$, then $-2 < a < 0$

II. $5 + a < 0 \rightarrow |5 + a| = -a - 5 > 3 \rightarrow a < -8$

Lets summarize all possible values of a on a number line.



-8, -2, 2 and 8 are not included.

As you can see on the number line, all three cases are possible. The answer is (E).

20. Answer: (C)
Since the order in which the students are picked is not important, the number of possibilities is

$$\frac{8!}{2! \cdot 6!} = 28$$

The answer is (C).

Section 2

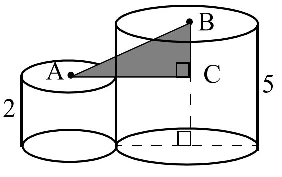
Part 1

1. Answer: (E)
Use your calculator to convert the fractions in the answer choices to decimal numbers. Write all the answers in decimal format between the two boundaries, 0.015 and 0.045, aligning the decimal points:

0.015
(A) 0.02
(B) 0.025
(C) 0.025
(D) 0.0333
(E) 0.05
0.045

The hundredth digit is the first non-zero digit after the decimal point for all seven numbers. You need to compare this digit only. The minimum and maximum numbers for this digit is 1 (of 0.015) and 4 (of 0.045). The only hundredth digit which is not in between 1 and 4 is 5 in Case (E).

2. Answer: (D)
Lets draw a right triangle ABC as shown in the figure. Note that $\overline{BC}$ is vertical, i.e. $\overline{BC}$ is perpendicular to the flat surface on which the cylinders are standing.



$BC = h_2 - h_1 = 5 - 2 = 3$
where h_1 and h_2 are the heights of the small and large cylinders, respectively.

$AC = r_1 + r_2 = 3/2 + 5/2 = 4$
where r_1 and r_2 are the radii of the bases of the small and large cylinders, respectively.

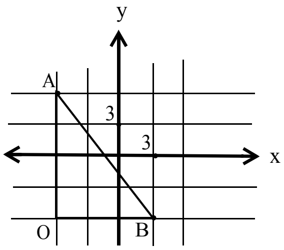
Using the Pythagorean Theorem:

$$AB = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

The answer is (D).

3. Answer: (C)
$3^a = 27 = 3^3 \rightarrow a = 3$
$5^b = 125 = 5^3 \rightarrow b = 3$
$a - b = 3 - 3 = 0$
The answer is (C).

4. Answer: (D)
Consider the right triangle AOB. Since each grid in the figure is 3 units long and 3 units wide,



$AO = 4 \cdot 3 = 12$
$OB = 3 \cdot 3 = 9$

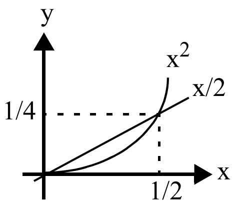
Using the Pythagorean Theorem:

$$AB = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = 15$$

The answer is (D).

5. Answer: (C)
$x^2 - x/2 = x(x - 1/2) = 2x(2x - 1)$
$2x < 1 \rightarrow 2x - 1 < 0 \rightarrow x^2 - x/2 < 0 \rightarrow x^2 < x/2$

The graph of both $x/2$ and x^2 are displayed in the figure.



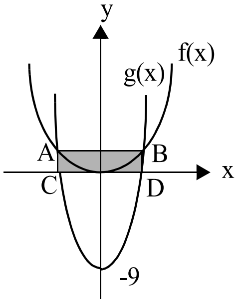
As you can see, x^2 is always less than $x/2$ for $0 < 2x < 1$ (or equivalently $0 < x < 1/2$)

The answer is (C).

6. Answer: (E)
If the probability of picking a card with number 18 or less is 2/3, then 2/3 of the cards have number 18 or less on them. ➔ 3/3 - 2/3 = 1/3 of the cards have a number more than 18 on them. Since there are 10 cards that have numbers more than 18, 1/3 of all cards is 10. ➔ Total number of cards is $3 \cdot 10 = 30$. The answer is (E).

7. Answer: (D)
Let j, m and s be the amounts of money John, Mary and Sue have respectively.
Mary has $6.00 more than John. ➔ $m = j + 6$ ➔ Mary has more than $6.
Sue has two-thirds of the money that Mary has. ➔ $s = 2m/3$ ➔ Sue has more than $(2 \cdot 6)/3$ = $4.00
The answer is (D).

8. Answer: (D)
Figure shows f(x) and g(x). The area in question is the shaded rectangle in the figure.
To find the area of this rectangle, you need to find the coordinates of A or B.
At these two points $f(x) = g(x)$ ➔



$x^2 = 5x^2 - 9$ ➔ $4x^2 = 9$ ➔ $x = 3/2$ or $x = -3/2$
-3/2 and 3/2 are the x coordinates of A and B respectively. ➔ $AB = 3/2 - (-3/2) = 3$

When $x = 3/2$, $f(x) = (3/2)^2 = 9/4$
This is the y-coordinate of both A and B and it is the length of $\overline{AC}$.
Area of the rectangle is

$$AC \times AB = \frac{9}{4} \times 3 = 27/4$$

The answer is (D).

Part 2

9. Answer: 30
Let x be the number in question.
Three less than 1/2 of a number is 12. ➔
$x/2 - 3 = 12$ ➔ $x/2 = 15$ ➔ $x = 30$

10. Answer: 320
Employees who worked less than 5 years will get (100 - 20) = 80% of $400.00.

80% of 400 is $400 \times \frac{80}{100} = \$320$

11. Answer: 9
Figure shows ΔABC and ΔXYZ.
$XZ/AC = XY/AB = 2$ and $\angle X = \angle A$ ➔
$\Delta ABC \sim \Delta XYZ$ ➔
$YZ/BC = YZ/4.5 = 2$ ➔
$YZ = 9$

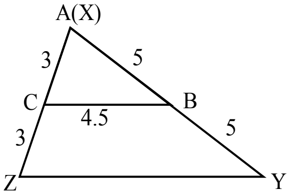


Figure is not drawn to scale.

12. Answer: 75
Employer will save $\frac{20}{100} \times 400 = \$80.00$ from each employee.
Since the employer's total saving is $6000.00, there are 6000/80 = 75 employees in the company.

13. Answer: 100
The remainders of 100 consecutive integers starting with 7 are 1, 2, 0, 1, 2, 0, 1, 2, 0, ...1, 2, 0, 1. The pattern of (1, 2, 0) is repeated 100/3 = 33 + 1/3 times. So there are 33 of (1, 2, 0) and one 1 in the set of remainders. Hence the addition is 33(1 + 2 + 0) + 1 = 100.

14. Answer: 2.10

Date/Day	Time	Number Called	Duration
5/25/Tue.			
5/26/Wed.			
5/27/Thu.	1:03 AM	333-333-3333	10 min., 59 sec.
5/28/Fri.			
5/29/Sat.			
5/30/Sun.			
5/31/Mon.			
6/01/Tue.			
6/02/Wed.			
6/03/Thu.			
6/04/Fri.	2:30 PM	333-333-3333	4 min., 03 sec.

In the above table, we simplify the original table and show only the calls made to 333-333-3333. We also include the days of the week. To make it clear, the days that John did not make any calls to this number is left blank.

The first call John made was on Thursday, at 1:00 AM. He will be charged $0.10 per minute, because it is outside the 8 AM-6 PM interval. He talked 10 minutes and 59 seconds. The duration is approximated up to 11 minutes. So the total charge for this call is $11 \times 0.10 = \$1.10$.

The second call John made was on Friday, at 2:30 PM. He will be charged $0.20 per minute, because it is inside the 8 AM-6 PM interval. He talked 4 minutes and 3 seconds. The duration is approximated up to 5 minutes. So total charge for this call is $5 \times 0.20 = \$1.00$.

The total amount John paid is 1.10 + 1.00 = $2.10

15. Answer: 160
Lines p and q are crossing each other at point A. ➔
$\angle DAC = u°$
Lines p and m are crossing each other at point A. ➔
$\angle BAC = v°$
Lines p and n are crossing each other at point C. ➔
$\angle ACE = y°$

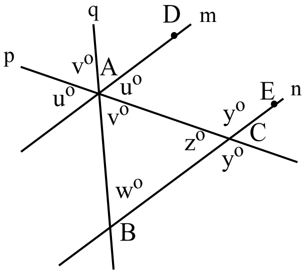


Figure is not drawn to scale.

$u + v + y = 200$ ➔ $v = 200 - u - y$

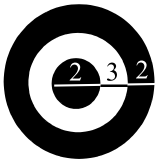
Since ΔABC is a triangle, $w + z = 180 - v$ ➔
$w + z = 180 - (200 - y - u) = y + u - 20$

$m \parallel n$ ➔ $u + y = 180$ ➔ $w + z = 180 - 20 = 160^o$

16. Answer: 6.82
Since the distinct integers are in increasing order, if you remove the smallest two numbers (1 and 2), there will be four numbers smaller than 8 and four numbers greater than 8. Hence the median will be 8. Since the arithmetic mean of these 9 numbers is equal to their median, the arithmetic mean will also be 8. ➔ The addition of these 9 numbers will be $8 \times 9 = 72$ ➔ The arithmetic mean of the original 11 numbers is (72 + 1 + 2)/11 = 6.82

1	2					8				

17. Answer: 92
If there are 9 white rings, there are also 9 black rings. Each white ring is 3″ and each black ring is 2″ Hence the radius is 9(2 + 3) + radius of the central black circle. = 45 + 1 = 46″ ➔ The diameter is $46 \cdot 2 = 92''$.



18. Answer: 2304
Since s is already an even number, for s + r to be even, r has to be even as well. r is already divisible by 3. To make it even, it has to be divisible by 2 also. So r has to be divisible by $3 \cdot 2 = 6$. ➔ r = 6m, where m is a positive integer.

Since r is already divisible by 3, for s + r to be divisible by 3, s has to be divisible by 3 as well. s is divisible by 2. To make it divisible by 3, s has to be divisible by $3 \cdot 2 = 6$. ➔ s = 6n, where n is a positive integer.

Substituting these values of s and r,
s + r = 6n + 6m = 6(n + m) ➔
For s + r to be an integer, less than 100, n + m has to be an integer less than 100/6 = 16.67. Hence the maximum value that m + n can take is 16.

$sr = 6m \cdot 6n = 36mn$ ➔ Greatest value of sr is obtained when mn is the greatest. Since m + n can be maximum of 16, you need to determine the maximum value of mn, when m + n = 16.

Below table shows the possible values of m, n, m + n and nm.
The 3rd row shows that m + n = 16 for all pairs of m and n.
The 4th row shows that mn is maximum when m = n = 8. ➔ $sr = 36 \cdot 8 \cdot 8 = 2304$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
m+n	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
mn	15	28	39	48	55	60	63	64	63	60	55	48	39	28	15

Section 3

1. Answer: (D)
$12.85/10^2 = 12.85/100 = 0.1285$
The answer is (D).

2. Answer: (B)
If m > 2n - 8 and n = 7, then $m > 2 \cdot 7 - 8$ ➔ m > 6 ➔ m - 10 > 6 - 10 ➔ m - 10 > -4
The answer is (B).

3. Answer: (E)
If you pay the same price for the equal amount, price per pound must be equal. In the below table, we show the price per pound for each type of bread. Price per pound = Price/Amount. For example for wheat bread, price per pound is 1.50/2 = $0.75. It is clear that Oat, Rye and Multi-Grain breads have the same price per pound. The answer is (E).

Bread Type	Amount	Price	Price per Pound
Wheat	2 lbs.	$1.50	$0.75
Oat	3 lbs.	$2.50	$0.83
Rye	4.5 lbs.	$3.75	$0.83
Multi-Grain	4.5 lbs.	$3.75	$0.83

4. Answer: (C)
Let's calculate each of the answer choices until we find the correct answer.

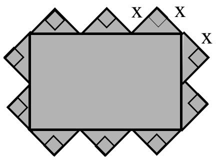
(A) $(3, 2) = 2^3/3 = 8/3$ - Not 9
(B) $(2, 3) = 3^2/2 = 9/2$ - Not 9
(C) $(3, 3) = 3^3/3 = 9$ - It is 9

The answer is (C).

5. Answer: (B)
If she uses 2 dozen eggs she is using $2 \cdot 12 = 24$ eggs. If the recipe uses 3 eggs to make x cookies, then she is making 24/3 = 8x cookies. If each box has 10 cookies, then she makes 8x/10 = 4x/5 boxes of cookies. The answer is (B).

6. Answer: (C)
The diagonal of each triangle surrounding the rectangle is
$\sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}x$

The length and the width of the rectangle is $3\sqrt{2}x$ and $2\sqrt{2}x$, respectively.



Hence the area of the rectangle is $3\sqrt{2}x \cdot 2\sqrt{2}x = 6 \cdot 2x^2 = 12x^2$

The area of each triangle is $x^2/2$. Since there are 10 triangles, the total area of the triangles surrounding the rectangle is $10x^2/2 = 5x^2$.

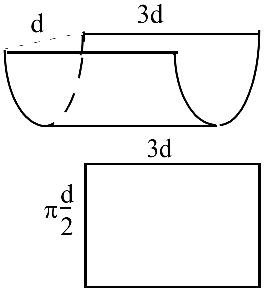
The total area of the shaded region = $12x^2 + 5x^2 = 17x^2$

The answer is (C).

7. Answer: (A)
Since 2' from the edges is covered by black tiles, the area covered by white tiles is 8'x8', which is 64 square feet.

Since the room is $10 \times 10 = 100$ square feet, 100 - 64 = 36 square feet is covered by black tiles. Hence the white-tile, black-tile ratio is 64/36 = 16/9. The answer is (A).

8. Answer: (D)
The circumference of a circle with diameter d is πd. Since the sheet of metal is half cylinder, the length of the curved edge is $\pi d/2$. Hence the half cylinder in the figure is a rectangle of length 3d and width $\pi d/2$ as shown in the figure.



The total area to be cleaned is

$$2\left(\pi\frac{d}{2} \cdot 3d\right) = 3\pi d^2$$

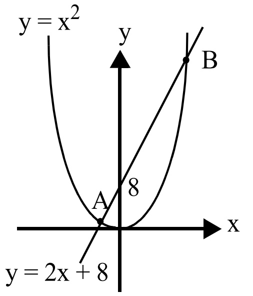
Note that the area of the rectangle is multiplied by 2, because both sides of the brass sheet needs to be cleaned.

Since it costs \$2.00 per square foot to clean the brass surface, the total amount required is

$2 \cdot 3\pi d^2 = 6\pi d^2$ dollars.

The answer is (D).

9. Answer: (C)
You can answer this question in two ways. You can draw a graph of both equations as shown in the figure. You don't need to be precise in your drawing. As long as you can see that these 2 functions cross each points (A and B), you have your answer, (C).

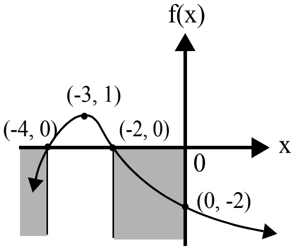


You can also solve the equations algebraically.
$y - 2x = 8 \Rightarrow y = 2x + 8$
$x^2 = 2x + 8 \Rightarrow x^2 - 2x - 8 = (x - 4)(x + 2) = 0 \Rightarrow$
$x = 4 \Rightarrow y = 4^2 = 16$ or $x = -2 \Rightarrow y = (-2)^2 = 4$

Once again, there are two set of values of (x, y) pairs that satisfy these two equations.

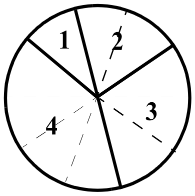
The answer is (C).

10. Answer: (E)
For negative values of x, f(x) is negative in the shaded regions shown in the figure. x values for these two regions are
$x < -4$ and $-2 < x < 0$
The answer is (E).



11. Answer: (E)
If $a@b = \sqrt{a + b}$, then (1@2)(3@3) =
$\sqrt{1 + 2}\,\sqrt{3 + 3} = \sqrt{18} = \sqrt{x + y} \Rightarrow$
The correct answer has to be in the form of x@y, where x + y = 18. All the answer choices from (A) to (D) are in this form. The answer is (E).

12. Answer: (A)
If the sales volume from the 4 departments are in ratio of 4:3:2:1, the total number of equal size slices is 1 + 2 + 3 + 4 = 10.



The smallest department will have 1 slice of the pie as shown in the figure.

Second department will have 2 slices on the pie. The third one will have 3 slices of the pie. The largest one will take 4 slices of the pie.

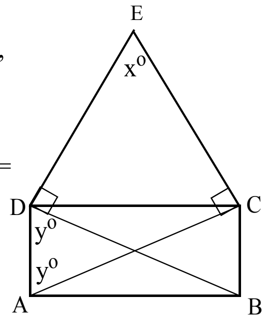
Out of 10 slices, the second and the third largest departments combined will have 2 + 3 = 5 slices. The total area of the pie chart is $\pi 3^2 = 9\pi$.

The area in question is $(9\pi)\left(\frac{5}{10}\right) = 4.5\pi$.

The answer is (A).

13. Answer: (D)
The decrease in speed is proportional to time, t, with a proportionality constant, a. ➔ The decrease in speed is $a \cdot t$. If the initial speed is 30 miles/hour, it will decrease by $a \cdot t$ once the car comes to a hill. ➔ s = 30 - at. The answer is (D).

14. Answer: (C)
Since ABCD is a rectangle, $\angle CAD = \angle ADB = y°$
Consider the quadrangle ACED.
The inner angles of ACDE =
$x + 90 + y + y + 90 =$
$x + 180 + 2y = 360 \Rightarrow$
$y = 90 - x/2$
The answer is (C).



15. Answer: (B

$$\frac{(b-1)^2}{b(b+1)} \cdot b^2 \cdot \frac{(b+1)^2}{b(b-1)} = (b-1)\cdot(b+1) =$$

$b^2 - 1$

$$2^a = 8^{\frac{(b-1)^2}{b(b+1)} \cdot b^2 \cdot \frac{(b+1)^2}{b(b-1)}} =$$

$$(2^3)^{(b^2-1)} = 2^{3(b^2-1)} \Rightarrow$$

$a = 3(b^2 - 1) = 3b^2 - 3 \Rightarrow b^2 = \frac{a+3}{3}$

$\Rightarrow b = \sqrt{\frac{a+3}{3}} = \sqrt{\frac{a}{3}+1}$

The answer is (B).

16. Answer: (E)

Let r and m be the total amount of rye bread and the cost of each bread, respectively.

x percent of the rye bread is $\frac{rx}{100}$

x percent of profit from each bread is $\frac{mx}{100}$

Hence the total profit from the sale of x percent of bread is

$$\frac{rx}{100} \cdot \frac{mx}{100} = \frac{mrx^2}{10000}$$

Similarly, the total loss from the sale of y percent of bread is

$$\frac{ry}{100} \cdot \frac{my}{100} = \frac{mry^2}{10000}$$

The total profit from the sale of all bread is

$$\frac{mrx^2}{10000} - \frac{mry^2}{10000} = \frac{mr}{10000}(x^2 - y^2)$$

Substituting y = 100 - x into the above equation:
The total profit from the sale of all bread is

$$\frac{mr}{10000}(x^2 - (100-x)^2) =$$

$$\frac{mr}{10000}(200x - 10000) =$$

$$(mr)\left(\frac{x}{50} - 1\right)$$

The total cost of the bread is mr.

Hence the total percentage profit is

$$\frac{(mr)\left(\frac{x}{50} - 1\right)}{mr} \cdot 100 = 2x - 100$$

The answer is (E).

Alternate Solution:

You can use common sense to eliminate the wrong answer choices. Here is how:

x and y are both numbers between 0 and 100. If all the bread is sold with profit, x = 100 (y = 0) and the profit is 100%. If all the bread is sold with a loss, then y = 100 (x = 0) and the profit is -100%. In other words, the minimum and maximum limits of the correct answer must be -100 and 100, respectively. Let's examine each case and find the correct answer.

Case (A):
x = 0 ➔ min. is -100, x = 100 ➔ max. is 0
Minimum limit is correct, but maximum limit is wrong. This means that if bread is sold with 100% profit, no profit is made. So this is the wrong answer.

Case (B):
x = 0 ➔ min. is 100, x = 100 max. is 0
This makes no sense at all. Both limits are wrong. These limits mean that when all bread is sold at 100% loss, total profit is 100%, and when all the bread is sold with 100% profit, total profit is 0. So this case is also wrong.

Case (C):
x = 0 ➔ min. is 50, x = 100 ➔ max. is -50
Again wrong answer.

Case (D):
x = 0 ➔ min. is 100, x = 100 ➔ max. is 300
Again wrong answer.

Case (E):
x = 0 ➔ min. is -100, x = 100 ➔ max. 100
Correct answer. When all the bread is sold at 100% loss, the total profit is -100% and when all the bread is sold for 100% profit, the total profit is also 100%.

Test 2

Date:

General Directions:

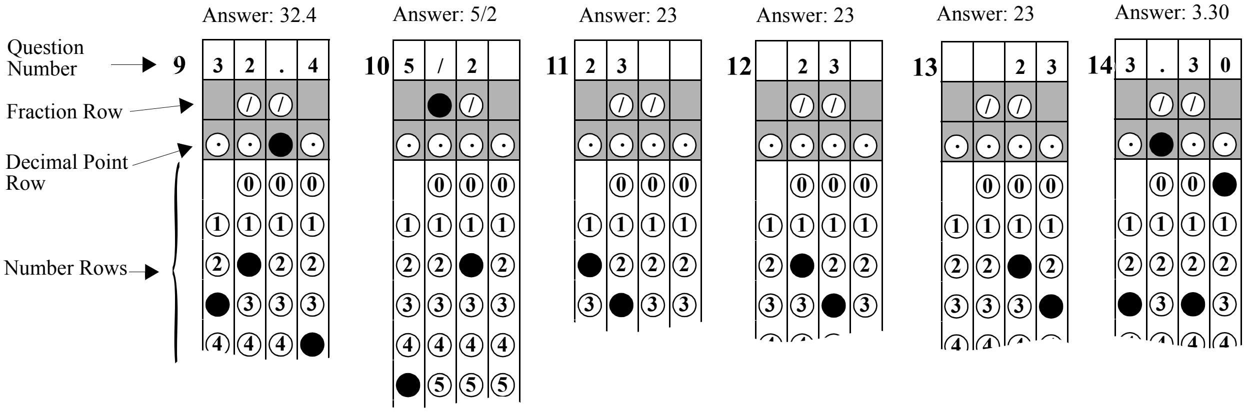
- The test is 70 minutes long.
- It has 3 sections.
  Section 1: 20 multiple choice questions, 25 minutes.
  Section 2: 8 multiple choice and 10 grid-in questions, 25 minutes.
  Section 3: 10 multiple choice questions, 20 minutes.
- You can only work on one section at a time.
- You are not allowed to transfer your answers from the test to the Answer Sheet after the time is up for each section. Therefore mark your answer on the Answer Sheet as soon as you finish answering a question.
- Make sure that the question number in the test matches the question number on the answer sheet.
- Mark only one answer for each question.
- If you want to change your answer, erase the old answer completely.
- You receive one point for each correct answer.
- You lose one point for 4 incorrect multiple choice answers.
- You don't lose any points for incorrect grid-in questions.
- You neither gain nor loose points for missing answers.

Directions for the Grid-In Questions

- You can only have zero or positive numbers as answers for the grid-in questions. Therefore, if your answer is negative it is wrong.
- The upper limit of your answer is 9999. If your answer is greater than 9999, it is wrong.
- You can express your answers as integers, fractions or decimal numbers. Do not spend any time to convert fractions to decimals or decimals to fractions.
- Mixed numbers have to be converted to fractions or decimals. For instance $3\frac{1}{2}$ has to be converted to $\frac{7}{2}$ or 3.5 before it is marked on the answer sheet.
- Write your answer in the first row. Note that this is optional and in real SAT it will not count as an answer. The only answer that counts is the answer that you mark on Fraction, Decimal Point and Number rows.
- If your answer has less than 4 characters, you can start from any column you wish. Leave the rest of the columns empty.
- If your answer has more than 4 characters, you can either mark the first 4 characters or you can round it to an appropriate 4-character number.

  For example, 10/3 = 3.33333333 can be marked in as 10/3, 3.33 or 3.30, but not as 3, 3.0, 3.3 or 3.4.
  200/3 can be marked in as 66.6 or 66.7. But 66 or 67 are not the correct answers.

Here are several examples to clarify the above statements.



Answer Sheet - Test 2

Section 1

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Section 2

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Section 3

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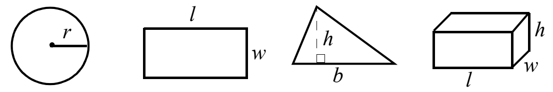
Test 2 - Section 1

25 Minutes, 20 Questions

Reference Information

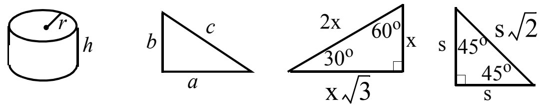


$A = \pi r^2$
$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

1. (m - 1)/(p + 1) = 1/m, m = 2, then p = ?

(A) -3
(B) -1
(C) 0
(D) 1
(E) 3

2. If John has $3.00 and Eric has $4.00, John's money is what percent of Eric's money?

(A) 3%
(B) 25%
(C) 33.3%
(D) 75%
(E) 133.3%

3. In the figure, what is the value of x?

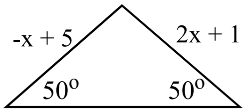


Figure is not drawn to scale.

(A) 3/4
(B) 4/3
(C) 3/2
(D) 2
(E) 5/2

4. In an eating contest, Mary eats 20 meatballs, x pound each, and drinks 8 glasses of lemonade, y pound each. How many pounds of food she consumed all together?

(A) 20x + 8y
(B) 20/x + 8/y
(C) x/20 + y/8
(D) 8x + 20y
(E) x/8 + y/20

5. The average of three numbers is 4. The average of two of them is 7. What is the value of the third number?

(A) 2
(B) 1
(C) 0
(D) -1
(E) -2

1/4

6. If an equilateral hexagon with side length a is inscribed inside a circle of radius 7, what is the value of a?

(A) 7
(B) $7\sqrt{2}$
(C) $7/(\sqrt{2})$
(D) $(\sqrt{205})/4$
(E) $7(\sqrt{3})/2$

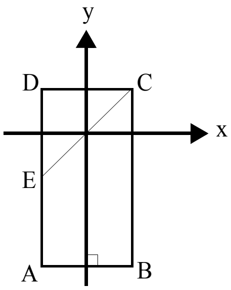
7. If ab = 1, then $\frac{1-a}{1-b} \cdot \frac{b}{a} = ?$

(A) -1/a
(B) -a
(C) b
(D) $-a^3$
(E) -a/b

1/3

8. In the figure, ABCD is a rectangle with AB = BC/2.

y-axis bisects $\overline{DC}$ and E bisects $\overline{AD}$. If DC = 6, what are the coordinates of B?



(A) (3, -9)
(B) (3, -6)
(C) (3, 9)
(D) (3, -12)
(E) (-3, -9)

9. In the figure, what is the area of the shaded region?

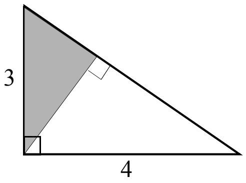


Figure is not drawn to scale.

(A) 1.51
(B) 1.92
(C) 2.00
(D) 2.08
(E) 2.16

10. Susan woke up at x AM. One hour after she woke up, she had breakfast at y AM, and had lunch at z PM. How many hours passed between the breakfast and lunch?

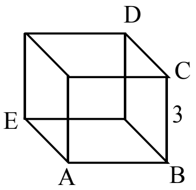
(A) z - x
(B) z - x - 1
(C) 11 + z - x
(D) 12 + z - x
(E) 13 + z - x

1/2

11. If you change the positions of the letters in the word "WHERE" randomly, how many different meaningful or meaningless words you can have in which both "E"s are side by side?

(A) 4
(B) 12
(C) 24
(D) 60
(E) 120

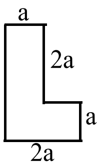
12. The cube in the figure is first cut by two distinct planes, both parallel to the bottom of the cube (the plane with points A, B and E on it) to obtain three identical prisms. It is then cut by two distinct planes which are parallel to the side of the cube (the plane with points B, C and D on it) to obtain additional identical prisms.



What is the volume of each prism obtained?

(A) 1
(B) 3
(C) 4.5
(D) 6
(E) 9

13. In the figure, all the angles of the "L" shaped object are 90^o. How many of these objects are required to create the smallest possible square tile, by putting these objects together without cutting or overlapping them?



(A) 2
(B) 4
(C) 6
(D) 8
(E) 10

14. Which of the following is the multiplication of 9 consecutive even integers between -8 and 8?

(A) -147456
(B) -2304
(C) 0
(D) 2304
(E) 147456

2/3

15. In the figure, h = x + 2 and the area of the triangle is 24. What is the value of $x^2 - 4$?

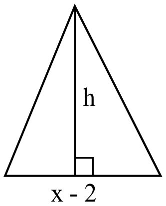


Figure is not drawn to scale.

(A) 12
(B) 24
(C) 48
(D) $\sqrt{24} - 2$
(E) $\sqrt{24} + 2$

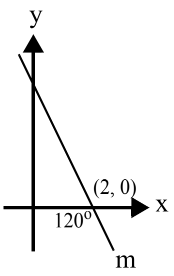
16. What is the value of x at which the quadratic function $f(x) = x^2 + 6x + 9$ has a minimum?

(A) x = 9
(B) x = 6
(C) x = 3
(D) x = 0
(E) x = -3

17. If $\frac{4x^2 - 2x - 2a}{(2x - a)} = 2(x + 1)$, then a = ?

(A) 4
(B) 3
(C) 2
(D) -2
(E) -3

18. Which of the following is the equation of the line m in the figure?



(A) $y = \sqrt{3}(x+2)$
(B) $y = -\sqrt{3}(x+2)$
(C) $y = -\sqrt{3}(x-2)$
(D) $y = -(1/\sqrt{3})x - 2$
(E) $y = -(1/\sqrt{3})x - \frac{1}{2}$

19. $f(x) = x^2 - 2x + 1$ and $g(s) = s^3$. If $g(f(x)) = 1$, then which of the following can be the value of x?

(A) -3
(B) -2
(C) -1
(D) 0
(E) 1

20. m and n are two positive integers. When m is divided by 3 the result is s, where s is an integer, and when n is divided by 3, the remainder is 2.
$m \cdot n = ?$

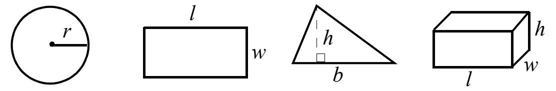
(A) k + 2s, where k is any positive integer.
(B) k + 2s, where k is any non-negative integer.
(C) 2k + 3s, where $k \geq 0$ and divisible by 2.
(D) k - 2s, where $k \geq 0$.
(E) k + 2s, where $k \geq 0$ and divisible by 3.

L

Test 2 - Section 2

25 Minutes, 18 Questions

Reference Information



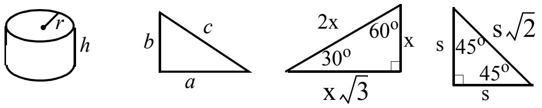
$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

Part 1

1. If $x - 5 + y = 3$, then $x + y + 3 = ?$

(A) 8
(B) 9
(C) 10
(D) 11
(E) 12

2. Which of the following (x, y) pairs satisfies the equation:

$$\frac{x-y}{x+y} \div \frac{y-x}{y+x} = 1$$

(A) (1, 1)
(B) (2, 2)
(C) (1, -1)
(D) (-1, 1)
(E) None of the above

1/4

3. In the figure, ABCD is a rectangle. What is the value of y?

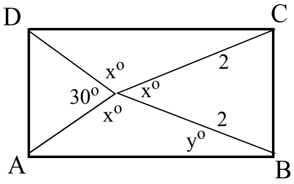


Figure is not drawn to scale.

(A) 30
(B) 35
(C) 50
(D) 55
(E) 70

1/3

4. The weight of a new born baby is given by the function $w(m) = 7.5 + 2.5m$, where m is the age of the baby in months. At what month the baby's weight is doubled?

(A) 2
(B) 2.5
(C) 3
(D) 3.5
(E) 4

1/2

5. Mary's mother is three times older than Mary. Mary was born two years after his brother Joe. If Joe is 20 years old, how old is Mary's mother?

(A) 18
(B) 36
(C) 54
(D) 60
(E) 66

6. If "p" percent of "n" is 6, and 10 percent of "p" is 3, what is n?

(A) 1.8
(B) 5
(C) 10
(D) 20
(E) 28

2/3

7. Two circles with radii r_1 and r_2 intersect each other. If $r_1 > r_2$ and the distance between their centers is d, what is the minimum value of d?

(A) r_1
(B) r_2
(C) $r_1 + r_2$
(D) $r_1 - r_2$
(E) $\sqrt{(r_1)^2 + (r_2)^2}$

8. If f(x) is a quadratic function with a minimum at point A(2, -1), which of the following has a minimum value at point O (0, 0)?

(A) $f(x) + 1$
(B) $f(x - 2) + 1$
(C) $f(x + 2) + 1$
(D) $f(x) - 2$
(E) $f(2 - x) + 1$

L

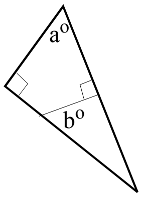
Part 2

9. Mr. McDonald has three sons. The youngest son’s age is 1/3 of the age of the oldest. The middle son is 2 years older than the youngest one. If the oldest son is 12 years old, how old is the middle son?

10. $|a + b| = 8$. If $a = 3$, what is the maximum value b^2 can take?

1/4

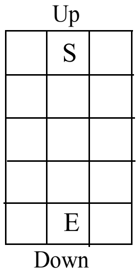
11. In the figure, $a - b = ?$



1/3

12. $a = b/2$, $b = 2c/3$. c is what percent of a?

13. In the figure, if you start from square labeled “S” and if you are allowed to go down only by moving diagonally or by jumping over the neighboring square just below you, in how many different ways can you reach the square labeled “E”? You are <u>not</u> allowed to pass through the same square more than once.



1/2

14. If $g(x) = (x + 2)^2$ and $f(x) = 2x - 1$, then $f(g(3)) = ?$

15. In the figure, CB = ? Approximate your answer to the nearest 100ths digit.

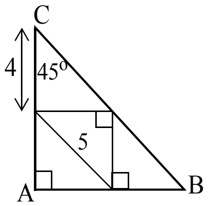


Figure is not drawn to scale.

2/3

16. You are traveling from town A to town B. If it takes 10 minutes to travel the first half of the distance and another 10 minutes to travel the half of the remaining half and another 10 minutes to travel half of the remaining distance, what percent of the original distance remains after 30 minutes of travel?

17. If x is 30% of y and y is 4/3 of z, what is z/x?

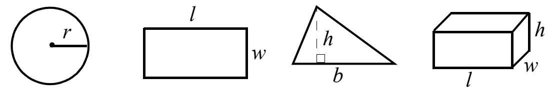
18. On the xy-plane, the center of circle O is (1, 2) and its radius is 5. Line p is perpendicular to y axis and passes through point (0, 3). What is the multiplication of the x-coordinates of the two points where line p intersects the circle?

L

Test 2 - Section 3

16 Questions, 20 minutes.

Reference Information



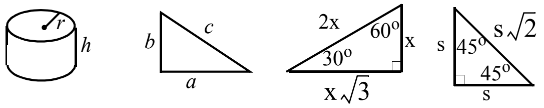
$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

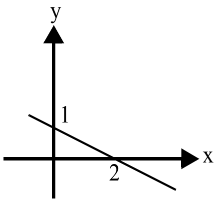
The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

1. If 3x + 7 = 100, then x - 5 = ?

(A) 26
(B) 31
(C) 36
(D) 41
(E) 92

2. Which of the following linear equations represents the graph of the line shown in the figure?



(A) y = -2x + 2
(B) y = -2x + 1
(C) y = x/2 + 1
(D) y = -x/2 + 2
(E) y = -x/2 + 1

3. ΔABC is a triangle with $\angle A < 45^o$ and $\angle B < \angle A$. Which of the following must be true?

(A) $\angle C = 90^o$
(B) $\angle C < 90^o$
(C) $\angle C > 90^o$
(D) $\angle C \leq 89°$
(E) $\angle C > 91^o$

4. In the figure, $l \parallel m$.
AB = BC = 2
What is the value of a?

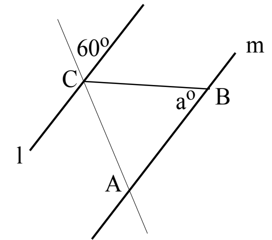


Figure is not drawn to scale.

(A) 15°
(B) 30°
(C) 45°
(D) 60°
(E) 120°

1/4

5. x @ z means xz and x & z means x/z.
If a @ b = a & b, then which of the following pairs can not be the values of a and b?

(A) (1, 1)
(B) (2, 1)
(C) 2, -1)
(D) (1, -1)
(E) (1, 2)

1/3

6. Gas mileage of a car is inversely proportional to the speed of the car. If the gas millage is 15 miles/gallon when the car is moving at 36 miles/hour, what is the speed of the car if the gas millage is 18 miles/gallon?

(A) 24 miles/hour
(B) 25 miles/hour
(C) 28 miles/hour
(D) 30 miles/hour
(E) 32 miles/hour

7. For an n-sided polygon, the average inside angle is 180(n - 2)/n. What is the average of the combined inner angles of a 3-sided and a 6-sided polygons combined?

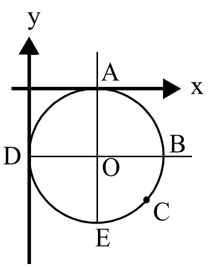
(A) 90^{o}
(B) 100^{o}
(C) 110^{o}
(D) 120^{o}
(E) 125^{o}

8. If $x < x^3 < x^2$, which of the following can be the value of x?

(A) -2
(B) -3/2
(C) -1
(D) -1/2
(E) 1/2

1/2

9. In the figure, x and y axes are tangent to the circle O at the points A and D, respectively. If the diameter of the circle O is 4 and point C bisects arc $\overset{\frown}{EB}$, what are the coordinates of point C?



(A) (2.5, 2.5)
(B) $(\sqrt{2}, -\sqrt{2})$
(C) $(2 + \sqrt{2}, 2 - \sqrt{2})$
(D) $(2 + \sqrt{2}, -2 - \sqrt{2})$
(E) $(1 + \sqrt{2}, -1 - \sqrt{2})$

10. $| -2x + 1 | = 3$ and $| 4y - 6 | = 12$. If $x > 0$ and $y < 0$, what is the value of $x + y$?

(A) 1/2
(B) 7/2
(C) -5/2
(D) 11/2
(E) -1/2

2/3

11. Once in every 5 minutes, the population of a certain bacteria triples. After how many minutes the population of the bacteria in a petri dish will increase to 81 times of its original size?

- **(A)** 5 minutes.
- **(B)** 10 minutes.
- **(C)** 15 minutes.
- **(D)** 20 minutes.
- **(E)** 25 minutes.

12. Two triangles in the figure are similar. $y > 90^{o}$ and $z > 90^{o}$. What is the value of a/b?

- **(A)** 3/5
- **(B)** 2
- **(C)** 1/2
- **(D)** 6/5
- **(E)** 5/6

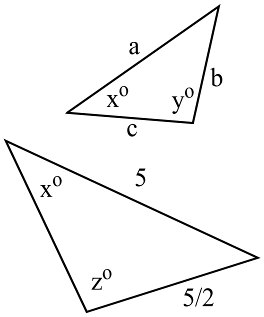


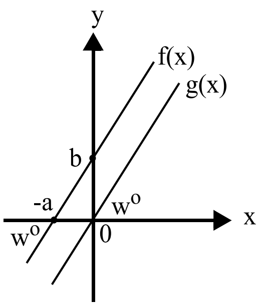
Figure is not drawn to scale.

13. If u(x) = sin(x) and v(x) = cos(x), then $u^2(x) + v^2(x) = ?$

- **(A)** -1
- **(B)** 1
- **(C)** 0
- **(D)** tan(x)
- **(E)** It depends on x. It is different for each value of x.

14. f(x) and g(x) are two functions, representing two parallel lines as shown in the figure. Which of the following statements is true?

- **I.** g(x) = f(x) - b
- **II.** g(x) = f(x - a)
- **III.** g(x) = f(x + a)



- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I and II
- **(E)** I and III

15. Karen is on a diet. The first month, she lost 3% of her weight. Second month she gained 1/3 of what she has lost in the first month. In the third month she lost 5% of the weight she had in the beginning of the third month. If she weighs 140 lbs. at the end of the third month, what was her original weight before she started dieting? Approximate your answer to the nearest whole number.

(A) 142
(B) 145
(C) 150
(D) 153
(E) 155

16. A circle is inscribed inside an isosceles right triangle as shown in the figure. If the area of the triangle is 16, what is the radius of the circle?

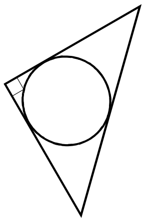


Figure is not drawn to scale.

(A) 1/3
(B) $\sqrt{2/3}$
(C) 4/3
(D) $\sqrt{8/\pi}$
(E) $4/(1+\sqrt{2})$

L

Answer Key - Test 2

Section 1

1.	(D)
2.	(D)
3.	(B)
4.	(A)
5.	(E)
6.	(A)
7.	(A)
8.	(A)
9.	(E)
10.	(C)
11.	(C)
12.	(B)
13.	(B)
14.	(C)
15.	(C)
16.	(E)
17.	(B)
18.	(C)
19.	(D)
20.	(E)

Section 2

Part 1

1.	(D)
2.	(E)
3.	(D)
4.	(C)
5.	(C)
6.	(D)
7.	(D)
8.	(C)

Part 2

9.	6
10.	25 or 121
11.	0
12.	300%
13.	11
14.	49
15.	9.9
16.	12.5%
17.	2.5 or 5/2
18.	-23

Section 3

1.	(A)
2.	(E)
3.	(C)
4.	(D)
5.	(E)
6.	(D)
7.	(B)
8.	(D)
9.	(D)
10.	(A)
11.	(D)
12.	(B)
13.	(B)
14.	(D)
15.	(C)
16.	(E)

Calculate Your Score

Calculate Your Raw Score

1. Count the number of correct answers and write it in Line 1of the following work sheet.

2. Count the number of incorrect (not missing) answers in Section 1, Section 2, Part 1 and Section 3. Divide this count by 4 and write it in the line 2.

Note that Part 2 of Section 2 has grid-in questions. No penalty is given for the incorrect answers in this part. That is why you don't need to count the incorrect answers here.

Also note that there is no penalty for missing any answers. That is why you don't need to count the missing answers.

3. Subtract Line 2 from Line 1 and write the result in Line 3.

4. Round the score on line 3 to the nearest whole number and write the result on Line 4. This is your raw score.

Work Sheet to Calculate the Raw Score

1.	Correct Answers	
2.	(Incorrect Answers)/4	
3.	Not rounded Raw Score	
4.	Raw Score	

Calculate Your SAT Score

Find your raw score in the below table and read the corresponding SAT score. Note that the actual SAT scores are expressed in ranges, not definite numbers. For our purposes, we assign only one SAT score for each raw score.

Raw Score	**SAT Score**	**Raw Score**	**SAT Score**	**Raw Score**	**SAT Score**
-6 or less	200	15	430	36	590
-5	205	16	435	37	595
-4	215	17	440	38	600
-3	225	18	450	39	610
-2	235	19	460	40	620
-1	245	20	465	41	630
0	260	21	470	42	640
1	280	22	480	43	650
2	290	23	490	44	655
3	310	24	495	45	660
4	320	25	500	46	670
5	330	26	510	47	690
6	340	27	520	48	700
7	360	28	525	49	710
8	370	29	530	50	730
9	380	30	540	51	740
10	385	31	550	52	760
11	390	32	560	53	775
12	400	33	565	54	800
13	410	34	570		
14	420	35	580		

Name:

Subject Table - Test 2

Date:

Question Number	Categories - Subjects				Difficulty Level
1.1	S. A - Mult. Unknowns				Easy
1.2	Arith. - Percentages	WQ - Regular			Easy
1.3	Geo. - Triangles	S. A - One Variable Equations			Easy
1.4	Arith. - Basic Arithmetic	WQ - Formulation Only			Easy
1.5	WQ -Statistics				Easy
1.6	Geo. - Polygons	Geo. - Triangles	Geo. - Circles	WQ - Desc. Figures	Easy
1.7	S. A - Mult. Unknowns				Medium
1.8	Geo. - Rectangles	Geo. - Squares	Geo. - Coordinate Geo.		Medium
1.9	Geo. - Triangles				Medium
1.10	WQ - Formulation Only				Medium
1.11	Others - Perm., Comb.	Others - Independent Events			Medium
1.12	Geo. - 3-D Objects	Others - Basic Counting			Medium
1.13	Others - Basic Counting				Medium
1.14	Arith. - Basic Arithmetic				Medium
1.15	Geo. - Triangles	S. A - Mult. Unknowns			Medium
1.16	A. A - Quadratic Functions				Hard
1.17	S. A - Mult. Unknowns				Hard
1.18	Geo. - Triangles	Geo. - Trigonometry	Geo. - Coordinate Geo.	A. A - Linear Func.	Hard
1.19	S. A - Exp. with Powers	A. A - Functions			Hard
1.20	Arith. - Divisibility				Hard
2.1	S. A - Mult. Unknowns				Easy
2.2	Arith. - Basic Arithmetic	S. A - Mult. Unknowns			Easy
2.3	Geo. - Points and Lines	Geo. - Angles	Geo. - Triangles	Geo. - Rectangles	Easy
2.4	S. A - Mult. Unknowns	A. A - Linear Functions	WQ - Regular		Medium
2.5	Arith. - Basic Arithmetic	WQ - Regular			Medium
2.6	Arith. - Percentages				Medium
2.7	Geo. - Circles	WQ - Describing Figures			Hard
2.8	Geo. - Coordinate Geo.	A. A - Functions			Hard
2.9	Arith. - Mult. Unknowns	WQ - Regular			Easy
2.10	S. A - Exp. with Abs. Value				Medium
2.11	Geo. - Triangles	S. A. - Mult. Unknowns			Medium
2.12	Arith. - Fractions, Ratios	Arith. - Percentages	S. A - Mult. Unknowns		Medium
2.13	Others - Basic Counting				Medium
2.14	A. A - Functions				Medium
2.15	Geo. - Points and Lines	Geo. - Triangles	Geo. - Squares	Others - Rounding	Medium
2.16	Arith. - Fractions, Ratios	Arith. - Percentages	Others - Basic Counting	WQ - Regular	Hard
2.17	Arith. - Fractions, Ratios	Arith. - Percentages	S. A - Mult. Unknowns		Hard
2.18	Geo. - Triangles	Geo. - Circles	Geo. - Coordinate Geo.	WQ - Desc. Figures	Hard
3.1	S. A - One Variable Eqs.				Easy
3.2	A. A - Linear Functions				Easy
3.3	Geo. - Triangles	WQ - Describing Figures			Easy
3.4	Geo. - Points and Lines	Geo. - Angles	Geo. - Triangles		Easy
3.5	S. A - Mult. Unknowns	Others - Defined Operators			Easy
3.6	S. A - One Variable Eqs.	S. A - Proportionality	WQ - Regular		Medium
3.7	Geo. - Polygons	Others - Statistics			Medium
3.8	Arith. - Negative Numbers	Arith. - Nmb. beween -1 and 1			Medium
3.9	Geo. - Circles	Geo. - Trigonometry	Geo. - Coordinate Geo.		Medium
3.10	S. A - One Variable Eqs.	S. A - Exp. with Abs. Value			Medium
3.11	Arith. - Powers	Others - Sequences	Others - Basic Counting	WQ - Regular	Medium
3.12	Geo. - Triangles				Hard
3.13	Geo. - Trigonometry				Hard
3.14	A. A - Functions	A. A - Linear Functions			Hard
3.15	Arith. - Basic Arithmetic	Arith. - Fractions, Ratios	Arith. - Percentages	WQ - Regular	Hard
3.16	Geometry - Triangles	Geometry - Circles			Hard

Skipped:

Wrong:

Analysis Chart - Test 2

Name: **Date:**

Category	Subject	Easy	Medium	Hard
Arithmetic	Basic Arithmetic	1.4, 2.2	1.14, 2.5	3.15
	Decimals			
	Fractions, Ratios		2.12	2.16, 2.17, 3.15
	Percentages	1.2	2.6, 2.12	2.16, 2.17, 3.15
	Powers		3.11	
	Square Root			
	Radicals			
	Negative Numbers		3.8	
	Numbers Between –1 and 1		3.8	
	Divisibility			1.20
	Even & Odd Numbers			
	Absolute Value			
Geometry	Points and Lines	2.3, 3.4	2.15	
	Angles	2.3, 3.4		
	Polygons	1.6	3.7	
	Triangles	1.3, 1.6, 2.3, 3.3, 3.4	1.9, 1.15, 2.11, 2.15	1.18, 2.18, 3.12, 3.16
	Quadrangles			
	Rectangles	2.3	1.8	
	Squares		1.8, 1.13, 2.15	
	Circles	1.6	3.9	2.7, 2.18, 3.16
	Trigonometry		3.9	1.18, 3.13
	Coordinate Geometry		1.8, 3.9	1.18, 2.8, 2.18
	Symmetry			
	3-D Objects		1.12	
Simple Algebra (S. A.)	One Variable Equations	1.3, 3.1	3.6, 3.10	
	Multiple Unknowns	1.1, 2.1, 2.2, 2.9, 3.5	1.7, 1.15, 2.4, 2.6, 2.11, 2.12	1.17, 2.17
	Equations with Powers			1.19
	Radical Equations			
	Inequalities			
	Expressions with Absolute Value		2.10, 3.10	
	Proportionality		3.6	
Advanced Algebra (A. A.)	Functions		2.14	1.19, 2.8, 3.14
	Linear functions	3.2	2.4	1.18, 3.14
	Quadratic Functions			1.16
Others	Rounding		2.15	
	Tables, Charts, Graphs			
	Sets			
	Defined Operators	3.5		
	Logic			
	Statistics	1.5	3.7	
	Sequences		3.11	
	Sums			
	Basic Counting		1.12, 1.13, 2.13, 3.11	2.16
	Permutations, Combinations		1.11	
	Mutually Exclusive Events			
	Independent Events		1.11	
	Probability			
Word Questions (WQ)	Regular	1.2, 2.9	2.4, 2.5, 3.6, 3.11	2.16, 3.15
	Formulation Only	1.4	1.10	
	Describing Figures	1.6, 3.3		2.7, 2.18

Skipped: **Wrong:**

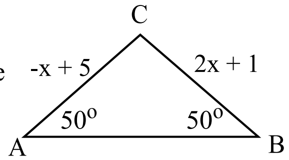
Solutions - Test 2

Section 1

1. Answer: (D)
(m - 1)/(p + 1) = 1/m
Substituting m = 2 into the equation:
(2 - 1)/(p + 1) = 1/2 ➔ 1/(p + 1) = 1/2 ➔
p + 1 = 2 ➔ p = 1
The answer is (D).

2. Answer: (D)
If John has $3.00 and Eric has $4.00, John's money is $\frac{3}{4} \times 100 = 75$ percent of Eric's money.
The answer is (D).

3. Answer: (B)

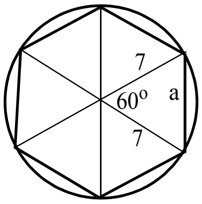


Since two of the inner angles of the triangle are the same, ΔABC is an isosceles triangle. ➔
AC = CB ➔
-x + 5 = 2x + 1 ➔
3x = 4 ➔ x = 4/3
The answer is (B).

4. Answer: (A)
If Mary eats 20 meatballs, x pound each, it means she ate 20x pounds of meatballs.
If she drinks 8 glasses of lemonade, y pound each, it means she drank 8y pounds of lemonade.
Hence she consumed a total of 20x + 8y pounds of food. The answer is (A).

5. Answer: (E)
Let a, b and c be three numbers. The average of these three numbers is 4 ➔ a + b + c = 3 · 4 = 12
The average of two of them (a and b) is 7. ➔
a + b = 2 · 7 = 14 ➔
c = (a + b + c) - (a + b) = 12 - 14 = -2
The answer is (E).

6. Answer: (A)



The equilateral hexagon inscribed inside a circle is shown in the figure. Since a hexagon has 6 sides, each of the 6 angles formed at the cross section of the diagonals in the center of the circle is 360/6 = 60°, as shown in the figure.

Since each of the triangles are already isosceles with two of their sides equal to the radius of the circle (which is 7) the other two angles of the triangles must be (180 - 60)/2 = 60°. Hence all 6 triangles are equilateral triangles with a = 7

The answer is (A).

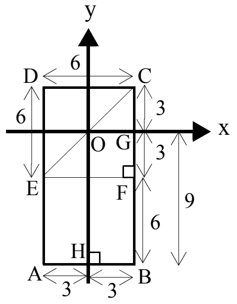
7. Answer: (A)
If ab = 1, then $\frac{1-a}{1-b} \cdot \frac{b}{a} = \frac{b-ab}{a-ab} = \frac{b-1}{a-1}$

ab = 1 ➔ b = 1/a ➔ $\frac{b-1}{a-1} = \frac{\frac{1}{a}-1}{a-1} = \frac{\frac{1-a}{a}}{a-1} =$

$-\frac{a-1}{a} \cdot \frac{1}{a-1} = -\frac{1}{a}$

The answer is (A).

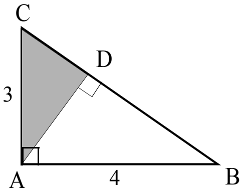
8. Answer: (A)



Let's label some of the points as shown in the figure. E bisects $\overline{AD}$ and DC = 6 ➔
BC = AD = 2 · 6 = 12 and
CF = ED = DC = 6 ➔ EFCD is a square. EC is a diagonal of this square, passing through the origin, O (0, 0) of the xy-plane.
EO = OC ➔
O is the center of the square EFCD ➔
AH = HB = CG = GF = 6/2 = 3
Since $\overline{OH} \perp \overline{AB}$, the long sides, $\overline{AD}$ and $\overline{BC}$, of the rectangle are parallel to the y-axis and the short sides, $\overline{DC}$ and $\overline{AB}$ of the rectangle are parallel to the x-axis. ➔
x-coordinate of B is HB = 3 and
y-coordinate of B is -(GF + FB) = -(3 + 6) = -9
The answer is (A).

Alternate Solution:
The above solution is long and tedious, but it is necessary for a formal solution. On the other hand, the figure is drawn to scale. The dimensions in it are quite obvious. You don't need to prove that they are indeed the correct dimensions.
x-coordinate of B is HB = 3 and
y-coordinate of B is -(GF + FB) = -(3 + 6) = -9
The answer is (A).

9. Answer: (E)



In the figure, ΔABC is a right triangle. By using the Pythagorean theorem
CB = $\sqrt{3^2 + 4^2}$ = 5

The area of ΔABC =
$\frac{4 \cdot 3}{2} = 6 = \frac{CB \cdot AD}{2} = \frac{5AD}{2}$ ➔ AD = 12/5

ΔADC is a right triangle. Using the Pythagorean Theorem:
CD = $\sqrt{3^2 - (12/5)^2}$ = 9/5
The area of $\Delta ADC = \frac{CD \cdot AD}{2} =$
$\frac{1}{2}\left(\frac{9}{5} \cdot \frac{12}{5}\right) = \frac{54}{25} = 2.16$

10. Answer: (C)
Susan woke up at x AM. One hour after she woke up, she had breakfast at y AM. ➔ $y = x + 1$
$12 - y = 12 - x - 1 = 11 - x$ hours passed between breakfast and noon.
She had lunch at z PM. ➔ z hours passed between noon and lunch.
So the total hours passed between breakfast and lunch is $11 - x + z = 11 + z - x$
The answer is (C).

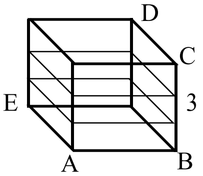
11. Answer: (C)
In the word WHERE, there are 5 positions for 5 letters. Let's label them as 1, 2, 3, 4, 5. There are only 4 ways that both "E"s will be side by side. They are when "E"s occupy the positions (1,2) or (2, 3) or (3, 4) or (4, 5). For each of these cases, the remaining 3 letters can arrange themselves in the remaining 3 positions in 3! = 6 different ways. Since this can be done for each of the 4 cases mentioned above, the total number of words that can be obtained is $4 \cdot 6 = 24$. The answer is (C).

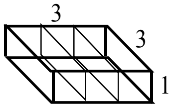
12. Answer: (B)
The first cut will create 3 identical prisms with dimensions (3, 3, 1) as shown in the middle figure.

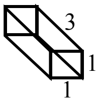
The second cut will create 9 identical prisms with dimensions (3, 1, 1), as shown in the last figure.

The volume of prisms with dimensions (3, 1, 1) is
$3 \times 1 \times 1 = 3$

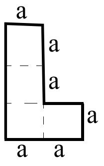
The answer is (B).



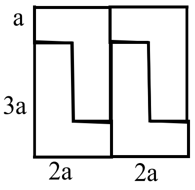




13. Answer: (B)
If the final tile is a square, its area must be a multiple of the area of the L-shaped object shown in the figure. The area of this object is $4a^2$. So the square has to have area of $8a^2$, $12a^2$, $16a^2$, etc..
On the other hand, because of the no cutting or overlapping rule, the edges of the square tile has to be a multiple of a. If the area is $8a^2$, then the edge is $2\sqrt{2}a$, which is not a multiple of a. Similarly, if the area is $12a^2$, then the edge is $2\sqrt{3}a$

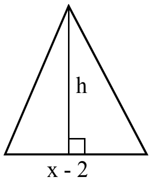


However, if the area is $16a^2$, then the edge length is 4a, which is a multiple of a. It will take 4 of the L-shaped objects to have a square tile with area $16a^2$. The figure shows this arrangement of the 4 L-shaped objects.



14. Answer: (C)
0 is an even integer between -8 and 8, and the multiplication of any number by 0 is 0.

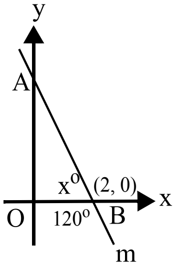
15. Answer: (C)
In the figure, $h = x + 2$ and the area of the triangle is 24. ➔
$(x + 2)(x - 2)/2 = (x^2 - 4) /2 = 24$ ➔
$(x^2 - 4) = 48$.
The answer is (C).



16. Answer: (E)
$f(x) = x^2 + 6x + 9 = (x + 3)^2$
Since f(x) is a square of a term, it is 0 or positive for all the values of x. ➔
The minimum is when $f(x) = 0$ ➔ $x + 3 = 0$ ➔ $x = -3$
The answer is (E).

17. Answer: (B)
$\frac{4x^2 - 2x - 2a}{(2x - a)} = 2(x + 1)$ ➔
$2(2x^2 - x - a) = 2(x + 1)(2x - a)$ ➔
$2x^2 - x - a = (x + 1)(2x - a) = 2x^2 - ax + 2x - a$ ➔
$3x - ax = 0$ ➔ $x(3 - a) = 0$ ➔ $(3 - a) = 0$ ➔ $a = 3$
The answer is (B).

18. Answer: (C)
In the figure
$x^o =$
$\angle ABO = 180 - 120 = 60°$
The slope of line m = -tan(x) =
$-\tan(60) = -\frac{\sin(60)}{\cos(60)} =$
$-\frac{(\sqrt{3})/2}{1/2} = -\sqrt{3}$ ➔



$\tan(x) = OA/OB = OA/2 = \sqrt{3}$ ➔ $OA = 2\sqrt{3}$ ➔
y-intercept of the line is $2\sqrt{3}$
Hence the equation of the line is
$y = -\sqrt{3}x + 2\sqrt{3} = -\sqrt{3}(x - 2)$
The answer is (C).

19. Answer; (D)
$f(x) = x^2 - 2x + 1 = (x - 1)^2$, $g(s) = s^3$ ➔
$g(f(x)) = [(x - 1)^2]^3 = (x - 1)^6 = 1$ ➔
$x - 1 = 1$ ➔ $x = 2$ (None of the answer choices is 2) or
$x - 1 = -1$ ➔ $x = 0$. The answer is (D).

20. Answer: (E)
m is a positive integer. When it is divided by 3 the result is s. ➔ m = 3s, where s is an positive integer.
n is a positive integer. When n is divided by 3, the remainder is 2. ➔ n = p + 2/3, where p is a non-negative integer.
$m \cdot n = 3s \cdot (p + 2/3) = 3p \cdot s + 2s = k + 2s$,
where k = 3ps. Since s is positive and p is non-negative, k is non-negative and divisible by 3.
The answer is (E).

Section 2

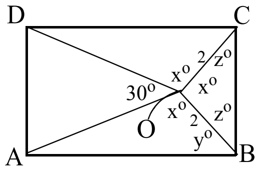
Part 1

1. Answer: (D)
$x - 5 + y = 3$ ➔ $x + y = 3 + 5 = 8$ ➔
$x + y + 3 = 8 + 3 = 11$
The answer is (D).

2. Answer: (E)

$$\frac{x-y}{x+y} \div \frac{y-x}{y+x} = \frac{x-y}{x+y} \cdot \frac{x+y}{-(x-y)} = -1$$

For all values of x and y, the expression is -1, not 1. The answer is (E).



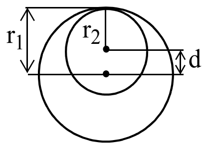
3. Answer: (D)
Let's redraw the figure close to its scale so that the 3 equal angles with measures of x^o look equal as shown in the figure.
$3x + 30 = 360$ ➔ $x = 110$

$OC = OB$ ➔ ΔOCB is an isosceles triangle. ➔
$\angle OCB = \angle OBC = z^\circ = (180 - x)/2 = (180 - 110)/2 = 35^\circ$
Since ABCD is a rectangle, $y + z = y + 35 = 90$ ➔
$y = 55$
The answer is (D).

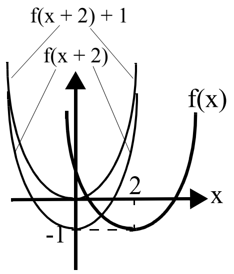
4. Answer: (C)
When a baby is born at month m = 0, her weight is w(0) = 7.5 lbs. If the baby's weight is doubled and becomes 15 lbs. months later, then
$w(m) = 7.5 + 2.5m = 15$ ➔
$m = (15 - 7.5)/2.5 = 3$ months
The answer is (C).

5. Answer: (C)
Mary was born two years after his brother Joe. ➔
Mary is 2 years younger than Joe. ➔
If Joe is 20 years old, then Mary is 18 years old. ➔
Mary's mother is three times older than Mary. ➔
Her mother is $3 \times 18 = 54$ years old.
The answer is (C).

6. Answer: (D)
10 percent of "p" is 3 ➔ $10p/100 = p = 30$
"p" percent of "n" is 6 ➔ $pn/100 = 6$ ➔
$30n/100 = 6$ ➔ $n = 20$
The answer is (D).



7. Answer: (D)
The minimum value of d occurs when the smaller circle is inside of the larger circle and tangent to it (crosses it at only one point), as shown in the figure. It is clear that $d = r_1 - r_2$
The answer is (D).

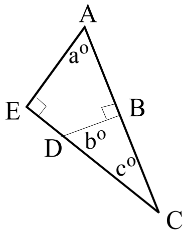


8. Answer: (C)
f(x) is shown in the figure.
f(x + 2) will shift f(x) by 2 toward left to a minimum (0, -1) as shown in figure.
f(x + 2) + 1 will raise f(x + 2) by 1 and hence move its minimum to (0, 0) as shown in the figure.
The answer is (C).

Part 2

9. Answer: 6 years old
Let y, m and o be the ages of Mr. McDonald's 3 sons from youngest to the oldest.
The youngest son's age is 1/3 of the oldest one. ➔
$y = o/3$
The middle son is 2 years older than the youngest one.
➔ $m = 2 + y$
If $o = 12$, then $y = o/3 = 12/3 = 4$ years old.
$m = 2 + y = 2 + 4 = 6$ years old.

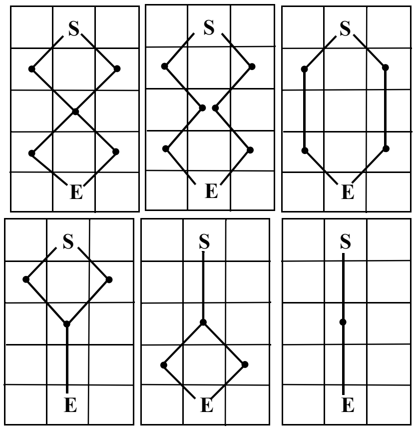
10. Answer: 25 or 121
$|a + b| = 8$ and $a = 3$ ➔ $|3 + b| = 8$
3 + b can have 2 values:
Case 1: $3 + b = 8$ ➔ $b = 8 - 3 = 5$ ➔ $b^2 = 25$
Case 2: $3 + b = -8$ ➔ $b = -11$ ➔ $b^2 = 121$
The answer is 121.



11. Answer: 0
In the figure, the addition of the inner angles of the triangles ΔACE and ΔBCD are both 180^o. ➔
$a + 90 + c = 180$
$b + 90 + c = 180$
Hence $a = b$ ➔ $a - b = 0$

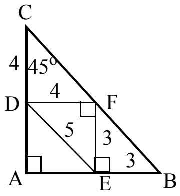
12. Answer: 300%
$b = 2c/3$ ➔ $c = 3b/2$
$a = b/2$ ➔ $c/a = \frac{3b}{2} \div \frac{b}{2} = \frac{3b}{2} \cdot \frac{2}{b} = 3$ ➔
c is $3 \cdot 100 = 300\,\%$ of a.

13. Answer: 11
There are 11different ways and they are shown in the figure.



14. Answer: 49
$g(x) = (x + 2)^2$ ➔ $g(3) = (3 + 2)^2 = 5^2 = 25$
$f(x) = 2x - 1$ ➔ $f(g(3)) = f(25) = 2 \cdot 25 - 1 = 49$

15. Answer: 9.90



In the figure,
$\angle CBA = 180 - 90 - 45 = 45^o$ ➔
$\overline{EF} \perp \overline{AB}$ and $\overline{EF} \perp \overline{DF}$ ➔
$\overline{DF} \parallel \overline{AB}$ ➔
$\angle CFD = \angle EFB = 45°$ ➔
ΔCFD and ΔEBF are isosceles. ➔ DF = DC = 4

ΔEFD is a right triangle. ➔

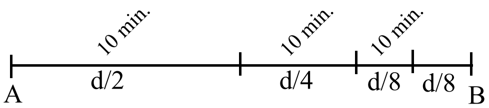
$EF^2 = 5^2 - DF^2 = 5^2 - 4^2 = 25 - 16 = 9$ ➔ EF = 3
ΔEBF ➔ EB = EF = 3 ➔

$CF^2 = 4^2 + 4^2$ ➔ $CF = 4\sqrt{2}$

$BF^2 = 3^2 + 3^2$ ➔ $BF = 3\sqrt{2}$

$CB = CF + BF = 7\sqrt{2} \cong 9.90$

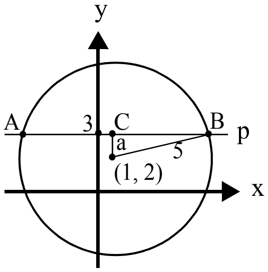
16. Answer: 12.5%



Let d represents the total distance between towns A and B as shown in the above figure. At the end of 30 minutes, the remaining distance is d/8. ➔
The remaining distance is:
$$\left(\frac{d}{8}\right) \cdot \frac{100}{d} = \frac{100}{8} = 12.5\,\%$$
of the original distance, d.

17. Answer: 2.5 or 5/2
x is 30% of y ➔ x = 30y/100 = 3y/10
y is 4/3 of z ➔ y = 4z/3 ➔
$x = \frac{3}{10} \cdot \frac{4z}{3} = \frac{4z}{10} = \frac{2z}{5}$ ➔ z/x = 5/2 = 2.5

18. Answer: -23



Let's first draw the circle and the line p on the xy-plane.

In the figure,
a = 3 - 2 = 1
CB = AC =
$\sqrt{5^2 - 1^2} = \sqrt{24}$ ➔

x-coordinate of A = $-(\sqrt{24} - 1)$ and

x-coordinate of B = $\sqrt{24} + 1$

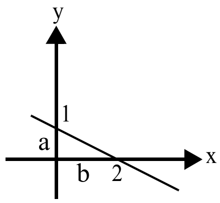
(x-coordinate of A)(x-coordinate of B) =

$-(\sqrt{24} - 1)(\sqrt{24} + 1) = -(24 - 1) = -23$

Section 3

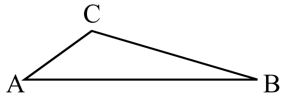
1. Answer: (A)
3x + 7 = 100 ➔ x = (100 - 7)/3 = 93/3 = 31 ➔
x - 5 = 31 - 5 = 26
The answer is (A).

2. Answer: (E)



The slope of the line is negative, because it is descending.
The slope is -a/b = -1/2
The y-intercept is 1.
Hence the equation of the line is y = -x/2 + 1
The answer is (E).

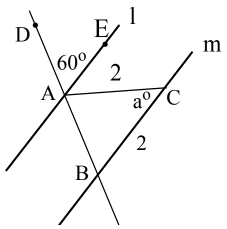
3. Answer (C)



Let's draw the figure as shown.

If $\angle A < 45^o$ and $\angle B < \angle A$, then $\angle B < 45^o$ ➔
$\angle A + \angle B < 45^o + 45^o = 90^o$ ➔
Since $\angle A + \angle B + \angle C = 180^o$, then
$\angle C = 180^o - (\angle A + \angle B) > 180^o - 90^o = 90^o$.
The answer is (C).

4. Answer: (D)



$l \parallel m$ ➔
$\angle ABC = \angle DAE = 60°$
AC = BC = 2 ➔
$\angle CAB = 60°$ ➔
a = 180 - 60 - 60 = 60

The answer is (D).

5. Answer: (E)
If a @ b = a & b, then ab = a/b ➔
b = 1/b ➔ b = 1 or b = -1
The only answer choice that does not have 1 or -1 as the value of b is (E).
The answer is (E).

6. Answer: (D)
Let m and s be the gas mileage and the speed of the car respectively.
Gas mileage of a car is inversely proportional to the speed of the car. ➔ m = a/s, where a is a constant.

m = 15 miles/gallon when s = 36 miles/hour ➔
15 = a/36 ➔
a = 15 × 36 = 540 mile^2/gallon-hour.
m = a/s ➔ s = a/m = 540/m

If m = 18 miles/gallon, then
s = 540/18 = 30 miles/hour

The answer is (D).

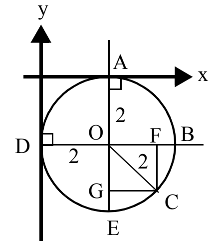
7. Answer: (B)
For an n-sided polygon, the average inside angle is 180(n - 2)/n. →
Addition of the inner angles of an n-sided polygon is 180(n - 2). →
The inner angles of a 3-sided polygon (triangle) = 180^o
The inner angles of a 6-sided polygon (hexagon) = $180(6 - 2) = 720^o$
There are 6 + 3 = 9 inner angles in a triangle and a hexagon combined with a total measure of
$180 + 720 = 900^o$ → The average measure of an angle is $900/9 = 100^o$.

The answer is (B).

8. Answer: (D)
$x < x^3 < x^2$ →
x is a negative number between -1 and 0.
The answer is (D).

To check the answer, let's substitute -1/2 for x into the above inequality: -1/2 < -1/8 < 1/4, which is true.

9. Answer: (D)
x and y axes are tangent to circle O at points A and D respectively. →
$\overline{AE} \perp x$ -axis and
$\overline{DB} \perp y$ -axis. →
$\angle EOB = 90°$



The diameter of circle O is 4 and C bisects $\overset{\frown}{EB}$ →
$\angle EOC = \angle COB = 90/2 = 45°$ →
OF = OG = 2cos(45) = $2 \cdot (\sqrt{2})/2 = \sqrt{2}$ →
x-coordinate of C = DF = 2 + $\sqrt{2}$
y-coordinate of C = -AG = -2 - $\sqrt{2}$
The answer is (D).

Alternate Solution:
The figure is drawn to scale. So the x- and y-coordinates of point C looks like equal in magnitude and opposite in direction. (x-coordinate is positive and y-coordinate is negative.) The three answer choices that satisfy these conditions are (B), (D) and (E). The x-coordinate of point C looks like more than 2 but a bit less than 4. Answer choice (B) is definitely not the right answer. The best choice is (D).

10. Answer: (A)
| -2x + 1 | = 3 → There are 2 cases:
Case 1: $-2x + 1 \geq 0$ → -2x + 1 = 3 → x = -1
Since x > 0, x = -1 is not a valid solution.
Case 2: -2x + 1 < 0 → 2x - 1 = 3 → x = 2
This is a valid solution because 2 is a positive number.
Similarly, | 4y - 6 | = 12 → There are 2 cases:
Case 1: $4y - 6 \geq 0$ → 4y - 6 = 12 → y = 9/2
Since y < 0, y = 9/2 is not a valid solution.
Case 2: 4y - 6 < 0 → 6 - 4y = 12 → y = -3/2
This is a valid solution because -3/2 is a negative number.
x + y = 2 - 3/2 = 1/2
The answer is (A).

11. Answer: (D)
Let p be the original population.
In the first 5 minutes, the population reaches 3p.

In the second 5 minutes, the population reaches
$3 \cdot 3p = 3^2p$

During the third 5 minutes, the population reaches
$3 \cdot 3^2p = 3^3p$

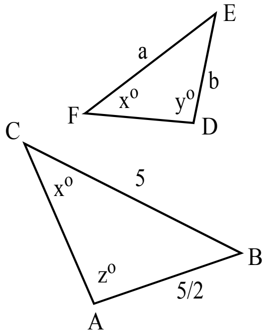
.

.

At the nth 5 minutes, the population reaches 3^np

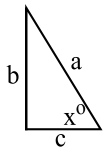
$81 = 3^4$ → The bacteria population has increased by 81 times during the fourth 5 minute interval, which is $4 \cdot 5 = 20$ minutes.
The answer is (D).

12. Answer: (B)
Since in any triangle there can be only one angle which is greater than 90^o, if these two triangles are similar, then z = y, because both y and z are greater than 90.
In the figure,
$\angle F = \angle C = x°$ →
$\Delta ABC \sim \Delta DEF$ →



a/b = EF/ED = BC/BA = 5/(5/2) = 2
The answer is (B).

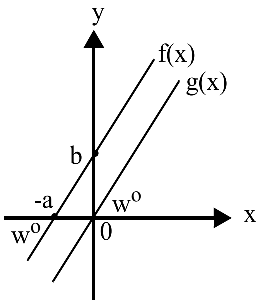
13. Answer: (B)
u(x) = sin(x) and v(x) = cos(x)
In the figure,
sin(x) = b/a and cos(x) = c/a



$u^2(x) + v^2(x) = \sin^2(x) + \cos^2(x) =$
$(b/a)^2 + (c/a)^2 = (b^2 + c^2)/a^2 = a^2/a^2 = 1$

Note that $\sin^2(x) + \cos^2(x)$ is always 1, regardless of the value of x. The triangle is a general right triangle. The values of a, b, c and x are not specified.

14. Answer: (D)
Since the two lines are parallel, they have the same slope, b/a. The two differences between the two lines are their x and y intercepts.



Since the difference between y-intercepts of f(x) and g(x) is b, g(x) = f(x) - b ➔ I is true.

Since the difference between x-intercepts f(x) and g(x) is -a, g(x) = f(x - a) ➔ II is true.

The answer is (D).

15. Answer: (C)
Let w be Karen's original weight.
First month, she lost 3% of her weight. ➔ She lost 3w/100 pound in the first month. Her weight at the end of the first month is (100 - 3)w/100 = 97w/100.

In the second month, she gained 1/3 of what she has lost in the first month. ➔
She gained (3w/100)/3 = 3w/300 = w/100 pounds.
Her weight at the end of the second month is 97w/100 + w/100 = 98w/100.

During the third month, she lost 5% of the weight she had in the beginning of the third month. ➔
She lost $\frac{5}{100}\left(\frac{98w}{100}\right)$ pounds.
Her weight at the end of third month is

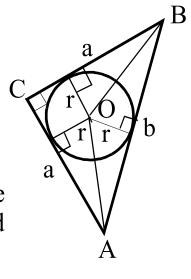
$$\frac{98w}{100} - \frac{5 \times 98w}{100 \times 100} = \frac{931w}{1000} = 140 \text{ pounds.} ➔$$

$$w = 140 \times \frac{1000}{931} \cong 150 \text{ lbs.,}$$

approximated to the nearest whole number.

The answer is (C).

16. Answer: (E)
ΔABC is isosceles. ➔
AC = BC = a, as shown in the figure.
r is the radius of the circle. Note that the 3 radii shown in the figure are perpendicular to the sides of the triangle since the circle is inscribed inside the triangle. Hence the sides of the circle are tangent to the circle.



The area of $\Delta ABC = a^2/2 = 16$ ➔
$a = \sqrt{32} = 4\sqrt{2}$
Using the Pythagorean Theorem,

$b^2 = a^2 + a^2 = 2a^2$ ➔ $b = \sqrt{2}a = 2 \cdot 4 = 8$

On the other hand, since the 3 radii shown in the figure are the heights of the triangles ΔABO, ΔACO and ΔBCO, the area of $\Delta ABC =$
Area of ΔABO +
Area of ΔACO +
Area of ΔBCO =
(1/2)(ar + ar + br) =
$(r/2)(2 \cdot 4\sqrt{2} + 8) = 4r(\sqrt{2} + 1) = 16$ ➔
$r = 4/(\sqrt{2} + 1)$

The answer is (E).

Test 3

Date:

General Directions:

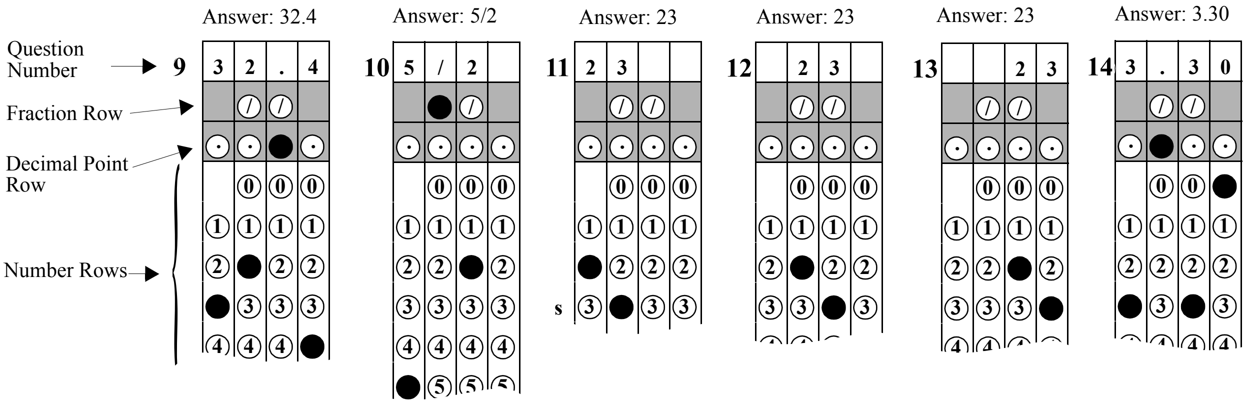
- The test is 70 minutes long.
- It has 3 sections.
  Section 1: 20 multiple choice questions, 25 minutes.
  Section 2: 8 multiple choice and 10 grid-in questions, 25 minutes.
  Section 3: 10 multiple choice questions, 20 minutes.
- You can only work on one section at a time.
- You are not allowed to transfer your answers from the test to the Answer Sheet after the time is up for each section. Therefore mark your answer on the Answer Sheet as soon as you finish answering a question.
- Make sure that the question number in the test matches the question number on the answer sheet.
- Mark only one answer for each question.
- If you want to change your answer, erase the old answer completely.
- You receive one point for each correct answer.
- You lose one point for 4 incorrect multiple choice answers.
- You don't lose any points for incorrect grid-in questions.
- You neither gain nor loose points for missing answers.

Directions for the Grid-In Questions

- You can only have zero or positive numbers as answers for the grid-in questions. Therefore, if your answer is negative it is wrong.
- The upper limit of your answer is 9999. If your answer is greater than 9999, it is wrong.
- You can express your answers as integers, fractions or decimal numbers. Do not spend the time to convert fractions to decimals or decimals to fractions.
- Mixed numbers have to be converted to fractions or decimals. For instance $3\frac{1}{2}$ has to be converted to $\frac{7}{2}$ or 3.5 before it is marked on the answer sheet.
- Write your answer in the first row. Note that this is optional and in real SAT it will not count as an answer. The only answer that counts is the answer that you mark on Fraction, Decimal Point and Number rows.
- If your answer has less than 4 characters, you can start from any column you wish. Leave the rest of the columns empty.
- If your answer has more than 4 characters, you can either mark the first 4 characters or you can round it to an appropriate 4-character number.

  For example, 10/3 = 3.33333333 can be marked in as 10/3, 3.33 or 3.30, but not as 3, 3.0, 3.3 or 3.4.
  200/3 can be marked in as 66.6 or 66.7. But 66 or 67 are not the correct answers.

Here are several examples to clarify the above statements.



Answer Sheet - Test 3

Section 1

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Section 2

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Section 3

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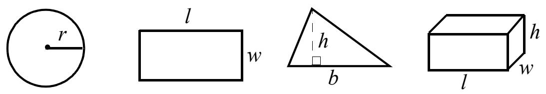
Test 3 - Section 1

25 Minutes, 20 Questions

Reference Information



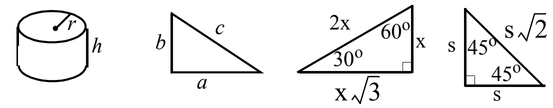
$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

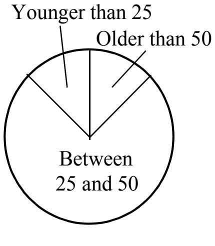
1. If 3/x - 8 = 4, then x = ?

- **(A)** 4
- **(B)** 1/4
- **(C)** 3
- **(D)** 1/3
- **(E)** -1/3

2. If $f(x) = 2^x + 1$, then f(5) = ?

- **(A)** 10
- **(B)** 11
- **(C)** 25
- **(D)** 26
- **(E)** 33

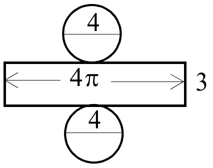
3. Pie chart shows the population of 3 different age groups in an adult education class; the students older than 50, the students younger than 25 and the students between 25 and 50. What is the approximate percentage of the students between 25 and 50 years of age?



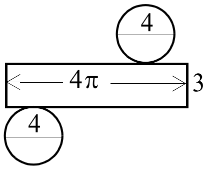
- **(A)** 20%
- **(B)** 25%
- **(C)** 60%
- **(D)** 75%
- **(E)** 90%

4. Which of the following can NOT be folded into a cylinder without cutting or overlapping?

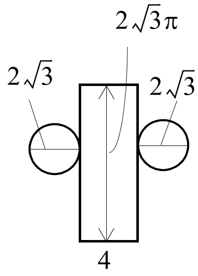
(A)



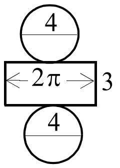
(B)



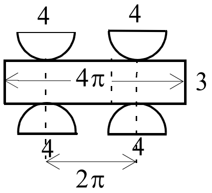
(C)



(D)



(E)



5. The distance traveled by an object with a constant acceleration is given by the equation: $d = vt + at^2$ where d is the distance traveled, v is the initial velocity, a is the acceleration and t is the time. If it takes 5 and 10 seconds for an object to travel 6 and 15 meters, respectively, what is the acceleration of the object?

- **(A)** 6/25 m/sec.^{2}
- **(B)** 15/100 m/sec.^{2}
- **(C)** 9/10 m/sec.^{2}
- **(D)** 3/50 m/sec.^{2}
- **(E)** 3/2 m/sec.^{2}

6. What is the maximum number of whole tiles, measuring $15'' \times 15''$, to fit a $97.5'' \times 180''$ room?

- **(A)** 70
- **(B)** 72
- **(C)** 74
- **(D)** 76
- **(E)** 78

7. Joe and John's ages add up to 21. Joe and Mary's ages add up to 15. Mary and John's ages add up to 12. How old is Mary?

- **(A)** 3
- **(B)** 6
- **(C)** 9
- **(D)** 12
- **(E)** 15

1/3

8. If you use 3 lbs. of chlorine to clean the water of a 1000 m^3 pool, approximately how much chlorine is necessary to clean the water in a 17,500 cubic feet pool? 1m = 3.28 feet.

- **(A)** 1 lbs.
- **(B)** 1.5 lbs.
- **(C)** 2 lbs.
- **(D)** 4.8 lbs
- **(E)** 15.9 lbs.

9. What are the coordinates of the cross section of $y = x^2 + 16$ and $y = 8x$?

- **(A)** (1, 8)
- **(B)** (2, 16)
- **(C)** (2, -16)
- **(D)** (4, 16)
- **(E)** (4, 32)

10. A 36,000 cubic feet swimming pool is filled by 2 different hoses, each connected to a water faucet. One hose supplies 1 cubic feet of water every minute, and the other one supplies 1.5 cubic feet of water per minute. How long will it take to fill the pool?

- **(A)** 5 days
- **(B)** 6 days
- **(C)** 10 days
- **(D)** 12 days
- **(E)** 12.5 days

1/2

11. Which of the below tables represents a linear function, $f(x) = -3x + 1$?

(A)

x	1	2	3	4
f(x)	4	5	10	13

(B)

x	1	2	3	4
f(x)	2	5	8	11

(C)

x	1	2	3	4
f(x)	-4	-5	-10	-13

(D)

x	1	2	3	4
f(x)	-2	-5	-8	-11

(E)

x	1	2	3	4
f(x)	-4	-5	-10	-13

12. In the figure, $y > 90^{o}$. Which of the following could be the value of a?

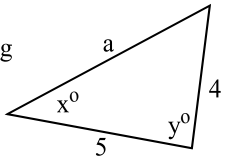


Figure is not drawn to scale.

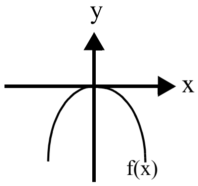
- **(A)** 10
- **(B)** 9
- **(C)** 6.5
- **(D)** 6
- **(E)** Cannot be determined from the information given.

13. The sum of the first 50 positive, even integers equals:

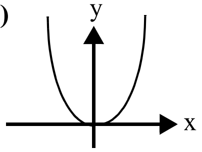
- **I.** The sum of the first 50 positive and odd integers.
- **II.** (The sum of the first 100 positive integers) - (The sum of the first 50 positive and odd integers)
- **III.** (The sum of the first 100 positive integers.)/2

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** II and III only
- **(E)** I and II and III

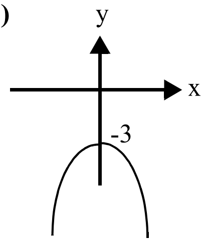
14. A function f(x) is shown in the figure. Which of the following is f(x - 3)?



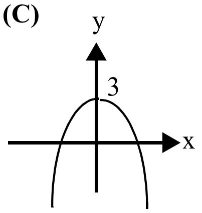
(A)



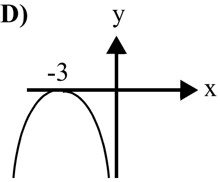
(B)



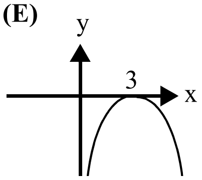
(C)



(D)



(E)



2/3

15. A sphere is inscribed inside a cube. If the volume of the cube is 27, what is the radius of the sphere?

- **(A)** 3
- **(B)** 1.5
- **(C)** $3/(\sqrt{2})$
- **(D)** $3(\sqrt{3})/2$
- **(E)** $3\sqrt{3}$

16. z depends on both x and y. When x is doubled and y is kept constant, the value of z increases by four times. When y is doubled and x kept constant, the value of z becomes half as much. Which of the following formulates the relationship between x, y and z.

- **(A)** $z = cx^4y^{1/2}$, where c is a constant.
- **(B)** $z = cx^2y/2$, where c is a constant.
- **(C)** $z = cx^2/y$, where c is a constant.
- **(D)** $z = cx^4/y^2$, where c is a constant.
- **(E)** $z = c(2x + y/2)$, where c is a constant.

17. In a geometric sequence, each term is 3/2 of the previous term. If the first term is 8, and if the nth term is 27, what is the value of n?

- **(A)** 2
- **(B)** 3
- **(C)** 4
- **(D)** 5
- **(E)** 6

18. In the figure, each square can assume two states, A and B. Originally all the squares are in state A. If the state of a square changes from A to B, it forces the neighboring, up, down, right and left squares to convert to the state B in exactly one second.

A	A	A	A	A
A	A	A	A	A
A	A	A	A	A
A	A	A	A	A
A	A	A	A	A

Once the state of a square becomes B, it can not revert to state A.

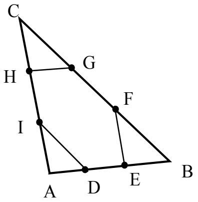
If you change the state of the lower left corner square from A to B, how many seconds will it take for all the squares to switch to state B?

- **(A)** 18 seconds
- **(B)** 14 seconds
- **(C)** 9 seconds
- **(D)** 8 seconds
- **(E)** 4 seconds

19. In a class of 20, the minimum and maximum test grades in a math exam is 64 and 98, respectively. Only one student gets 64 and only one student gets 98. If you exclude the highest and the lowest grades, the average and the median grades end up the same, 71. What is the difference between the average and the median grades when you include all the 20 grades?

- **(A)** 0
- **(B)** 1
- **(C)** 2
- **(D)** 5
- **(E)** 10

20. In the figure,
CH = HI = IA
CG = GF = FB
AD = DE = EB
If the area of ΔABC is 1, what is the area of the hexagon DEFGHI?



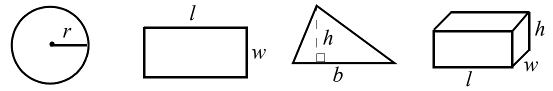
- **(A)** 1/2
- **(B)** 1/3
- **(C)** 4/7
- **(D)** 5/9
- **(E)** 2/3

L

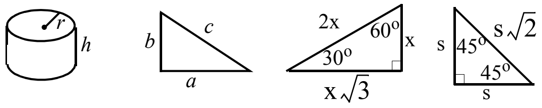
Test 3 - Section 2

25 Minutes, 18 Questions

Reference Information



$A = \pi r^2$ $C = 2\pi r$ $A = lw$ $A = \frac{1}{2}bh$ $V = lwh$



$V = \pi r^2 h$ $c^2 = a^2 + b^2$ Special Right Triangles

The number of degrees of arc in a circle is 360.

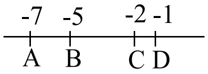
The sum of the measures in degrees of the angles of a triangle is 180.

Part 1

1. If y/3 = 3 - 2y/3, then 2y - 1 = ?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

2. Consider the four points on the number line in the figure. Which of the following is true?



(A) AB = BC/2
(B) AB = CD/2
(C) AC = BD
(D) AD = 2BC
(E) AD = AB + BC

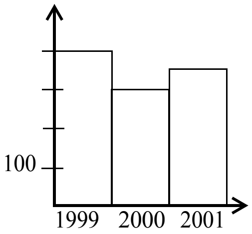
1/4

3. If a and b are two negative numbers, then $||a + b| + 3| = ?$

(A) -a - b + 3
(B) a + b + 3
(C) -a - b - 3
(D) a - b + 3
(E) a + b - 3

1/3

4. The graph shows the number of students graduated from RM High. Which of the following best represents the average number of students graduated in years 1999, 2000 and 2001?

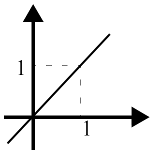


(A) 300
(B) 350
(C) 400
(D) 900
(E) 1050

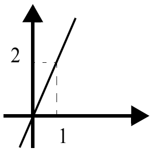
1/2

5. If f(x) = -2x + 3, which of the following is the graph of 2f(-x/2) - 3?

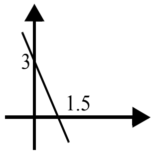
(A)



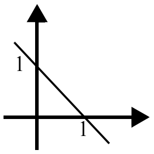
(B)



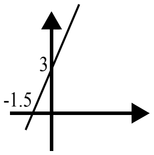
(C)



(D)



(E)



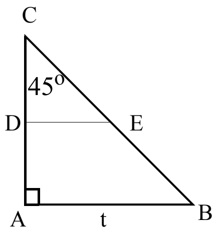
2/3

6. x, y, z and w are four consecutive even integers. If x < y < z < w and if their average is "a", what is the median of x, y, z, w and w + 2?

(A) a
(B) a + 1
(C) a + 2
(D) a + 3
(E) (a + w + 2)/2

7. In the figure, $\overline{DE} \parallel \overline{AB}$ and $\overline{DE}$ bisects $\overline{AC}$.

What is the perimeter of ABED?



(A) 2t
(B) 3t
(C) $t(2 + \sqrt{2})$
(D) $t(2 + (\sqrt{2})/2)$
(E) 2.5t

8. If x/y = 8, then y/(x + y) = ?

(A) 0.1
(B) 1/9
(C) 0.125
(D) 1/7
(E) 0.17

L

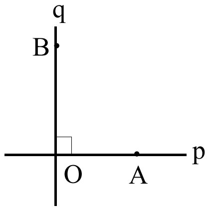
Part 2

9. 20% of the students in a class have grade A in math. If 12 students do not have A in math, how many students are there in the class?

10. John is preparing himself for a 10K race. He starts running 2 miles every day in the first week. In the second week he increases the distance and he runs 20% more miles every day. How many miles he runs during the second week?

1/4

11. In the figure, line p and line q are perpendicular to each other. If OB = 4 and AB = 5, what is the area of ΔABO ?



1/3

12. If $|x - y| = 4$ and $0 \le y \le 3$, how many integer values x can have?

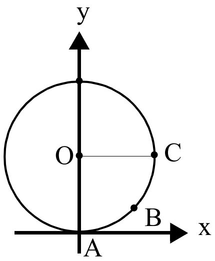
13. 10 cards with numbers from 0 to 9 inscribed on them, are placed in a box. Two cards are picked randomly to construct a two digit integer. The first card picked is used as the units digit and the second card is used as the tens digit. What is the probability of the two digit number being less than or equal to 50? Assume that the first card is placed back in the box before the second card is picked.

1/2

14. If $8^{2x + 1} = 128$, what is the value of x?

15. In the figure, O is the center of the circle, $\overline{OC}$ is parallel to x-axis and B bisects $\overset{\frown}{AC}$.

What is the slope of a line tangent to the circle O at point B?

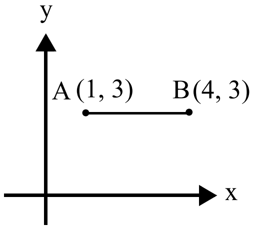


2/3

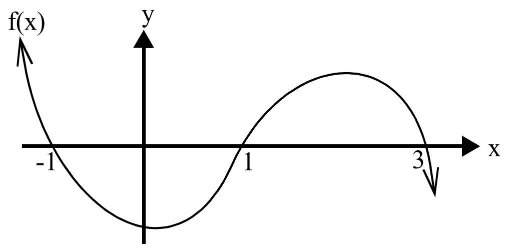
16. The speed of an object is the distance traveled in unit time. If the relation between the distance, d, and the time, t, is $d(t) = 5t^2 + 2t$, where d is in meters and t is in seconds, what is the average speed in meter/second, while the object is traveling between 5 and 5.5 seconds? Don't include the units in your answer.

17. V is the volume of the 3-D object, obtained by turning $\overline{AB}$ along y = 1 line by 360°.

What is the value of V/π ?



18.



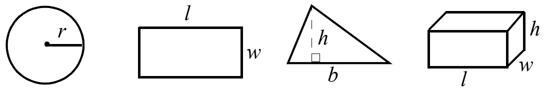
The above figure shows function f(x). What is the multiplication of x-intercepts of f(x + 3)?

L

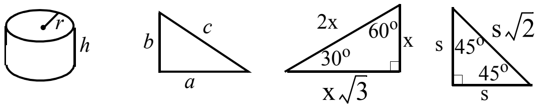
Test 3 - Section 3

20 Minutes, 16 Questions

Reference Information



$A = \pi r^2$ $C = 2\pi r$ $A = lw$ $A = \frac{1}{2}bh$ $V = lwh$



$V = \pi r^2 h$ $c^2 = a^2 + b^2$ Special Right Triangles

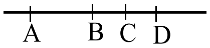
The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

1. a = 2t and t = v/2. If v = 8, then a = ?

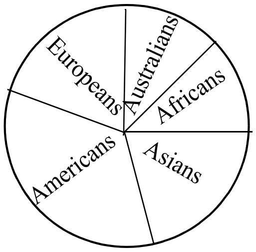
(A) 2
(B) 4
(C) 6
(D) 8
(E) 16

2. Consider the four points on the number line in the figure. B bisects AD. C bisects BD. If the coordinate of B is -5 and CD = 2, what is the coordinate of A?



(A) -9
(B) -7
(C) -1
(D) 7
(E) 9

3. The pie chart shows athletes from different continents in a sport competition. Which of the following best represents the percentage of athletes from Africa?



(A) 7.5%
(B) 12.5%
(C) 25%
(D) 50%
(E) 90%

4. If the remainder of a/b is c, where a, b and c are integers, which of the following must be correct?

(A) a/b = n + c, where n is an integer.
(B) a = nb + c, where n is an integer.
(C) a = nc + b, where n is an integer.
(D) a = b/a + c/a
(E) a/b = 1 + c/a

1/4

5. In an arithmetic sequence, the first term is 5 and each term after the first is 3 more than the previous term. What is the value of 39th term?

(A) 222
(B) 119
(C) 118
(D) 117
(E) 114

1/3

6. In the figure, what is the value of a?

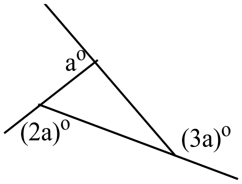


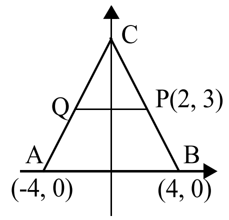
Figure is not drawn to scale.

(A) 30
(B) 40
(C) 50
(D) 60
(E) 70

7. x, y and z are three positive integers. If $y^{-z} = 1/27$ and $(x - y)^{1/2} = 4$, x = ?

(A) 13
(B) 14
(C) 16
(D) 18
(E) 19

8. In the figure, $\overline{QP} \parallel \overline{AB}$. If triangle ABC is shifted along the x-axis towards the left by 7, what would be the coordinates of point Q?



(A) (2, 10)
(B) (-2, 3)
(C) (-9, 3)
(D) (5, 3)
(E) (2, -4)

1/2

9. a is a positive prime number and b is a positive odd integer. Which of the following can be the remainder of (ab)/2?

(A) 0 only.
(B) 1 only.
(C) 2 only.
(D) 0 or 1
(E) 0 or 2

10. The distance between town A and town B is d. Ken is traveling from town A to town B at a speed of x miles/hour. At the end of two hours he still has 3 miles to reach town B. Which of the following is <u>wrong</u>?

(A) $d = 2x + 3$
(B) $x = (d - 3)/2$
(C) $2x - d = -3$
(D) $3x + d = 2$
(E) $d/2 = x + 1.5$

2/3

11. In the figure, the quadratic function $f(x) = ax^2 + bx + c$ is displayed. Which of the following could be a, b and c?

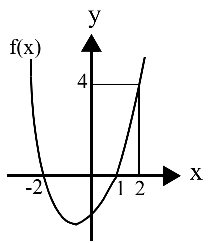
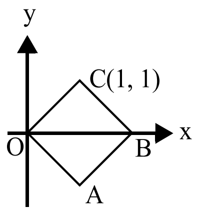


Figure is not drawn to scale.

(A) a = 1, b = 1, c = 2
(B) a = -1, b = 1, c = 2
(C) a = 1, b = 1, c = -2
(D) a = 1, b = -1, c = -2
(E) a = -1, b = -1, c = -2

12. In the figure, ABCO is a square. It is <u>not</u> symmetric around which of the following lines?



(A) $y = 0$
(B) $x = 1$
(C) $y = x - 1$
(D) $y = -x + 1$
(E) $y = -x - 1$

13. Mary's hair is 60% longer than Julie's hair. If both Mary and Julie shortened their hair by 20%, what is the percentage by which Mary's hair is longer than Julie's hair?

(A) 60%
(B) 54%
(C) 28%
(D) 24%
(E) 2.4%

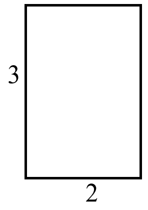
14. x is a non-zero number. Which of the following equals (21 less than (7 times x)) divided by (3 less than x)?

(A) 3
(B) -3
(C) 7
(D) -7
(E) 9

15. The side lengths of a right rectangular prism are three consecutive integers. If the longest of them is z, which of the following is the surface area, A, of the prism?

(A) $z(z - 1)(z - 2)$
(B) $z(z - 1) + z(z - 2)$
(C) $z(z - 1) + z(z - 2) + (z - 1)(z - 2)$
(D) $2(z(z - 1) + z(z - 2) + (z - 1)(z - 2))$
(E) $2(z^2 + (z-1)^2 + (z-2)^2)$

16. 2' x 3' vertical wall (as shown in the figure) will be covered by red, blue and white, 1' x 2' tiles. If each color tile is used once, in how many different ways can you cover the wall?



(A) 3
(B) 6
(C) 9
(D) 12
(E) 18

L

Answer Key - Test 3

Section 1

1. (B)
2. (E)
3. (D)
4. (D)
5. (D)
6. (B)
7. (A)
8. (B)
9. (E)
10. (C)
11. (D)
12. (C)
13. (B)
14. (E)
15. (B)
16. (C)
17. (C)
18. (D)
19. (B)
20. (E)

Section 2

Part 1

1. (C)
2. (D)
3. (A)
4. (B)
5. (E)
6. (B)
7. (D)
8. (B)

Part 2

9. 15
10. 16.8
11. 6
12. 8
13. 0.51
14. 2/3
15. 1
16. 54.5
17. 12
18. 0

Section 3

1. (D)
2. (A)
3. (B)
4. (B)
5. (B)
6. (D)
7. (E)
8. (C)
9. (D)
10. (D)
11. (C)
12. (E)
13. (A)
14. (C)
15. (D)
16. (E)

Calculate Your Score

Calculate Your Raw Score

1. Count the number of correct answers and write it in Line 1 of the following work sheet.

2. Count the number of incorrect (<u>not</u> missing) answers in Section 1, Section 2, Part 1 and Section 3. Divide this count by 4 and write it in Line 2.

Note that Part 2 of Section 2 has grid-in questions. No penalty is given for the incorrect answers in this part. That is why you don't need to count the incorrect answers here.

Also note that there is no penalty for missing any answers. That is why you don't need to count the missing answers.

3. Subtract Line 2 from Line 1 and write the result in Line 3.

4. Round the score on Line 3 to the nearest whole number and write the result on Line 4. This is your raw score.

Work Sheet to Calculate the Raw Score

1.	Correct Answers	
2.	(Incorrect Answers)/4	
3.	Not rounded Raw Score	
4.	Raw Score	

Calculate Your SAT Score

Find your raw score in the below table and read the corresponding SAT score. Note that the actual SAT scores are expressed in ranges, not definite numbers. For our purposes, we assign only one SAT score to each raw score.

Raw Score	**SAT Score**	**Raw Score**	**SAT Score**	**Raw Score**	**SAT Score**
-6 or less	200	15	430	36	590
-5	205	16	435	37	595
-4	215	17	440	38	600
-3	225	18	450	39	610
-2	235	19	460	40	620
-1	245	20	465	41	630
0	260	21	470	42	640
1	280	22	480	43	650
2	290	23	490	44	655
3	310	24	495	45	660
4	320	25	500	46	670
5	330	26	510	47	690
6	340	27	520	48	700
7	360	28	525	49	710
8	370	29	530	50	730
9	380	30	540	51	740
10	385	31	550	52	760
11	390	32	560	53	775
12	400	33	565	54	800
13	410	34	570		
14	420	35	580		

Name:

Subject Table - Test 3

Date:

Question Number	Categories - Subjects				Difficulty Level
1.1	S. A - One Variable Eqs.				Easy
1.2	S. A - Exprs. with Powers	A. A - Functions			Easy
1.3	Others - Rounding	Others - Tbls, Chrts, Grphs			Easy
1.4	Geo. - Circles	Geo. - 3-D Objects			Medium
1.5	S. A - Multiple Unknowns	WQ - Regular			Easy
1.6	Geo. - Rectangles	Geo. - Squares	Others - Basic Counting		Easy
1.7	Arith. - Basic Arithmetic	S. A - Multiple Unknowns	WQ - Regular		Medium
1.8	S. A - Proportionality	WQ - Regular			Medium
1.9	Geo. - Coordinate Geo.	S. A - Multiple Unknowns	A. A - Functions		Medium
1.10	Arith. - Basic Arithmetic	WQ - Regular			Medium
1.11	A. A - Linear Functions	Others - Tbls, Chrts, Grphs			Medium
1.12	Geo. - Triangles				Medium
1.13	Others - Sums				Medium
1.14	A. A - Functions				Medium
1.15	Geo. - 3-D Objects	S.A - Exprs. with Powers			Hard
1.16	S. A - Proportionality	WQ - Formulation Only			Hard
1.17	S.A - Exprs. with Powers	Others - Sequences			Hard
1.18	Others - Logic	Others - Basic Counting			Hard
1.19	Others - Statistics	WQ - Regular			Hard
1.20	Geo. - Triangles				Hard
2.1	S. A - One Variable Eqs.				Easy
2.2	Arith. - Basic Arithmetic	Arith. - Abs. Value	Geo. - Points and Lines		Easy
2.3	Arith. - Negative Numbers	Arith. - Abs. Value			Medium
2.4	Others - Rounding	Others - Tbles, Chrts, Grphs	Others - Statistics		Easy
2.5	A. A - Linear Functions				Medium
2.6	Arith. - Even Odd Numbers	Others - Statistics			Hard
2.7	Geo. - Triangles	WQ - Formulation Only			Hard
2.8	Arith. - Fractions, Ratios				Hard
2.9	Arith. - Percentages	WQ - Regular			Easy
2.10	Arith. - Percentages	WQ - Regular			Easy
2.11	Geo. - Triangles	Geo. - Coordinate Geo.			Easy
2.12	S. A - Multiple Unknowns	S.A - Exprs. with Abs. Value			Medium
2.13	Others - M. Ex. Events	Others - Independent Events	Others - Probability	WQ - Regular	Hard
2.14	S. A - One Variable Eqs.	S.A - Exprs. with Powers			Medium
2.15	Geo. - Points and Lines	Geo. - Angles	Geo. - Circles	A. A - Linear Funcs.	Medium
2.16	A. A - Functions	Others - Statistics			Hard
2.17	Geo. - Circles	Geo. - Coordinate Geo.	Geo. - 3D Objects		Hard
2.18	A. A - Functions				Hard
3.1	S. A - Multiple Unknowns				Easy
3.2	Arith. - Abs. Value	Geo. - Points and Lines			Easy
3.3	Others - Rounding	Others - Tbls, Chrts, Grphs			Easy
3.4	Arihmetic - Divisibility	WQ - Formulation Only			Easy
3.5	Others - Sequences				Medium
3.6	Geo. - Angles	Geo. - Triangles	S. A - One Variable Eqs.		Medium
3.7	S. A - Multiple Unknowns	S.A - Exprs. with Powers			Medium
3.8	Geo. - Coordinate Geo.	Geo. - Symetry			Medium
3.9	Arihmetic - Divisibility	Arith. - Even Odd Numbers			Medium
3.10	S. A - Multiple Unknowns	WQ - Formulation Only			Medium
3.11	Geo. - Coordinate Geo.	A. A - Functions			Hard
3.12	Geo. - Squares	Geo. - Coordinate Geo.	Geo. - Symetry	A. A - Linear Funcs.	Hard
3.13	Arith. - Fractions, Ratios	Arith. - Percentages	WQ - Regular		Hard
3.14	S. A - One Variable Eqs.	WQ - Regular			Hard
3.15	Geo. - Rectangles	Geo. - 3-D Objects	WQ - Desc. Figures		Hard
3.16	Others - Basic Counting	Others - Perm., Comb.	Others - M. Exclusive Events		Hard

Skipped:

Wrong:

Analysis Chart - Test 3

Name: **Date:**

Category	Subject	Easy	Medium	Hard
Arithmetic	Basic Arithmetic	2.2	1.7, 1.10	
	Decimals			
	Fractions, Ratios			2.8, 3.13
	Percentages	2.9, 2.10		3.13
	Powers			
	Square Root			
	Radicals			
	Negative Numbers		2.3	
	Numbers Between –1 and 1			
	Divisibility	3.4	3.9	
	Even & Odd Numbers		3.9	2.6
	Absolute Value	2.2, 3.2	2.3	
Geometry	Points and Lines	2.2, 3.2	2.15	
	Angles		2.15, 3.6	
	Polygons			
	Triangles	2.11	1.12, 3.6	1.20, 2.7
	Quadrangles			
	Rectangles	1.6		3.15
	Squares	1.6		3.12
	Circles		1.4, 2.15	2.17
	Trigonometry			
	Coordinate Geometry	2.11	1.9, 3.8	2.17, 3.11, 3.12
	Symmetry		3.8	3.12
	3-D Objects		1.4	1.15, 2.17, 3.15
Simple Algebra (S. A.)	One Variable Equations	1.1, 2.1	2.14, 3.6	3.14
	Multiple unknowns	1.5, 3.1	1.7, 1.9, 2.12, 3.7, 3.10	
	Equations with Powers	1.2	2.14, 3.7	1.15, 1.17
	Radical Equations			
	Inequalities			
	Expressions with Absolute Value		2.12	
	Proportionality		1.8	1.16
Advanced Algebra (A. A.)	Functions	1.2	1.9, 1.14	2.16, 2.18, 3.11
	Linear functions		1.11, 2.5, 2.15	3.12
	Quadratic Functions			
Others	Rounding	1.3, 2.4, 3.3		
	Tables, Charts, Graphs	1.3, 2.4, 3.3	1.11	
	Sets			
	Defined Operators			
	Logic			1.18
	Statistics	2.4		1.19, 2.6, 2.16
	Sequences		3.5	1.17
	Sums		1.13	
	Basic Counting	1.6		1.18, 3.16
	Permutations, Combinations			3.16
	Mutually Exclusive Events			2.13, 3.16
	Independent Events			2.13
	Probability			2.13
Word Questions (WQ)	Regular	1.5, 2.9, 2.10	1.7, 1.8, 1.10	1.19, 2.13, 3.13, 3.14
	Formulation Only	3.4	3.10	1.16, 2.7
	Describing Figures			3.15

Skipped: **Wrong:**

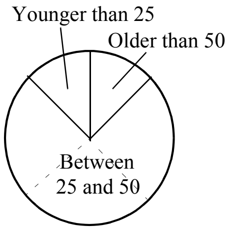
Solutions - Test 3

Section 1

1. Answer: (B)
$3/x - 8 = 4$ ➔ $3/x = 4 + 8 = 12$ ➔ $x = 3/12 = 1/4$
The answer is (B).

2. Answer: (E)
If $f(x) = 2^x + 1$, then $f(5) = 2^5 + 1 = 32 + 1 = 33$
The answer is (E).

3. Answer: (D)
The students older than 50 and the students younger than 25 combined make up about 1/4 of the pie chart. 1/4 of the pie chart is 25% of the all the students.



Hence the rest of the students between 25 and 50 years old, is 100 - 25 = 75% of all the students.
The answer is (D).

4. Answer: (D)
For these figures to be folded into a cylinder, the length of the rectangle must be equal to the circumferences of the circles. (A), (B), (C) and (E) satisfy this condition, but (D) doesn't. In this case, the circumferences of the circles are 4π, but the length of the rectangle is 2π. The answer is (D).

5. Answer: (D)
If it takes 5 seconds for the object to travel 6 meters, then $6 = 5v + 5^2a = 5v + 25a$ ➔ $a = (6 - 5v)/25$

If it takes 10 seconds for the object to travel 15 meters, then
$15 = 10v + 100a = 10v + 100(6 - 5v)/25 =$
$10v + 24 - 20v = 24 - 10v$ ➔
$v = (24 - 15)/10 = 9/10 = 0.9$ m/sec. ➔

$$a = \frac{6-5v}{25} = \frac{6-(5\times 0.9)}{25} = \frac{1.5}{25} = \frac{3}{50} \text{ m/sec.}^2$$

The answer is (D).

6. Answer: (B)
You can fit 97.5/15 = 6.5 tiles along the width of the room. However, you are not allowed to cut a tile, so you can only fit 6 tiles along the width of the room. You can fit 180/15 = 12 tiles along the length of the room. So all together, $6 \times 12 = 72$ tiles you can fit into the room without cutting or overlapping.
The answer is (B).

7. Answer: (A)
Let the ages of Joe, John and Mary be e, n and m respectively.
Total age of Joe and John is 21. ➔ $e + n = 21$
Total age of Joe and Mary is 15. ➔ $e + m = 15$
Total age of Mary and John is 12. ➔ $m + n = 12$

$$\begin{array}{rl} & e + n = 21 \\ - & e + m = 15 \\ \hline & n - m = 21 - 15 \Rightarrow \\ & n - m = 6 \\ + & m + n = 12 \\ \hline & 2n = 6 + 12 \Rightarrow 2n = 18 \Rightarrow n = 18/2 = 9 \end{array}$$

$m + n = m + 9 = 12$ ➔ $m = 12 - 9 = 3$
The answer is (A).

8. Answer: (B)
1m = 3.28 feet ➔ $1m^3 = 3.28^3 \cong 35.29$ feet^3 ➔
17500 cubic feet = $\frac{17500}{35.29} \cong 495.9$ 17500/35.29 = 495.89 m^3

If you use 3 lbs. of chlorine to clean the water of a 1000 m^3 pool, you will need 3/1000 = 0.003 lb. chlorine to clean 1m^3 water.

To clean 495.89m^3 (= 17,500 cubic feet) water you will need $0.003 \times 495.89 \cong 1.5$ lbs. of chlorine.
The answer is (B).

Alternate Solution:
As you calculated above,
17,500 feet^3 = 495.89m^3 $\cong$ 500 m^3
which is 1/2 of 1000.

Since it takes 3 lbs. of chlorine to clean 1000 m^3 of water, it will take 3/2 = 1.5 lbs. of chlorine to clean 500 m^3 water.

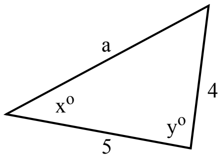
The answer is (B).

9. Answer: (E)
They cross each other when $x^2 + 16 = 8x$ ➔
$x^2 - 8x + 16 = (x - 4)^2 = 0$ ➔ $x - 4 = 0$ ➔ $x = 4$
Substitute $x = 4$ to $y = 8x$ ➔ $y = 8 \times 4 = 32$
Hence the coordinates of the cross section are (4, 32).
The answer is (E).

10. Answer: (C)
In one minute, 1 + 1.5 = 2.5 cubic feet water is supplied by both hoses combined. Since the pool is 36,000 cubic feet, it will take
36000/2.5 = 14400 minutes = 14400/60 = 240 hours = 240/24 = 10 days to fill the swimming pool.
The answer is (C).

11. Answer: (D)
$f(x) = -3x + 1$ ➔ $f(1) = -3 + 1 = -2$
Only in Case (D), $f(1) = -2$
The answer is (D).

12. Answer: (C)
In the figure, $y^o > 90^o$ ➔ y is the largest angle and the side across it, a, is the longest side of the triangle.



If y were 90^o, then
$a = \sqrt{5^2 + 4^2} = \sqrt{41} = 6.4$
Since $y^o > 90^o$, then $a > 6.4$

On the other hand, a has to be less than $4 + 5 = 9$
The only answer choice between 6.4 and 9 is 6.5.
The answer is (C).

13. Answer: (B)
There are 50 positive, even integers and 50 positive, odd integers between 1and 100. Hence
(The sum of the first 100 positive integers) =
(The sum of the first 50 positive, even integers) +
(The sum of the first 50 positive, odd integers) ➔
(The sum of the first 50 positive, even integers) =
(The sum of the first 100 positive integers) -
(The sum of the first 50 positive, odd integers)

The answer is (B).

In fact:
The sum of the first 50 positive, even integers =
$(2 + 100)(50/2) = 2550$
The sum of the first 50 positive, odd integers =
$(1 + 99)(50/2) = 2500$
The sum of the first 100 positive integers =
$(1 + 100)(100/2) = 5050$
(The sum of the first 100 positive integers)/2 =
5050/2 = 2525

14. Answer: (E)
f(x - 3) shifts f(x) to right by 3.
The answer is (E).

15. Answer: (B)
Let c be the side length of the cube. The volume of the cube = $c^3 = 27$ ➔ c = 3.
If a sphere is inscribed inside a cube, then the diameter of the sphere equals the side length of the cube. Hence the radius of the sphere = 3/2 = 1.5
The answer is (B).

16. Answer: (C)
When x is doubled and y is kept constant, the value of z increases by four times. ➔ $z \propto x^2$

When y is doubled and x kept constant, the value of z becomes half as much. ➔ $z \propto 1/y$

Hence $z \propto x^2/y$ ➔ $z = c(x^2/y)$, where c is a constant.
The answer is (C).

17. Answer: (C)
n^{th} term $= 8(3/2)^{n-1} = 27$ ➔ $(3/2)^{n-1} = 27/8 = 3^3/2^3$
➔ n - 1 = 3 ➔ n = 4
The answer is (C).

18. Answer: (D)
In the figure, we indicated the time required for each square to convert to the state B, after the lower-left corner square's state is converted to B. The last square to be converted to the state B is the upper-right square. It takes 8 seconds to convert its state.
The answer is (D).

4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5
B	1	2	3	4

19. Answer: (B)
If you exclude the highest and the lowest grades, there will be 20 - 2 = 18 grades with an average grade of 71.
➔ The sum of these 18 grades is $71 \cdot 18 = 1278$ ➔

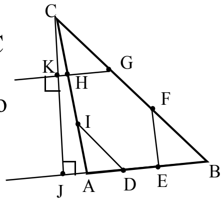
The sum of 20 grades = 1278 + 64 + 98 = 1440 ➔
The average of 20 grades = 1440/20 = 72

On the other hand, the median remains the same, 71, between 18 grades and 20 grades, because the median does not change when one removes the lowest and the highest elements.

Hence the difference between the average and median grades is 72 - 71 = 1

The answer is (B).

20. Answer: (E)
Consider two triangles, ΔABC and ΔHGC.
$\angle ACB$ is an angle common to both triangles.
CH/CA = CG/CB = 1/3 ➔
$\Delta ABC \sim \Delta HGC$ ➔



$\overline{GH} \parallel \overline{AB}$ and $\frac{AB}{HG} = \frac{AC}{HC} = 3$ ➔ AB = 3HG
Let $\overline{CJ}$ be the height of ΔABC as shown in the figure.

Since $\overline{GH} \parallel \overline{AB}$, then $\overline{CK}$ is perpendicular to $\overline{GK}$ and the height of ΔHGC ➔
$\Delta JBC \sim \Delta KGC$ ➔
CK/CJ = CG/CB = 1/3 ➔ CJ = 3CK

The area of $\Delta ABC = (AB \cdot CJ)/2 =$
$(3GH \cdot 3CK)/2 = 9$ x (The area of ΔHGC)

Similarly,
The area of $\Delta ABC = 9$ x (The area of ΔEBF) =
9 x (The area of ΔADI) ➔
Since the area of $\Delta ABC = 1$, then
The area of ΔHGC = The area of ΔEBF =
The area of ΔADI = 1/9

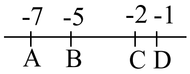
The area of DEFGHI =
The area of ΔABC -
The area of ΔHGC - The area of ΔEBF -
The area of ΔADI =
1 - 1/9 - 1/9 - 1/9 = 2/3
The answer is (E).

Section 2

Part 1

1. Answer: (C)
$y/3 = 3 - 2y/3$ ➔ $y/3 + 2y/3 = y = 3$ ➔
$2y - 1 = 2 \cdot 3 - 1 = 5$
The answer is (C).

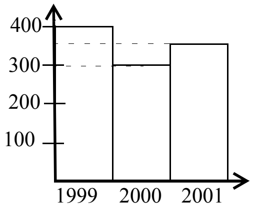
2. Answer: (D)



AB = -5 - (-7) = 2
BC = -2 - (-5) = 3
CD = -1 - (-2) = 1
AD = -1 - (-7) = 6 ➔ AD = 2BC
The answer is (D).

3. Answer: (A)
If a and b are two negative numbers,
then $| a + b | = -a - b$ ➔ $||a + b| + 3| = | -a - b + 3 |$
Since a and b are negative, -a - b is positive ➔
-a - b + 3 is positive ➔ $| -a - b + 3 | = -a - b + 3$
The answer is (A).

4. Answer: (B)
Let's read the number of students graduated in each of the 3 years from the chart. They are 400, 300 and 350 students for 1999, 2000 and 2001, respectively.

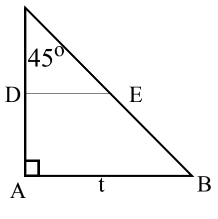


The average of these 3 numbers is
(400 + 300 + 350)/3 = 350
The answer is (B).

5. Answer: (E)
f(x) = -2x + 3 ➔
2f(-x/2) - 3 = 2[-2(-x/2) + 3] - 3 = 2x + 6 - 3 = 2x + 3
This function has a positive slope, 2, and a positive y-intercept, 3. The answer choice that matches is (E).

6. Answer: (B)
If x, y, z and w are 4 consecutive even integers in increasing order, their average is the average, a, of the two numbers in the middle:
a = (y + z)/2 = z - 1 (or y + 1) ➔ z = a + 1
The median of x, y, z, w and w + 2 is z = a + 1, because x and y are smaller than z, and, w and w + 2 are greater than z. The answer is (B).

7. Answer: (D)
Since $\angle C = 45°$, ΔABC is an isosceles right triangle with AC = AB = t ➔



$BC^2 = t^2 + t^2 = 2t^2$ ➔
$BC = \sqrt{2}t$
$\overline{DE}$ bisects $\overline{AC}$ ➔
$\overline{DA} = t/2$
$\overline{DE} \parallel \overline{AB}$ ➔
$\angle CDE = \angle CAB$ and $\angle CED = \angle CBA$ ➔
$\Delta ABC \sim \Delta DEC$ ➔

Since $\overline{DE}$ bisects $\overline{AC}$, DE = AB/2 = t/2 and
$EB = CB/2 = \sqrt{2}t/2 = t\sqrt{2}/2$

The perimeter of ABED =
AB + BE + ED + DA =
$t + t\sqrt{2}/2 + t/2 + t/2 = t(2 + \sqrt{2}/2)$

The answer is (D).

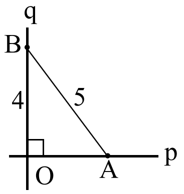
8. Answer: (B)
x/y = 8 ➔ x/y + 1 = 8 + 1 ➔ (x + y)/y = 9 ➔
y/(x + y) = 1/9
The answer is (B).

Part 2

9. Answer: 15
If 20% of the students in a class get grade A in math, then 100 - 20 = 80% of the students don't have an A. Since 12 students do not get A as math grade, 80% of the students is 12. Hence there are (12/80)100 = 15 students in the class.

10. Answer: 16.8 miles
In the first week John ran $2 \cdot 7 = 14$ miles.
In the second week, he ran 20% more each day. ➔
He ran 20% more in the second week. ➔
He ran $14 \cdot 20/100 = 2.8$ miles more in the second week. ➔
He ran 14 + 2.8 = 16.8 miles in the second week.

11. Answer: 6
If OB = 4 and AB = 5, then
$OA = \sqrt{5^2 - 4^2} = 3$ ➔
The area of ΔABO =
$(OA \cdot BO)/2 = 3 \cdot 4/2 = 6$



12. Answer: 8
If | x - y | = 4, then there are 2 cases:

Case 1: $x - y \geq 0$ ➔ x - y = 4 ➔ x = 4 + y ➔
If $0 \leq y \leq 3$, the integer values that x can take are:
4 + 0 = 4, 4 + 1 = 5, 4 + 2 = 6 or 4 + 3 = 7

Case 2:x - y < 0 ➔ x - y = -4 ➔ x = -4 + y ➔
If $0 \leq y \leq 3$, the integer values that x can take are:
-4 + 0 = -4, -4 + 1 = -3, -4 + 2 = -2 or -4 + 3 = -1

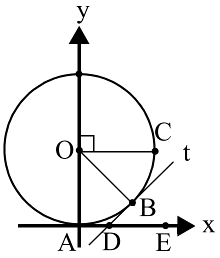
So x can have 8 different integer values, -4, -3, -2, -1, 4, 5, 6, 7.

13. Answer: 0.51
As long as the second card is 4 or less, the number will be less than 50. Example: 49, 37, 01 etc. Since there are 5 numbers between 0 and 4, the probability of the second card being between 0 and 4 is 5/10 = 1/2
However, the number can also be 50. The probability of the first number's being 0 and the second number's being 5 is (1/10)(1/10) = 1/100.

Combined probability of these two cases is
1/2 + 1/100 = 51/100 = 0.51

14. Answer: 2/3
$128 = 2^7$ and $8^{2x+1} = (2^3)^{2x+1} = 2^{6x+3} \Rightarrow$
$2^{6x+3} = 2^7 \Rightarrow 6x + 3 = 7 \Rightarrow x = 4/6 = 2/3$

15. Answer: 1
Let's modify the figure and display the tangent line, t, in question.



$\overline{OC}$ is parallel to x-axis $\Rightarrow$
$\angle AOC = 90°$

B bisects $\overset{\frown}{AC} \Rightarrow$
$\angle BOA = 90/2 = 45° \Rightarrow$

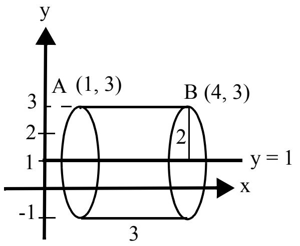
$\angle BOA = \angle BDE = 45°$ (Both arms of the two angles are perpendicular to each other.)
Slope of the line t is tan(45) = 1 $\Rightarrow$

16. Answer: 54.5
The distance the object travels between 5 and 5.5 seconds is d(5.5) - d(5) =

$5 \cdot (5.5)^2 + 2 \cdot (5.5) - (5 \cdot 5^2 + 2 \cdot 5) =$
$162.25 - 135 = 27.25$ meters.

Since it takes 5.5 - 5 = 0.5 seconds for the object to travel 27.25 meters, the average speed of the object is 27.25/0.5 = 54.5 m/sec.

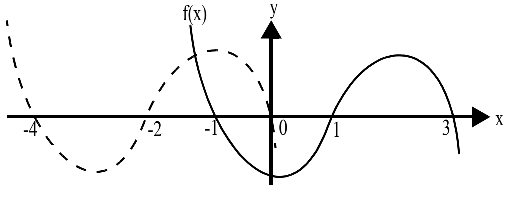
17. Answer: 12
As shown in the figure,



when you rotate $\overline{AB}$ around y = 1 line by 360^o, you will get a cylinder of height, h = 4 - 1 = 3 and base radius, r = 3 - 1 = 2

The volume, V, of this cylinder is
$\pi r^2 \cdot h = \pi \cdot 2^2 \cdot 3 = 12\pi \Rightarrow V/\pi = 12$

18. Answer: 0

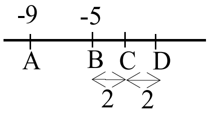


f(x + 3) is similar to f(x), except that it is shifted to left by 3 as shown in the figure by the dotted curve. The x-intercepts of f(x) are -1, 1 and 3. Because f(x + 3) is shifted to left by 3, the x intercepts of f(x + 3) are -1 - 3 = -4, 1 - 3 = -2 and 3 - 3 = 0. Hence the multiplication of x-intercepts is $(-4) \cdot (-2) \cdot 0 = 0$

Section 3

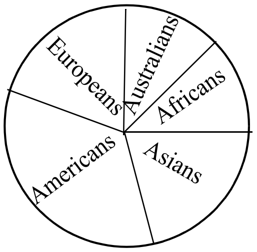
1. Answer: (D)
$v = 8 \Rightarrow t = v/2 = 8/2 = 4 \Rightarrow a = 2t = 2 \cdot 4 = 8$

2. Answer: (A)
Since C is the midpoint between B and D and CD = 2, then BC = 2 $\Rightarrow$



BD = 2 + 2 = 4 $\Rightarrow$
Since the coordinate of B is -5 and B is the midpoint between A and D, AB = BD and the coordinate of A is -5 - 4 = -9
The answer is (A).

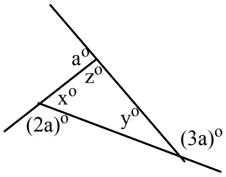
3. Answer: (B)
Australians and Africans together are 1/4 of the pie chart. It means that 25% of all the athletes are Australians and Africans. Since they are approximately the same in quantity in the pie chart, only Africans are 25/2 = 12/5% of all the athletes.
The answer is (B)



4. Answer: (B)
The remainder of a/b is c. $\Rightarrow a = n \cdot b + c$, where n is an integer. The answer is (B).

5. Answer: (B)
In an arithmetic sequence, the nth term = f + d(n-1), where f is the first term and d is the difference between the consecutive terms.
In this case f = 5 and d = 3 $\Rightarrow$
39th term = 5 + 3(39 - 1) = 119
The answer is (B).

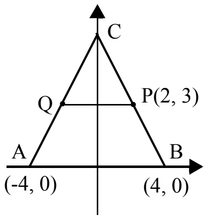
6. Answer: (D)
In the figure,
$x + 2a = 180 \Rightarrow x = 180 - 2a$
$y + 3a = 180 \Rightarrow y = 180 - 3a$
$z + a = 180 \Rightarrow z = 180 - a$
$x + y + z = 180 \Rightarrow$



$180 - 2a + 180 - 3a + 180 - a = 540 - 6a = 180 \Rightarrow$
$a = (540 - 180)/6 = 60$
The answer is (D).

7. Answer: (E)
$y^{-z} = 1/27 \Rightarrow y^z = 27 \Rightarrow y = z = 3$
$(x - y)^{1/2} = 4 \Rightarrow x - y = 4^2 = 16 \Rightarrow x - 3 = 16 \Rightarrow$
$x = 19$
The answer is (E).

8. Answer: (C)
In the figure, the points A and B are symmetric around y-axis with coordinates (-4, 0) and (4, 0). $\Rightarrow$
$\overline{AC}$ and $\overline{BC}$ are symmetrical around y- axis.



$\overline{QP} \parallel \overline{AB}$ ➔ points P and Q are symmetrical around y-axis. ➔

The coordinates of Q is (-2, 3).
If ΔABC is shifted along x-axis towards left by 7, the x-coordinates of all the points will decrease by 7. So the new coordinates of Q will be
(-2 - 7, 3) = (-9, 3)

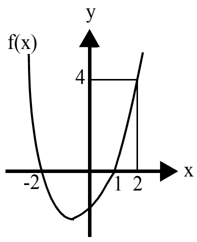
The answer is (C).

9. Answer: (D)
If b is a positive odd integer, then b = 2n + 1, where n is an positive integer. ➔ ab/2 = a(2n + 1)/2 = an + a/2
Since a is a positive prime number, it can not be divided by 2 unless a = 2.
If a ≠ 2 , the remainder of a/2 is 1.
If a = 2, the remainder of a/2 is 0.
The answer is (D).

10. Answer: (D)
If the distance between town A and town B is d, and if Ken is traveling from town A to town B at a speed of x miles/hour, then at the end of two hours he traveled 2x miles. Hence he still has d - 2x = 3 miles to reach town B. By using this equation, you can drive all the equations in answer choices, except (D). So (D) is the answer.

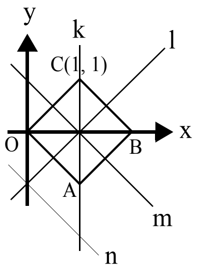
Alternate Solution:
All the answer choices except (D) are equivalent.
The answer is (D).

11. Answer: (C)
From the graph,
the two x-intercepts of f(x) are -2 and 1. ➔
f(x) = m(x - (-2))(x - 1) = m(x^2 + x - 2)
From the graph,
f(2) = 4 ➔ m(2^2 + 2 - 2) = 4m = 4➔ m = 1



Hence, f(x) = x^2 + x - 2 ➔ a = 1, b = 1 and c = -2
The answer is (C).

12. Answer: (E)
If ABCO is a square, then it is symmetric around:
1. y = 0, which is x-axis.
2. x = 1, line k in the figure.
3. y = x - 1, line l in the figure.
4. y = -x + 1, line m in the figure.
You can also easily see that it is not symmetric around x = -x - 1, line n in the figure.
The answer is (E).



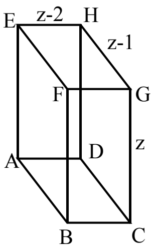
13. Answer: (A)
Let m and j be the lengths of Mary's and Julie's hair respectively.
Mary's hair is 60% longer than Julie's hair. ➔
m = (60j/100) + j ➔ m/j = (3/5) + 1 = 8/5
Let m_r and j_r be the lengths of Mary's and Julie's hair after they shortened their hair by 20%. ➔
m_r = 80m/100 and j_r = 80j/100 ➔
m_r/j_r = m/j = 8/5
Since m_r/j_r = m/j, the percentage remains the same after they both cut their hair. Hence Mary's hair is still 60% longer than Julie's hair after both reduce their hair length by 20%.The answer is (A).

14. Answer: (C)
Start with the innermost parenthesis for each term:
7 times x = 7x
21 less than (7 times x) = 7x - 21
(3 less than x) = x - 3
(21 less than (7 times x)) divided by (3 less than x) =

$$\frac{7x-21}{x-3} = \frac{7(x-3)}{x-3} = 7$$

The answer is (C).

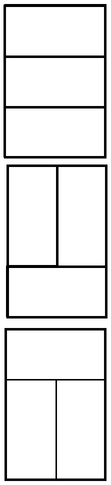
15. Answer: (D)
If the side lengths of a rectangular right prism are three consecutive integers and if the longest of them is z, then the other two lengths are z - 1 and z - 2 as shown in the figure.
The surface area of the prism is the addition of the areas of all 6 surfaces.



These areas are:
The area of the rectangle ABCD =
the area of the rectangle EFGH = (z - 1)(z - 2)
The area of the rectangle BCGF =
the area of the rectangle ADHE = z(z - 2)
The area of the rectangle ABFE =
the area of the rectangle DCGH = z(z - 1)
Hence the surface area of the prism is
2(z - 1)(z - 2) + 2z(z - 2) + 2z(z - 1) =
2(z(z - 1) + z(z - 2) + (z - 1)(z - 2))
The answer is (D).

16. Answer: (E)
Since the area of the tiles is 2 and the area of the wall is 6, only 6/2 = 3 tiles is necessary and sufficient to cover the whole wall. Since each color is used only once, there will be one tile of each color. We can put them in 3 different patterns, as shown in the figure. Within each pattern, red, blue and white colors can be arranged in 3! = 6 different ways. Hence the total number of ways the wall can be covered is $3 \cdot 6 = 18$.

The answer is (E).



Test 4

Date:

General Directions:

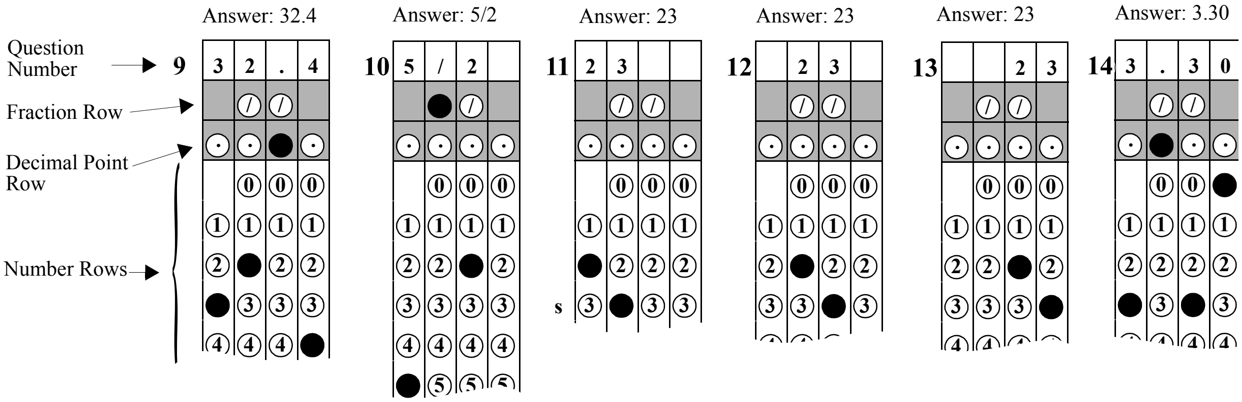
- The test is 70 minutes long.
- It has 3 sections.
  Section 1: 20 multiple choice questions, 25 minutes.
  Section 2: 8 multiple choice and 10 grid-in questions, 25 minutes.
  Section 3: 10 multiple choice questions, 20 minutes.
- You can only work on one section at a time.
- You are not allowed to transfer your answers from the test to the Answer Sheet after the time is up for each section. Therefore mark your answer on the Answer Sheet as soon as you finish answering a question.
- Make sure that the question number in the test matches the question number on the answer sheet
- Mark only one answer for each question.
- If you want to change your answer, erase the old answer completely.
- You receive one point for each correct answer.
- You lose one point for 4 incorrect multiple choice answers.
- You don't lose any points for incorrect grid-in questions.
- You neither gain nor loose points for missing answers.

Directions for the Grid-In Questions

- You can only have zero or positive numbers as answers for the grid-in questions. Therefore, if your answer is negative it is wrong.
- The upper limit of your answer is 9999. If your answer is greater than 9999, it is wrong.
- You can express your answers as integers, fractions or decimal numbers. Do not spend the time to convert fractions to decimals or decimals to fractions.
- Mixed numbers have to be converted to fractions or decimals. For instance $3\frac{1}{2}$ has to be converted to $\frac{7}{2}$ or 3.5 before it is marked on the answer sheet.
- Write your answer in the first row. Note that this is optional and in real SAT it will not count as an answer. The only answer that counts is the answer that you mark on Fraction, Decimal Point and Number rows.
- If your answer has less than 4 characters, you can start from any column you wish. Leave the rest of the columns empty.
- If your answer has more than 4 characters, you can either mark the first 4 characters or you can round it to an appropriate 4-character number.

  For example, 10/3 = 3.33333333 can be marked in as 10/3, 3.33 or 3.30, but not as 3, 3.0, 3.3 or 3.4.
  200/3 can be marked in as 66.6 or 66.7. But 66 or 67 are not the correct answers.

Here are several examples to clarify the above statements.



Answer Sheet - Test 4

Section 1

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Section 2

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Section 3

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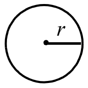
Test 4 - Section 1

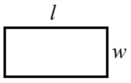
25 Minutes, 20 Questions

Reference Information

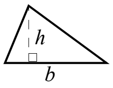


$A = \pi r^2$

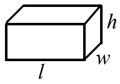
$C = 2\pi r$



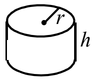
$A = lw$



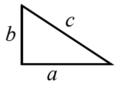
$A = \frac{1}{2}bh$



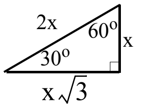
$V = lwh$

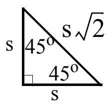


$V = \pi r^2 h$



$c^2 = a^2 + b^2$





Special Right Triangles

The number of degrees of arc in a circle is 360.

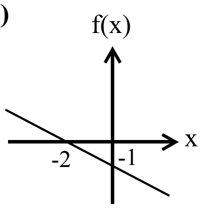
The sum of the measures in degrees of the angles of a triangle is 180.

1. If $2z + 17 = -1$, then $z + 2 = ?$

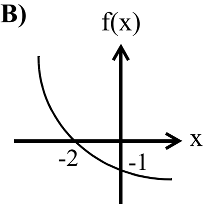
- **(A)** 10
- **(B)** 8
- **(C)** -7
- **(D)** -9
- **(E)** -11

2. Which of the following is a linear function with x-intercept = -2 and negative slope?

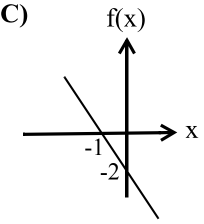
(A)



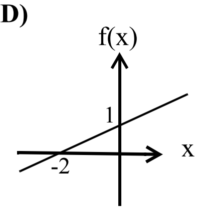
(B)



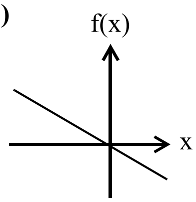
(C)



(D)



(E)



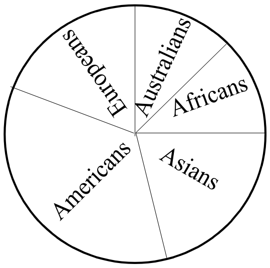
3. If $x/2 + 5y = -3$ and $y < -6$, which of the following must be true?

- **(A)** $x < -66$
- **(B)** $x > -66$
- **(C)** $x < 54$
- **(D)** $x > 54$
- **(E)** $x = 54$

4. If x and y are two consecutive integers, which of the following can be $x^2 + y^2$?

- **(A)** 1739
- **(B)** 1740
- **(C)** 1741
- **(D)** 1742
- **(E)** 1743

5. The pie chart shows athletes from different continents in a sport competition. If the total number of athletes is 550, which of the following best represents the number of athletes from Europe and America combined?



- **(A)** 250
- **(B)** 275
- **(C)** 300
- **(D)** 350
- **(E)** 400

6. If | x + 1 | = 1, which of the following may be correct?

I. $x + 1 = 1$
II. $x + 1 = -1$
III. $-x - 1 = -1$

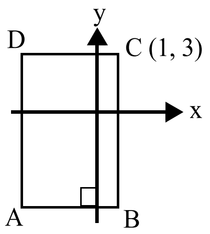
- **(A)** I only
- **(B)** II only
- **(C)** I and II only
- **(D)** I and III only
- **(E)** I and II and III

7. $$\frac{2}{1-\frac{1}{1-\frac{1}{3}}} - \frac{1}{1-3} = ?$$

- **(A)** -7/2
- **(B)** -3/2
- **(C)** -1/2
- **(D)** 7/6
- **(E)** 13/2

1/3

8. ABCD is a rectangle with length 8 and width 5. What are the coordinates of point A?



- **(A)** (-5, -8)
- **(B)** (-5, -5)
- **(C)** (-4, -5)
- **(D)** (-4, -6)
- **(E)** (-4, 5)

9. Bob is paid twice a month. How many times he is paid in y years and m months?

- **(A)** 12m + y
- **(B)** 12y + m
- **(C)** 24y + 2m
- **(D)** 24m + 2y
- **(E)** 24y + m

10. If $\frac{x+3}{x^2+x-1} = 1$, x = ?

I. 2
II. 0
III. -2

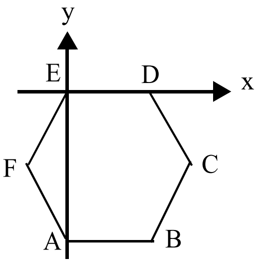
- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** I and II only
- **(E)** I and III only

1/2

11. $u^5 = v^{-15}$. What is v in terms of u?

- **(A)** u^{-3}
- **(B)** $1/(\sqrt[3]{u})$
- **(C)** $\sqrt[3]{u}$
- **(D)** u^3
- **(E)** $u^{1/3}$

12. ABCDEF is a regular hexagon with side length s. Which of the line segments have the slope tan(60^o)?



(A) $\overline{AB}$
(B) $\overline{BC}$
(C) $\overline{CD}$
(D) $\overline{FD}$ (not shown)
(E) $\overline{EB}$ (not shown)

13. Ahmet's password consist of three, 2-digit numbers. One number is even, the other is a multiple of 3 and the other is the month that Ahmet's mother is born. If Ahmet's password is 10-11-12, in what month his mother is born?

(A) January
(B) September
(C) October
(D) November
(E) December

14. If $\sqrt{u^2 - 4} = u - 2$, then which of the following is true?

(A) $u + 2 = u - 2$
(B) $u + 2 = (u - 2)^2$
(C) $-2u + 8 = 0$
(D) $8 - 4u = 0$
(E) None of the above.

2/3

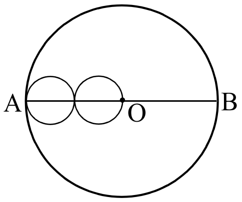
15. How many even integers are there between -30 and 30, inclusive?

(A) 61
(B) 60
(C) 31
(D) 30
(E) 29

16. If the diagonal of a square is x, what is its area in terms of x?

(A) $x^2/2$
(B) $3x^2/4$
(C) x^2
(D) $\sqrt{3}x^2/4$
(E) $x^2/(\sqrt{2})$

17. Two small circles of equal radius and the large circle in the figure are tangents to each other at the points of contact. O is the center of the large circle. The centers of all three circles are on $\overline{AB}$.



What is the ratio of the combined area of the two small circles to the area of the circle O?

(A) 1/16
(B) 1/8
(C) 3/16
(D) 1/4
(E) 5/16

18. S is an arithmetic sequence. First term is -20 and each term is 5 more than the previous term. If the sum of all the terms in S is 130, how many terms are there in S?

- **(A)** 10
- **(B)** 11
- **(C)** 12
- **(D)** 13
- **(E)** 14

19. a, b, c, d, e and f are positive numbers and

$$\left(\frac{a^{1/2}b^{2c/3}}{d^{e\cdot f}}\right)^3 = a^{-3/2}$$

If d = b/4 and ef = c, then for what value of c, ab = $(\sqrt{b/4})^3$?

- **(A)** 2
- **(B)** 2/3
- **(C)** 4
- **(D)** 5/2
- **(E)** 3/2

20. f(x) and g(x) are two functions. f(x) is symmetric around y-axis. Both are displayed in the figure. If g(x) = f(a(x + b)) + c, which of the following could be the value of abc?

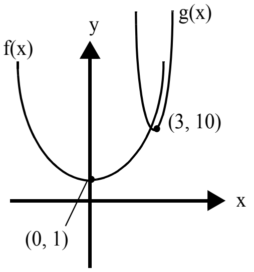


Figure is not drawn to scale.

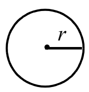
- **(A)** 54
- **(B)** 27
- **(C)** 24
- **(D)** 12
- **(E)** -12

L

Test 4 - Section 2

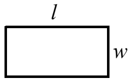
25 Minutes, 18 Questions

Reference Information

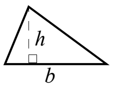


$A = \pi r^2$

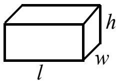
$C = 2\pi r$



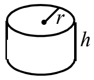
$A = lw$



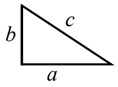
$A = \frac{1}{2}bh$



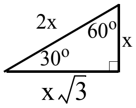
$V = lwh$

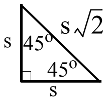


$V = \pi r^2 h$



$c^2 = a^2 + b^2$





Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

Part 1

1. If $c = \frac{u}{5u} + 7u$ and c = -4/5, then u - 6/7 = ?

(A) -4
(B) -4/7
(C) -1
(D) -5/7
(E) -4/3

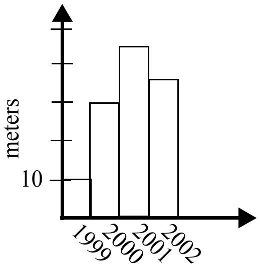
2. On a number line, the coordinate of point A is -3. If the distance between point A and point B is 8, what is the coordinate of point B?

I. -5
II. 5
III. -11

(A) I only
(B) II only
(C) III only
(D) I and II
(E) II and III

1/4

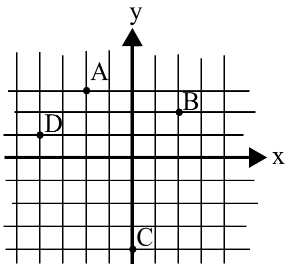
3. Graph shows the water level in a reservoir in four consecutive years. What is the average water level in these 4 years?



(A) 30
(B) 35
(C) 40
(D) 120
(E) 125

1/3

4. Which of the following two points satisfy the condition $|y_2 - y_1| > 5$, where y_1, y_2 are the y-coordinates of the first and second points respectively?



(A) (B, D)
(B) (C, D)
(C) (B, C)
(D) (A, B)
(E) (A, D)

1/2

5. If n = -1, then $\frac{(2n)^2}{3} \div (-2)^n$ = ?

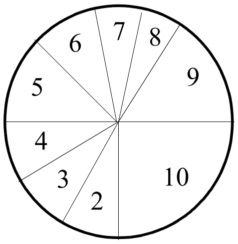
(A) -8/3
(B) 8/3
(C) -2/3
(D) 2/3
(E) -1/3

6. If the side length of an equilateral triangle is x, what is its area?

(A) $3x^2/4$
(B) $3x^2/2$
(C) $\sqrt{2}(x^2/2)$
(D) $\sqrt{3}(x^2/4)$
(E) $\sqrt{3}(x^2/2)$

2/3

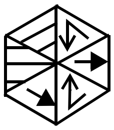
7. Performance of the employees in a company is measured by scores from 1 to 10 at the end of each year. Pie chart in the figure summarizes the results for 2005. Each slice of the pie represents the number of employees with the score shown in that slice.



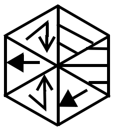
What is the median score?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

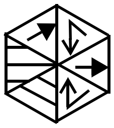
8. Which of the following can be obtained if the regular hexagon in the figure is rotated by 120°, counter clockwise?



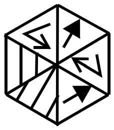
(A)



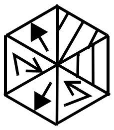
(B)



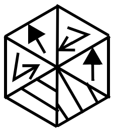
(C)



(D)



(E)



L

Part 2

9. The water level in a reservoir is 50 feet at the end of May, and drops 6 feet during June. If at the end of July, the level drops to 1/2 of its former level in the beginning of July, what is the water level at the end of July?

10. Jill spends 20% of her $20.00 for lunch and one quarter of it for a movie ticket. How much money does she have left?

1/4

11. The table lists three available types of flowers in four different colors. If you were to give one flower to each of your friends and if you want no two friends to have the same flower with the same color, what is the maximum number of friends who can receive your gift?

Type	**Color**
Rose	Red
Daisy	White
Tulip	Yellow
	Pink

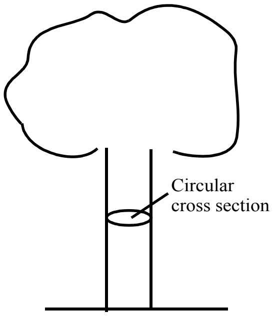
1/3

12. The median of nine consecutive odd integers is 7. What is the sum of these integers?

13. $f(x) = x^2 - 2x -1$ and $g(x) = 2f(3x) + 10$
What is $g(-2)$?

1/2

14. If the diameter of a tree trunk triples every 10 years, what is the ratio of the area of its circular cross section in the current year to the area of its circular cross section 10 years ago? Circular cross section is shown in the figure.



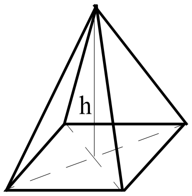
2/3

15. If n is an even number, divisible by 3, what is the remainder of (n + 7)/6?

16. Integers from 1 to 5 are written on seven identical balls. Each number is used at least once. The balls are placed in a box. If the probability of picking number 4 is three times greater than picking any other number, what is the average of the numbers on the balls?

17. Ben and Karen live in Baltimore, MD and Richmond, VA, respectively. They both start driving at 10:00 AM toward each other's city, using the same road. If their car passes through the same point 2/3 of the way from Richmond to Baltimore, what is the ratio of Karen's average speed to Ben's average speed?

18. All the sides of the right pyramid shown in the figure are equal in length and their addition is 8. h is the height of the pyramid. What is the value of h^2 ?

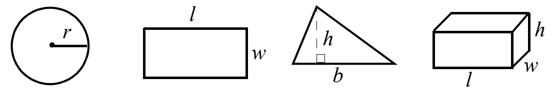


L

Test 4 - Section 3

20 Minutes, 16 Questions

Reference Information



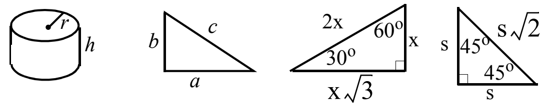
$A = \pi r^2$

$C = 2\pi r$

$A = lw$

$A = \frac{1}{2}bh$

$V = lwh$



$V = \pi r^2 h$

$c^2 = a^2 + b^2$

Special Right Triangles

The number of degrees of arc in a circle is 360.

The sum of the measures in degrees of the angles of a triangle is 180.

1. If a + 1 = 3, what is 2a - 10?

(A) 6
(B) 4
(C) 2
(D) -4
(E) -6

2. A sequence starts with 1 and each term is twice as great as the previous term. Which of the following are the first five terms of the sequence?

(A) {1, 2, 3, 4, 5, ...}
(B) {1, 2, 4, 8, 16, ...}
(C) {2, 3, 4, 5, 6, ...}
(D) {2, 4, 8, 16, 32, ...}
(E) {1, 3, 5, 7, 9, ...}

3. Three friends are on a diet to lose weight. After one week they lost a total of 3 pounds. If one of them gained one pound during the week, how many pounds the other two have lost on the average?

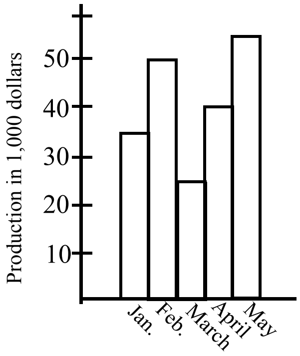
(A) 1
(B) 2
(C) 3
(D) 4/3
(E) 3/4

4. a and b are two consecutive integers.
If x = ab and y = a + b, which of the following is true?

(A) x and y are both odd integers.
(B) x and y are both even integers.
(C) x is odd and y is even integers.
(D) x is even and y is odd integers.
(E) If y is an even integer, then x is an odd integer.

1/4

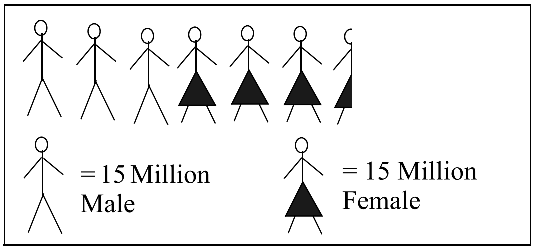
5. The chart shows the revenue of a company between January and May. What is the median revenue between January and May?



(A) $25,000
(B) $35,000
(C) $40,000
(D) $40
(E) $50

1/3

6. The below figure shows the male and female population of a country. What is the total population?

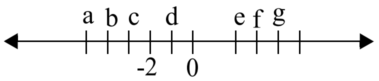


(A) 105 Million
(B) 97.5
(C) 90
(D) 70
(E) 65

7. If x and y are two negative integers and $x^3 + y = -21$, which of the following is a possible value of y?

(A) 13
(B) 6
(C) -6
(D) -12
(E) -13

8.



On the number line in the above figure, a, b, c, d, e, f and g are the coordinates of seven points. | a + f | = ?

- **(A)** c
- **(B)** d
- **(C)** 1
- **(D)** e
- **(E)** g

1/2

9. The price of the orange juice in a one-gallon container is p dollars. There is a 20% discount if one buys 6 of them. If you buy 6 one-gallon orange juices at the discounted price and sell each for $0.50 more than the original not-discounted price, how much profit you make after selling all six of them?

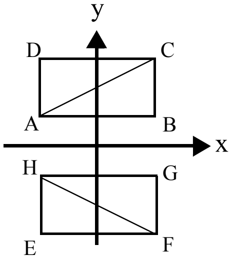
- **(A)** 0.2p + 0.5
- **(B)** p + 3
- **(C)** 1.2p + 3
- **(D)** p + 6
- **(E)** 6p + 3

10. r lbs. of rice and 3 lbs. of sugar cost $7.50. 4 lbs. of rice and 2 lbs. of sugar cost $9.00. How much is a pound of rice?

- **(A)** 6/(6 - r)
- **(B)** 6/(r - 6)
- **(C)** (6 - r)/6
- **(D)** (r - 6)/6
- **(E)** $2.50

2/3

11. Rectangle ABCD is a reflection of the rectangle HGFE along the x-axis, as shown in the figure. If BC = 2 and CD = 4, then what is the slope of $\overline{HF}$?



- **(A)** 1/2
- **(B)** 0
- **(C)** -1/4
- **(D)** -1/2
- **(E)** -1

12. Multiplication of 3 consecutive integers is zero. Which of the following could be the addition of these integers?

- **I.** -3
- **II.** 0
- **III.** 3

- **(A)** I only
- **(B)** II only
- **(C)** III only
- **(D)** II or III only
- **(E)** I or II or III

13. In the figure, BC = BD = AB What is the value of x?

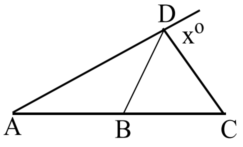
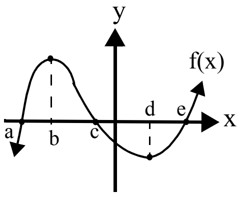


Figure is not drawn to scale.

- **(A)** 60°
- **(B)** 70°
- **(C)** 80°
- **(D)** 90°
- **(E)** 100°

14. Which of the following can be f(x)?



- **(A)** x(x - b)(x - d)
- **(B)** (x + a)(x + c) - d
- **(C)** (x - a)(x - c) - e
- **(D)** (x + a)(x + c)(x - e)
- **(E)** (x - a)(x - c)(x - e)

15. Consider a rectangle, ABCD, with CD = w and AB = 5. If you rotate the rectangle along $\overline{CD}$ by 180°, the volume of the resultant object is 50π . What is the value of w?

- **(A)** 1
- **(B)** 2
- **(C)** 4
- **(D)** 8
- **(E)** 16

16. $f(x) = x(x^2 - 16)(a - bx)^2$, where a and b are constants. Which of the following value of x is <u>not</u> the x-intercept of f(x)?

- **(A)** -4
- **(B)** 0
- **(C)** 4
- **(D)** a/b
- **(E)** b/a

L

Answer Key - Test 4

Section 1

1. (C)
2. (A)
3. (D)
4. (C)
5. (C)
6. (E)
7. (A)
8. (C)
9. (C)
10. (E)
11. (B)
12. (B)
13. (D)
14. (D)
15. (C)
16. (A)
17. (B)
18. (D)
19. (E)
20. (A)

Section 2

Part 1

1. (C)
2. (E)
3. (A)
4. (C)
5. (A)
6. (D)
7. (E)
8. (E)

Part 2

9. 22
10. 11
11. 12
12. 63
13. 104
14. 9
15. 1
16. 23/7 or 3.29
17. 2
18. 1/2

Section 3

1. (E)
2. (B)
3. (B)
4. (D)
5. (C)
6. (B)
7. (E)
8. (D)
9. (C)
10. (A)
11. (D)
12. (E)
13. (D)
14. (E)
15. (C)
16. (E)

Calculate Your Score

Calculate Your Raw Score

1. Count the number of correct answers and write it in Line 1 of the following work sheet.

2. Count the number of incorrect (<u>not</u> missing) answers in Section 1, Section 2, Part 1 and Section 3. Divide this count by 4 and write it in Line 2.

Note that Part 2 of Section 2 has grid-in questions. No penalty is given for the incorrect answers in this part. That is why you don't need to count the incorrect answers here.

Also note that there is no penalty for missing any answers. That is why you don't need to count the missing answers.

3. Subtract Line 2 from Line 1 and write the result in Line 3.

4. Round the score on line 3 to the nearest whole number and write the result on Line 4. This is your raw score.

Work Sheet to Calculate the Raw Score

1.	Correct Answers	
2.	(Incorrect Answers)/4	
3.	Not rounded Raw Score	
4.	Raw Score	

Calculate Your SAT Score

Find your raw score in the below table and read the corresponding SAT score. Note that the actual SAT scores are expressed in ranges, not definite numbers. For our purposes, we assign only one SAT score to each raw score.

Raw Score	SAT Score	Raw Score	SAT Score	Raw Score	SAT Score
-6 or less	200	15	430	36	590
-5	205	16	435	37	595
-4	215	17	440	38	600
-3	225	18	450	39	610
-2	235	19	460	40	620
-1	245	20	465	41	630
0	260	21	470	42	640
1	280	22	480	43	650
2	290	23	490	44	655
3	310	24	495	45	660
4	320	25	500	46	670
5	330	26	510	47	690
6	340	27	520	48	700
7	360	28	525	49	710
8	370	29	530	50	730
9	380	30	540	51	740
10	385	31	550	52	760
11	390	32	560	53	775
12	400	33	565	54	800
13	410	34	570		
14	420	35	580		

Name:

Subject Table - Test 4

Date:

Question Number	Categories - Subjects				Difficulty Level
1.1	S. A - One Variable Eqs.				Easy
1.2	A. A - Linear Funcs.				Easy
1.3	S. A - Multiple Unknowns	S. A - Inequalities			Medium
1.4	Arith. - Even, Odd Nums.	S. A - Multiple Unknowns			Medium
1.5	Others. - Rounding	Others - Tbls, Chrts, Grphs			Easy
1.6	S.A - Exprs. with Abs. Value				Medium
1.7	Arith. - Fractions, Ratios				Medium
1.8	Geo. - Rectangles	Geo. - Coordinate Geo.			Easy
1.9	WQ - Formulation Only				Medium
1.10	Arith. - Fractions, Ratios	Arith. - Powers	S. A - One Variable Eqs.		Medium
1.11	Arith. - Radicals	S.A - Exprs. with Powers			Medium
1.12	Geo. - Polygons	Geo. - Trigonometry	A. A - Linear Funcs.		Medium
1.13	Arith. - Divisibility	Arith. - Even, Odd Nums.	Others - Logic		Medium
1.14	S. A - One Variable Eqs.	S.A - Exprs. with Powers			Medium
1.15	Others - Basic Counting				Hard
1.16	Geo. - Triangles	Geo. - Squares	WQ - Desc. Figures		Hard
1.17	Arith. - Fractions, Ratios	Geo. - Circles			Hard
1.18	S. A - One Variable Eqs.	Others - Sequences	Others - Sums		Hard
1.19	Arith. - Square Root	S. A - Exprs. with Powers			Hard
1.20	A. A - Functions				Hard
2.1	S. A - Multiple Unknowns				Easy
2.2	Arith. - Abs. Value	Geo. - Points and Lines	WQ - Desc. Figures		Easy
2.3	Others. - Rounding	Others - Tbls, Chrts, Grphs	Others - Statistics		Easy
2.4	Arith. - Abs. Value	Geo. - Coordinate Geo.	S. A - Inequalities		Medium
2.5	Arith. - Fractions, Ratios	Arith. - Powers	Arith. - Negative Nums.		Medium
2.6	Geo. - Triangles	WQ - Desc. Figures			Medium
2.7	Others - Tbls, Chrts, Grphs	Others - Statistics			Hard
2.8	Geo. - Symetry				Hard
2.9	Arith. - Basic Arithmetic	WQ - Regular			Easy
2.10	Arith. - Basic Arithmetic	Arith. - Fractions, Ratios	Arith. - Percentages	WQ - Regular	Easy
2.11	Others - Tbls, Chrts, Grphs	Others - Basic Counting	Others - Perm., Comb.		Easy
2.12	Others - Statistics				Medium
2.13	S.A - Exprs. with Powers	A. A - Functions			Medium
2.14	Arith. - Powers	Geo. - Circles	Geo. - 3D Objects	WQ - Regular	Medium
2.15	Arith. - Divisibility				Medium
2.16	Others - Logic	Others - Statistics	Others - Probability	WQ - Regular	Hard
2.17	Arith. - Fractions, Ratios	WQ - Regular			Hard
2.18	Geo. - Triangles	Geo. - 3D Objects			Hard
3.1	Arith. - Negative Nums.	S. A - One Variable Eqs.			Easy
3.2	Others - Sequences				Easy
3.3	Arith. - Basic Arithmetic	Others - Statistics	WQ - Regular		Easy
3.4	Arith. - Even, Odd Nums.				Easy
3.5	Others - Tbls, Chrts, Grphs	Others - Statistics			Medium
3.6	Others - Tbls, Chrts, Grphs				Medium
3.7	Arith. - Powers	Arith. - Negative Nums.	S. A - Mult. Unknowns		Medium
3.8	Arith. - Abs. Value	Geo. - Points, Lines			Medium
3.9	Arith. - Basic Arithmetic	Arith. - Percentages	WQ - Form. Only		Medium
3.10	Arith. - Basic Arithmetic	WQ - Formulation Only			Medium
3.11	Geo. - Rectangles	Geo. - Symetry	A. A - Linear Funcs.		Medium
3.12	Arith. - Basic Arithmetic	S. A - One Variable Eqs.	Others - Basic Counting		Medium
3.13	Geo. - Angles	Geo. - Triangles	Geo. - Circles		Hard
3.14	A. A - Functions				Hard
3.15	Geo. - 3D Objects				Hard
3.16	A. A - Functions				Hard

Skipped:

Wrong:

Analysis Chart - Test 4

Name: **Date:**

Category	**Subject**	**Easy**	**Medium**	**Hard**
Arithmetic	Basic Arithmetic	2.9, 2.10, 3.3	3.9, 3.10, 3.12	
	Decimals			
	Fractions, Ratios	2.10	1.7, 1.10, 2.5	1.17, 2.17
	Percentages	2.10	3.9	
	Powers		1.10, 2.5, 2.14, 3.7	
	Square Root			1.19
	Radicals		1.11	
	Negative Numbers	3.1	2.5, 3.7	
	Numbers Between –1 and 1			
	Divisibility		1.13, 2.15	
	Even & Odd Numbers	3.4	1.4, 1.13	
	Absolute Value	2.2	2.4, 3.8	
Geometry	Points and Lines	2.2	3.8	
	Angles			3.13
	Polygons		1.12	
	Triangles		2.6	1.16, 2.18, 3.13
	Quadrangles			
	Rectangles	1.8	3.11	
	Squares			1.16
	Circles		2.14	1.17, 3.13
	Trigonometry		1.12	
	Coordinate Geometry	1.8	2.4	
	Symmetry		3.11	2.8
	3-D Objects		2.14	2.18, 3.15
Simple Algebra (S. A.)	One Variable Equations	1.1, 3.1	1.10, 1.14, 3.12	1.18
	Multiple unknowns	2.1	1.3, 1.4, 3.7	
	Equations with Powers		1.11, 1.14, 2.13	1.19
	Radical Equations			
	Inequalities		1.3, 2.4	
	Expressions with Absolute Value		1.6	
	Proportionality			
Advanced Algebra (A. A.)	Functions		2.13	1.20, 3.14, 3.16
	Linear functions	1.2	1.12, 3.11	
	Quadratic Functions			
Others	Rounding	1.5, 2.3		
	Tables, Charts, Graphs	1.5, 2.3, 2.11	3.5, 3.6	2.7
	Sets			
	Defined Operators			
	Logic		1.13	2.16
	Statistics	2.3, 3.3	2.12, 3.5	2.7, 2.16
	Sequences	3.2		1.18
	Sums			1.18
	Basic Counting	2.11	3.12	1.15
	Permutations, Combinations	2.11		
	Mutually Exclusive Events			
	Independent Events			
	Probability			2.16
Word Questions (WQ)	Regular	2.9, 2.10, 3.3	2.14	2.16, 2.17
	Formulation Only		1.9, 3.9, 3.10	
	Describing Figures	2.2	2.6	1.16

Skipped: **Wrong:**

Solutions - Test 4

Section 1

1. Answer: (C)
$2z + 17 = -1$ ➔ $2z = -1 - 17 = -18$ ➔
$z = -18/2 = -9$ ➔ $z + 2 = -9 + 2 = -7$
The answer is (C).

2. Answer: (A)
Answer choices (A), (C), (D) and (E) are all linear functions. However, only (A), (C) and (E) have negative slopes, because they are decreasing functions. Among these three, only (A) has x-intercept = -2.
The answer is (A)

3. Answer: (D)
$x/2 + 5y = -3$ and $y < -6$ ➔ $x/2 + 5 \cdot (-6) > -3$ ➔
$x/2 - 30 > -3$ ➔
$x/2 > -3 + 30$ ➔ $x/2 > 27$ ➔ $x > 54$
The answer is (D).

4. Answer: (C)
Let $a = x^2 + y^2$
Since x and y are two consecutive integers, one is odd and the other is even. ➔ If x^2 is odd, y^2 is even or vice versa. ➔ The answer, a, is the addition of an odd and an even integer. Hence a is an odd number. You can eliminate (B) and (D), because they are even integers.

x and y are two consecutive integers. ➔
$y = x + 1$ ➔ $a = x^2 + y^2 = x^2 + (x + 1)^2 =$
$x^2 + x^2 + 2x + 1 = 2x^2 + 2x + 1 = 2x(x + 1) + 1$ ➔
$(a - 1)/2 = x(x + 1) = xy$, which is always even, because either x or y is even.

Only (C) satisfies both of these two conditions. 1741 is an odd number and $(1741 - 1)/2 = 870$ is an even number.

In fact these two consecutive integers are 29 and 30, and $29^2 + 30^2 = 841 + 900 = 1741$

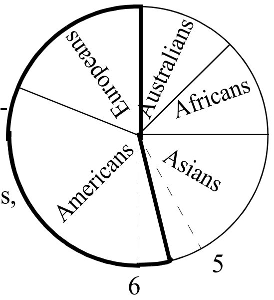
Alternate Solution:
Since x and y are consecutive, x^2 and y^2 are different but close to each other.

Hence $a = x^2 + y^2 \cong 2x^2 \cong 2y^2$ ➔

$x \cong y \cong \sqrt{\dfrac{1740}{2}} \cong 29$ ➔
The two consecutive numbers must be close to 29 and 30. You know from the previous solution that
$29^2 + 30^2 = 1741$
The answer is (C).

Note that all the answer choices are close to each other. You can replace 1740 in the above solution with any one of the answer choices and come up with the same approximate value for x and y.

5. Answer: (C)
In questions like this, it is easier to think about the pie-chart as if it is a clock-face. Imagine the total circle of the clock-face representing all the athletes, with each of the 12 hour-slices representing 1/12th of the athletes.



Thus half of this clock-face corresponds to the half of the athletes, as indicated by the dotted-line at the six-o-clock position.

The additional 1/12th is represented by the second dotted line at the five-o-clock position.

In the figure, the number of athletes from America and Europe combined is thus represented by the clock-surface that is larger than the half (6/12th), but smaller than 7/12th of the clock surface.

If the total number of athletes is 550, then half of them is $550/2 = 275$ and 7/12th of them is
$550 \times 7/12 \cong 321$
The only answer choice between 275 and 321 is 300, case (C).

6. Answer: (E)
If $|x + 1| = 1$, then $x + 1$ can be 1 or -1. So I and II may both be correct. In Case I, when you multiply both sides of the equation by -1, you will get Case III. Hence III may also be correct. The answer is (E).

7. Answer: (A)

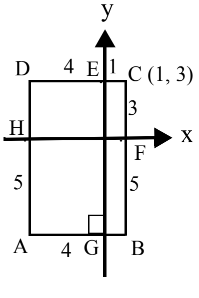
$$\frac{2}{1 - \frac{1}{\left(1 - \frac{1}{3}\right)}} - \frac{1}{1 - 3} = \frac{2}{1 - \frac{1}{\left(\frac{2}{3}\right)}} - \frac{1}{1 - 3} =$$

$$\frac{2}{\left(1 - \frac{3}{2}\right)} - \frac{1}{1 - 3} = \frac{2}{\left(-\frac{1}{2}\right)} - \frac{1}{1 - 3} = -2 \cdot 2 - \frac{1}{-2} =$$

$$-4 + \frac{1}{2} = -\frac{8}{2} + \frac{1}{2} = -7/2$$

The answer is (A).

8. Answer: (C)
The length of the rectangle ABCD is 8 and the coordinates of C is (1, 3) ➔ CF = 3 and AH = BF = 8 - 3 = 5 as shown in the figure.



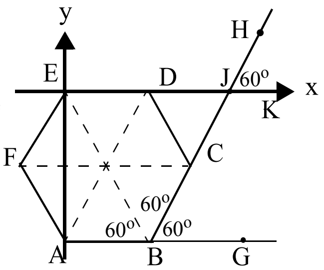
The width of the rectangle ABCD is 5 and the coordinates of C is (1, 3) ➔ CE = 1 and
GA = ED = 5 - 1 = 4
as shown in the figure. Hence the coordinates of A are (-4, -5). The answer is (C).

9. Answer: (C)
There are 12y + m months in y years and m months. If Bob is paid 2 times a month, then he is paid 2(12y + m) = 24y + 2m times in y years and m months.
The answer is (C).

10. Answer: (E)
$$\frac{x+3}{x^2+x-1} = 1 \Rightarrow x + 3 = x^2 + x - 1 \Rightarrow$$
$x^2 - 4 = 0$ ➔ x = 2 or x = -2
The answer is (E).

11. Answer: (B)
$u^5 = v^{-15} = (v^{-3})^5 \Rightarrow u = v^{-3} \Rightarrow v = u^{-1/3} = 1/(\sqrt[3]{u})$
The answer is (B).

12. Answer: (B)
Slope of a line is tan(a^o), where a^o is the angle between the line and the x-axis.



A regular hexagon contains six equilateral triangles as shown in the figure, separated by the dotted lines.

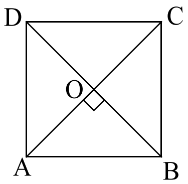
Hence $\angle CBG = 180 - 60 - 60 = 60°$ ➔ $\overline{BC}$ makes 60^o angle with x-axis. ➔
Slope of $\overline{BC}$ is tan(60^o). The answer is (B).

13. Answer: (D)
The only 2-digit number that is not even or not divisible by 3 is 11. Therefore it must be Ahmet's mother's birth month. The answer is (D).

14. Answer: (D)
$\sqrt{u^2 - 4} = u - 2 \Rightarrow u^2 - 4 = (u - 2)^2 \Rightarrow$
$u^2 - 4 = u^2 - 4u + 4 \Rightarrow 8 - 4u = 0$
The answer is (D).

15. Answer: (C)
There are (30 - (-30))/2 + 1 = 31 even integers between -30 and 30, inclusive. The answer is (C).

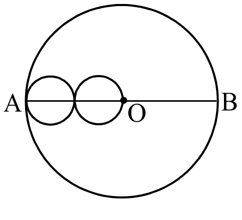
16. Answer: (A)
In the figure, square ABCD consists of 2 congruent right triangles, ΔABC and ΔADC. The bases, AC and heights, OB and OD, of these triangles are x and x/2, respectively. Hence the area of each triangle is:



$$\frac{1}{2}\left(x \cdot \frac{x}{2}\right) = \frac{x^2}{4} \Rightarrow \text{the area of ABCD} = 2 \cdot \frac{x^2}{4} = \frac{x^2}{2}$$

Notice that we didn't use the more traditional method in which one calculates the side length of the square as AB = $x/\sqrt{2}$ by using the Pythagorean Theorem and then calculates the area as $AB^2 = x^2/2$

17. Answer: (B)
Let r be the radius of circle O. Then the radii of the congruent small circles is r/4.



The area of circle O is πr^2 and the area of each small circle is
$\pi(r/4)^2 = \pi(r^2/16) \Rightarrow$

The combined area of the two small circles is $\pi(r^2/8)$ ➔ The ratio of the combined area of the two small circles to the area of the circle O is
$$\frac{\pi(r^2/8)}{\pi r^2} = 1/8$$
The answer is (B).

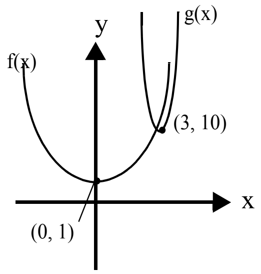
18. Answer: (D)
Let S has n terms. The nth term in an arithmetic series is a + (n - 1)d, where a is the first term and d is the difference between the consecutive terms. For sequence S, it is: -20 + 5(n -1).

The sum of first n terms is
-20n + 5n(n + 1)/2 -5n = -25n + 5n(n + 1)/2 =
(5n/2)(n + 1 - 10) = (5n/2)(n - 9) = 130 ➔
n(n - 9) = $(130 \cdot 2)/5$ = 52 ➔ $n^2 - 9n - 52 = 0$ ➔
(n - 13)(n + 4) = 0 ➔ n = 13
Note that n = -4 is not the solution, because n has to be positive. The answer is (D).

Alternate Solution:
If these formulas are too much, you can always add the terms, starting from the first term until you reach 130. However, you can use this method only if the sum is small and you can finish adding the terms in a reasonable time.

19. Answer: (E)
d = b/4 and ef = c ➔
$$\left(\frac{a^{1/2}b^{2c/3}}{d^{e \cdot f}}\right)^3 = \left(\frac{a^{1/2}b^{2c/3}}{(b/4)^c}\right)^3 = a^{-3/2} \Rightarrow$$
$$\frac{a^{3/2}b^{2c}}{(b/4)^{3c}} = a^{-3/2} \Rightarrow (b/4)^{3c} = a^3b^{2c} \Rightarrow$$
for c = 3/2, $(b/4)^{9/2} = a^3b^3$ ➔
$ab = (b/4)^{3/2} = (\sqrt{b/4})^3$. The answer is (E).

20. Answer: (A)
Since g(x) is shifted by 10 - 1 = 9, along the y-axis compared to f(x), c = 9



Since g(x) is shifted to right by 3, along the x-axis compared to f(x), b = -3

Hence, $abc = a \cdot (-3) \cdot 9 = -27a$

Since g(x) is "squeezed", rather than "expanded" compared to f(x), |a| has to be more than 1. ➔

$abc = a \cdot (-3) \cdot 9 = -27a$
If a > 1, then abc < -27 and
if a < -1, then abc > 27

The only answer choice within these limits is (A).

Note: Because f(x) is symmetric around y-axis, g(x) is symmetric around x = 3 line. Hence both negative and positive values of "a" will result in the same function, as long as |a| remains the same.

Section 2

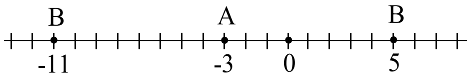
Part 1

1. Answer: (C)

$c = \frac{u}{5u} + 7u$ and c = -4/5 ➔ $-\frac{4}{5} = \frac{1}{5} + 7u$ ➔

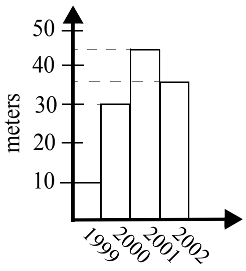
u = -1/7 ➔ u - 6/7 = -1/7 - 6/7 = -1

2. Answer: (E)



As shown on the number line above, if the distance between A and B is 8, B can be on either side of A, 8 units away. Hence the coordinate of B can be -3 + 8 = 5 or -3 - 8 = -11. The answer is (E).

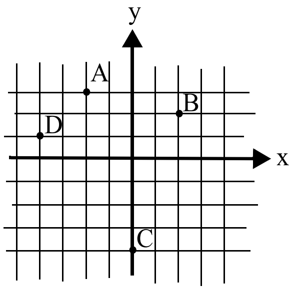
3. Answer: (A)
From the figure, let's read the water level for each year as follows:



Year	Water Level (m)
1999	10
2000	30
2001	45
2002	35

The average of these 4 years is
(10 + 30 + 45 + 35)/4 = 120/4 = 30 m
The answer is (A).

4. Answer: (C)
y-coordinates of A, B, C and D are 3, 2, -4 and 1, respectively.



Among the answer choices, $|y_2 - y_1| > 5$, only for points B and C. For these two points, $|y_2 - y_1| = |-4 - 2| = |-6| = 6 > 5$

The answer is (C).

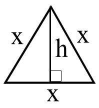
5. Answer: (A)

n = -1 ➔ $\frac{(2n)^2}{3} \div (-2)^n = \frac{(-2)^2}{3} \div (-2)^{-1} =$

$\frac{4}{3} \times (-2)^1 = -\frac{8}{3}$

The answer is (A).

6. Answer: (D)
From the figure, the height, h, of the triangle is

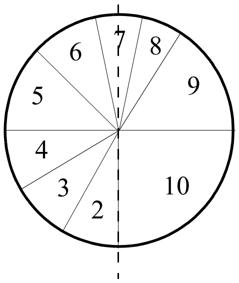


$\sqrt{x^2 - (x/2)^2} = \sqrt{x^2 - x^2/4} = \sqrt{3}x/2$

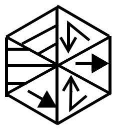
The area of the triangle is $\frac{1}{2}x\sqrt{3}\left(\frac{x}{2}\right) = \frac{\sqrt{3}x^2}{4}$

The answer is (D).

7. Answer: (E)
As is shown in the figure, score 7 divides the number of employees into half. The number of employees who scored more than 7 is equal to the number of employees who scored less than 7. Hence the median score is 7.
The answer is (E).



8. Answer: (E)
Since the figure is a regular hexagon, each triangle in the figure is an equilateral triangle with 60° angles.



When you rotate the figure counter clockwise by 120°, the arrow pointing East, in the right most triangle must now point North West, in the top-left triangle. Only (D) and (E) satisfy this condition. One of the two arrows in the figure doesn't point outward. Since in case (D), both of the arrows point outward, it can not be the answer. The answer is (E).

Part 2

9. Answer: 22
The water level in the reservoir is 50 feet at the end of May, and drops 6 feet during June. ➔
In the beginning of July the water level is
50 - 6 = 44 feet. ➔
At the end of July, the level drops to 1/2 of its former level it had in the beginning of July. ➔
The water level at the end of July is 44/2 = 22 feet.

10. Answer: \$11
Jill has \$20.00. She spent 20% of it for lunch. ➔
She spent $\frac{20 \cdot 20}{100}$ = \$4 for lunch.
If she has spent one quarter of her money for movie ticket, she spent 20/4 = \$5. Hence she's got 20 - 4 - 5 = \$11 left.

11. Answer: 12
Each flower comes in 4 different colors. So you can give the same flower to 4 different friends and each gets a different color.

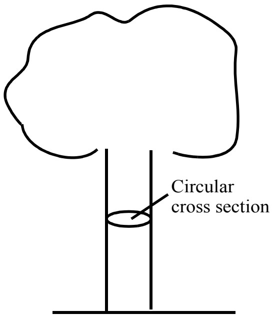
Type	**Color**
Rose	Red
Daisy	White
Tulip	Yellow
	Pink

Since there are 3 flowers available, $3 \cdot 4 = 12$ friends can receive your gift and each will have either a different type or a different color flower.

12. Answer: 63
The median of the nine consecutive odd integers is 7 ➔ The average of these integers is also 7, because they are regularly increasing numbers. So the sum of these integers is $9 \cdot 7 = 63$

13. Answer: 104
$f(x) = x^2 - 2x - 1$ and $g(x) = 2f(3x) + 10$ ➔
$g(x) = 2((3x)^2 - 2(3x) - 1) + 10 =$
$2(9x^2 - 6x - 1) + 10 = 18x^2 - 12x - 2 + 10 =$
$18x^2 - 12x + 8$ ➔
$g(-2) = 18(-2)^2 - 12(-2) + 8 = 72 + 24 + 8 = 104$

14. Answer: 9
If the diameter, d, of a tree trunk triples in 10 years, then the area of its circular cross section, which is $\pi(d^2/4)$, will increase by the square of the diameter. The answer is $3^2 = 9$

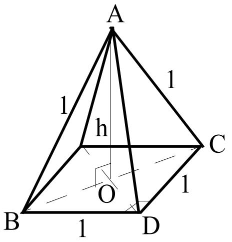


15. Answer: 1
n is an even number, divisible by 3 ➔
$n = (2 \cdot 3)m = 6m$, where m is an integer. ➔
$(n + 7)/6 = n/6 + 7/6 = (6m)/6 + 7/6 = m + 1 + 1/6$ ➔
The remainder of $(n + 7)/6$ is 1.

16. Answer: 23/7 or 3.29
If the probability of picking number 4 is three times greater than picking any other number, all the other numbers are used only once and 4 is used on 3 balls. Hence the numbers on 7 balls are: 1, 2, 3, 4, 4, 4, 5. The average of these numbers is
$(1 + 2 + 3 + 3 \cdot 4 + 5)/7 = 23/7$ or 3.29

17. Answer: 2
Let d be the distance in miles between Baltimore and Richmond, and t be the time in hours that each person travels before they pass through the same point. ➔
Ben travelled d/3 miles in t hours and Karen travels 2d/3 miles in t hours. ➔
Ben's average speed is d/(3t) miles/hour and Karen's average speed is 2d/(3t) miles/hour. ➔
The ratio of Karen's average speed to Ben's average speed is
$\frac{2d}{3t} \div \frac{d}{3t} = \frac{2d}{3t} \times \frac{3t}{d} = 2$

18. Answer: 1/2
This pyramid has 8 equal length edges. Since their addition is 8, the length of each edge is 1. Consider the triangle ΔBDC. It is an isosceles right triangle with two legs measuring 1. Hypotenuse, BC, of ΔBDC is



$\sqrt{BD^2 + DC^2} = \sqrt{2}$
Now consider the triangle ΔAOC. It is a right triangle with hypotenuse, 1, and two legs $\sqrt{2}/2$ and h. For this triangle
$h^2 = AC^2 - OC^2 = 1^2 - (\sqrt{2}/2)^2 = 1 - 2/4 = 1/2$

Section 3

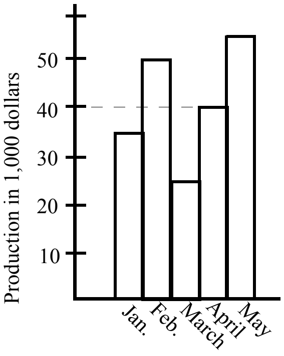
1. Answer: (E)
a + 1 = 3 ➔ a = 3 - 1 = 2 ➔
$2a - 10 = 2 \cdot 2 - 10 = 4 - 10 = -6$
The answer is (E).

2. Answer: (B)
Among the answer choices only (A), (B) and (E) starts with 1, but only in (B) each term is twice as great as the previous term. The answer is (B).

3. Answer: (B)
If they lost a total of 3 pounds and if one of them gained one pound during the week, the other two must have lost 3 + 1 = 4 lbs.
On the average, each one has lost 4/2 = 2 lbs.
The answer is (B).

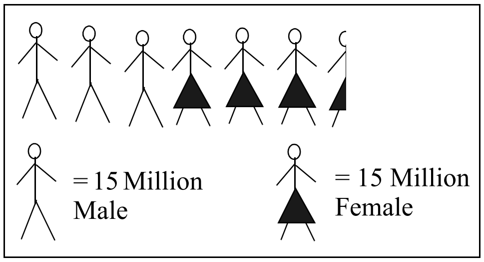
4. Answer: (D)
a and b are two consecutive integers. ➔
One of them is even and the other is odd. ➔
x = ab is even and y = a + b is odd.
The answer is (D).

5. Answer: (C)
Between January and May, the revenue in January and March is less than the revenue in April, and, the revenue in February and May is greater than the revenue in April. Hence the revenue, which is \$40,000, in April is the median revenue. The answer is (C).



Notice that you don't have to determine the revenues for all five months to answer the question. Just notice that the revenue in April is in the middle.

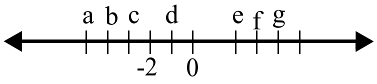
6. Answer: (B)



According to the figure, each male figure corresponds to 15 Million. Since there are 3 male figures, then there are $3 \cdot 15 = 45$ million males in the country. Similarly, each female figure corresponds to 15 million females. Since there are 3.5 female figures, there are $3.5 \cdot 15 = 52.5$ million females in the country. Total population is 45 + 52.5 = 97.5 million. The answer is (B).

7. Answer: (E)
x and y are two negative integers ➔
Both x^3 and y are negative.
$x^3 + y = -21$ ➔ x is either -1 or -2, because if $x < -2$, then $x^3 < -21$, which is not possible.
If x = -1, then y = -21 + 1 = -20, which is not one of the answer choices.
If x = -2, then $y = -21 - (-2)^3 = -21 + 8 = -13$
The answer is (E).

8. Answer: (D)



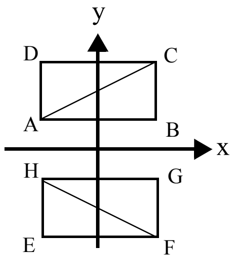
The scale of the figure is one unit per small interval.
➔ a = -5 and f = 3 ➔ | a + f | = |-5 + 3| = |-2| = 2 = e
The answer is (D).

9. Answer: (C)
With 20% discount, each one-gallon container costs p - 20p/100 = p - p/5 dollars.
If you sell each for \$0.50 more than the original price, then you sell it for p + 0.5 dollars.
Your profit from one container is p + 0.5 - (p - p/5) = p + 0.5 - p + p/5 = 0.2p + 0.5 dollars. ➔
Your profit from the sale of six of them is
6(0.2p + 0.5) = 1.2p + 3
The answer is (C).

10. Answer: (A)
Let the unit price of rice and sugar be x and y respectively.
r lbs. of rice and 3 lbs. of sugar is \$7.50. ➔
rx + 3y = 7.5
4 lbs. of rice and 2 lbs. of sugar is \$9.00. ➔
4x + 2y = 9 ➔ y = (9 - 4x)/2
Substituting y = (9 - 4x)/2 into rx + 3y = 7.5
rx + 3(9 - 4x)/2 = 7.5 ➔ rx + 27/2 - 6x = 7.5 ➔
(r - 6)x = 7.5 - 13.5 = -6 ➔ x = 6/(6 - r)
The answer is (A).

11. Answer: (D)
If BC = 2 and CD = 4, then
FG = 2 and GH = 4
The slope of
$\overline{HF}$ = -(FG/GF) = -1/2

The answer is (D).

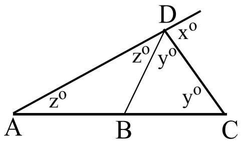


12. Answer: (E)
Since the multiplication of the three consecutive integers is zero, one of these integers must be zero. The below table shows the possible values of the three consecutive integers and their sums.

Integers	Addition
-2, -1, 0	-3
-1, 0, 1	0
0, 1, 2	3

Since all the values are possible, the answer is (E).

13. Answer: (D)
BC = BD ➔ ΔBCD is isosceles. ➔
$\angle BDC = \angle DCB = y^o$ ➔
$\angle DBC = 180 - 2y$



BD = AB ➔ ΔABD is isosceles. ➔
$\angle BAD = \angle BDA = z^o$ ➔ $\angle DBA = 180 - 2z$
$\angle DBA + \angle DBC = 180$ ➔
(180 - 2z) + (180 - 2y) = 180 ➔
180 - 2(y + z) = 0 ➔
y + z = 180/2 = 90
$\angle BDA + \angle BDC + x = 180$ ➔
z + y + x = 180 ➔ 90 + x = 180 ➔ x = 90
The answer is (D).

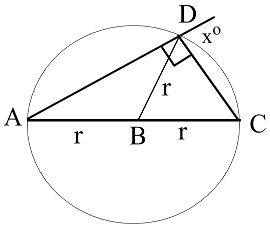
Alternate Solution:

Since AB = BC = BD, A, D and C are on the circle B.

Since $\overline{AC}$ is the diameter of this circle,

$\angle ADC = 90°$ ➔

x = 180 - 90 = 90



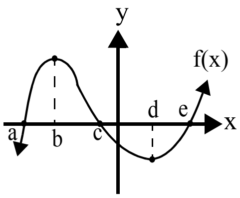
14. Answer: (E)

f(x) has three x-intercepts with x-coordinates, a, c, and e.

Since f(x) = 0 at these points, it has to be in the form of

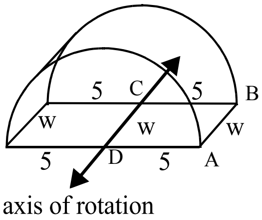
f(x) = (x - a)(x - c)(x - e)

The answer is (E).



15. Answer: (C)

As you can see in the figure, when you rotate the rectangle along its width, you will get a half cylinder with base radius 5 and height w.



The volume of this half cylinder is $\frac{w}{2}\pi 5^2 = 50\pi$

➔ w = 4

The answer is (C).

16. Answer: (E)

$f(x) = x(x^2 - 16)(a - bx)^2 = x(x - 4)(x + 4)(a - bx)^2$ ➔

The three x-intercepts of f(x) are

x = 0

$x^2 - 16 = 0$ ➔ x = 4 or x = -4

a - bx = 0 ➔ x = a/b

The answer is (E).

APPENDIX A

THE ANALYSIS CHART

What is it?

The Analysis Chart plays an important role in the Private Tutor Method. It allows you to gauge your overall performance, and your weak and strong points after each diagnostic test. The Analysis Chart is utilized in providing customized information to each student in Chapter 3.

Each question in all four diagnostic tests is examined and categorized by its subject(s) and difficulty level. The result is displayed on the Analysis Chart. To utilize the chart correctly, you need to find and highlight on this chart each question that you have missed or answered incorrectly.

If a question belongs to more than one subject, its number appears more that once in the Analysis Chart; once for each subject that it belongs to. It is important that you find and highlight all the occurrences of each missing and/or incorrectly answered question in this chart.

To help you find all the subjects categories of a question, we created the "**Subject Table**" for each Analysis Chart. This table displays all the subjects to which each question belongs. Before you highlight a question on the Analysis Chart, find all the subjects to which the question belongs in the Subject Table. Then make sure that you highlight the question number for each of these subjects in the Analysis Chart.

Each diagnostic test in this book comes with its own Analysis Chart and Subject Table, pre-filled with question numbers in all 47 subjects and three Word Question categories, and all three difficulty levels; Easy, Medium and Hard. Below are two step-by-step examples on how to fill and interpret the Analysis Chart.

How to Fill Your Analysis Chart

Pay attention to the following important points:

Write your name and the date on top of the Subject Table and the Analysis Chart. This information will help you track your progress later on. It will also remind you the tests you have already finished.

Notice that **the Analysis Chart already has all the question numbers and the Diagnostic Test Number printed on it.** You don't have to fill the question numbers on your own. All you have to do is find in this chart the questions that you have skipped or answered incorrectly.

Find the numbers of the questions you have skipped and answered incorrectly. Record these question numbers at the bottom of the Subject Table and the Analysis Chart, in "Skipped" and "Wrong" fields. This way you don't have to go back and forth between the Analysis Chart and your Answer Key. It is also useful when you have to solve the questions that you have skipped earlier or incorrectly answered. They are listed here all in one place.

Highlight all the missing and incorrectly answered questions on the Subject Table.

Highlight **in one color** all the question numbers that you have skipped. Highlight **in another color** all the question numbers that you have answered incorrectly on the Analysis Chart.

Since this book is printed in black and white, in the examples below, we have used bold and italic text to distinguish the missing and incorrectly answered questions. However, we strongly recommend that you highlight these two group of questions in two different colors.

Make sure to highlight all the occurrences of these questions on the Analysis Chart. Subject Table provides you all the subjects of each question and helps you to find all the occurrences on the Analysis Chart.

Example 1

After you have already taken a Private Tutor diagnostic test, make a list of the questions that you skipped and answered incorrectly.

Suppose that you have skipped questions 2.7, 2.14 and 3.5 and you have answered the questions 1.16, 1.18 and 3.15 incorrectly. The Subject Table and the Analysis Chart below illustrates this case. We would like to emphasize that:

- Name of the student and the date of the Diagnostic Test is filled in both in the Subject Table and Analysis Chart.
- The skipped and incorrectly answered questions are listed both in the Subject Table and Analysis Chart.
- Questions 2.7 and 3.5 are marked both in a math subject and also in "Word Questions - WQ". As you can easily see in the Subject Table, these questions are Word Questions in "Basic Arithmetic" and "Tables, Charts, Graphs" subjects respectively. Therefore, they are marked in both categories.
- Question 1.18 is marked in two different subject categories, "Coordinate Geometry" and "Functions," because it requires a knowledge of both of these subjects as can be seen in the Subject Table.

When you mark the questions that you've skipped or answered incorrectly, mark all occurrences of these questions.

At this level, your instructions in Chapter 3 include identifying the following two subject categories.

Subjects in which you missed or incorrectly answered at least one question:
These subjects are: Basic Arithmetic, Coordinate Geometry, 3-D objects, Functions, Tables, Charts, Graphs and Permutations. Pay special attention if the questions in these subjects are presented in Word Questions format.

Subjects that are not in the Diagnostic Test:
These subjects are: Powers, Square Root, Radicals, Negative Numbers, Even and Odd Numbers, Absolute Value, Polygons, Quadrangles, Trigonometry, Radical Equations, Rounding, Logic, Combinations, Exclusive Events and Independent Events.

Name: Almost Perfect

Subject Table - Test 1

Date: 6/ 6/ 2005

Question Number	Categories - Subjects			Difficulty Level
1.1	Algebra I-One Variable Equations			Easy
1.2	Arithmetic-Basic Arithmetic			Easy
1.3	Geometry-Triangles,	Geometry-Coordinate Geometry		Easy
1.4	Others-Tables, Charts, Graphs			Easy
1.5	Others-Statistics	WQ-Regular		Medium
1.6	Geometry-Squares			Easy
1.7	Arithmetic-Basic Arithmetic			Easy
1.8	Arithmetic-Fractions & Ratios			Easy
1.9	Algebra I-One Variable Equations	Algebra I- Equations with Powers		Medium
1.10	Algebra I-One Variable Equations	WQ-Regular		Medium
1.11	Geometry-Circles	Algebra II-Functions		Medium
1.12	Others-Tables, Charts, Graphs			Medium
1.13	Others-Sequences			Medium
1.14	Geometry-Points, Lines	Geometry- Coordinate Geometry	WQ-Formulation Only	Medium
1.15	Arithmetic-Percentages	Others-Tables, Charts, Graphs		Medium
1.16	**Geometry-3-D Objects**			**Medium**
1.17	Algebra I-Multiple Unknowns	WQ-Formulation Only		Medium
1.18	**Geometry-Coordinate Geometry**	**Algebra II-Functions**		**Hard**
1.19	Algebra I-Expressions with Absolute Value	WQ-Formulation Only		Hard
1.20	Others-Basic Counting			Hard
2.1	Algebra I-One Variable Equations			Easy
2.2	Geometry-Triangles			Easy
2.3	Arithmetic-Divisibility			Medium
2.4	Geometry-Rectangles	Geometry-Circles		Medium
2.5	Others-Probability			Medium
2.6	Others-Tables, Charts, Graphs	Others-Probability		Medium
2.7	**Arithmetic-Basic Arithmetic**	**WQ-Formulation Only**		**Hard**
2.8	Geometry-Coordinate Geometry			Medium
2.9	Algebra I-Multiple Unknowns			Medium
2.10	Arithmetic-Percentages	WQ-Regular		Easy
2.11	Geometry-Triangles			Medium
2.12	Others-Statistics			Medium
2.13	Algebra II-Functions			Medium
2.14	**Others-Permutations, Combinations**			**Hard**
2.15	Geometry-Angles, Triangles			Medium
2.16	Others-Defined Operators			Medium
2.17	Geometry-Coordinate Geometry	Algebra II- Linear Functions	Algebra II-Functions	Hard
2.18	Arithmetic-Basic Arithmetic	WQ-Regular		Hard
3.1	Algebra I-Multiple Unknowns			Easy
3.2	Algebra I-Proportionality	WQ-Regular		Easy
3.3	Algebra I-Multiple Unknowns			Easy
3.4	Geometry-Points, Lines			Easy
3.5	**Others-Tables, Charts, Graphs**	**WQ-Regular**		**Medium**
3.6	Algebra I-One Variable Equations	Others-Statistics		Medium
3.7	Algebra II-Linear Functions			Medium
3.8	Algebra II-Linear Functions			Medium
3.9	Geometry-Triangles			Medium
3.10	Arithmetic-Percentages	Algebra I-One Variable Equations	WQ-Regular	Medium
3.11	Arithmetic-Fractions & Ratios	Geometry-Squares	Others-Basic Counting	Medium
3.12	Arithmetic-Numbers Between –1 and 1	Algebra I-Multiple Unknowns		Medium
3.13	Others-Statistics	WQ-Regular		Medium
3.14	Algebra I-Equations with Powers			Hard
3.15	**Algebra II-Functions**			**Hard**
3.16	Others-Sets			Hard

Skipped: 2.7, 2.14, 3.5

Wrong: 1.16, 1.18, 3.15

Analysis Chart - Test 1

Name: Almost Perfect **Date: 6/ 6/ 2005**

Category	Subject	Easy	Medium	Hard
Arithmetic	Basic Arithmetic	1.2, 1.7		**2.7**,2.18
	Decimals			
	Fractions & Ratios	1.8	3.11	
	Percentages	2.10	1.15, 3.10	
	Powers			
	Square Root			
	Radicals			
	Negative Numbers			
	Numbers Between –1 and 1		3.12	
	Divisibility		2.3	
	Even & Odd Numbers			
	Absolute Value			
Geometry	Points, Lines	3.4	1.14	
	Angles		2.15	
	Polygons			
	Triangles	1.3, 2.2	2.11, 2.15, 3.9	
	Quadrangles			
	Rectangles		2.4	
	Squares	1.6	3.11	
	Circles		1.11, 2.4	
	Trigonometry			
	Coordinate Geometry	1.3	1.14, 2.8	***(1.18)***, 2.17
	Symmetry			
	3-D Objects		***(1.16)***	
Simple Algebra (S. A.)	One Variable Equations	1.1, 2.1	1.9, 1.10, 3.6, 3.10	
	Multiple Unknowns	3.1, 3.3	1.17, 2.9, 3.12	
	Equations with Powers		1.9	3.14
	Radical Equations			
	Inequalities			
	Expressions with Absolute Value			1.19
	Proportionality	3.2		
Advance Algebra (A. A)	Functions		1.11, 2.13, 1.18	***(1.18)***, 2.17, ***(3.15)***
	Linear Functions		3.7, 3.8	2.17
	Quadratic Functions			
Others	Rounding			
	Tables, Charts, Graphs	1.4	1.12, 1.15, 2.6, **3.5**	
	Sets			3.16
	Defined Operators		2.16	
	Logic			
	Statistics		1.5, 2.12, 3.6, 3.13	
	Sequences		1.13	
	Sums			
	Basic Counting		3.11	1.20
	Permutations, Combinations			**2.14**
	Exclusive Events			
	Independent Events			
	Probability		2.5, 2.6	
Word Questions (WQ)	Regular	2.10, 3.2	1.5, 1.10, **3.5,** 3.10, 3.13	2.18
	Formulation Only		1.14, 1.17	1.19, **2.7**
	Describing Figures			

Skipped: 2.7, 2.14, 3.5 ***Wrong: (1.16), (1.18), (3.15)***

Example 2

Consider the Analysis Chart shown in the figure below. In this case, the student has skipped questions 1.14, 2.7, 2.13, 2.14, 2.15, 3.5, 3.13, 3.14, 3.16 and answered questions 1.11, 1.16, 1.17, 1.18, 1.20, 2.5, 2.6, 2.12, 2.18, 3.3, 3.6, 3.9, 3.12, 3.15 incorrectly, mostly at the Hard and Medium levels.

We would like to emphasize that:

- Name of the student and the date of the Diagnostic Test is filled in both in the Subject Table and Analysis Chart.
- The skipped and incorrectly answered questions are listed both in the Subject Table and Analysis Chart.
- Questions 1.14, 1.17, 2.7, 2.18, 3.5 and 3.13 are marked both in a math subject and also in "Word Questions - WQ". As you can easily see on the Subject Table, these questions are Word Questions in different math subjects. Therefore, they are marked in both categories.
- Question 1.11, 1.14, 1.18, 2.6, 3.6 and 3.12 are marked in two different subject categories, because they require a knowledge of both of these subjects as can be seen in the Subject Table.

When you mark the questions that you've skipped or answered incorrectly, mark all occurrences of these questions.

At this level, your instructions in Chapter 3 include identifying the following two subject categories:

The subjects that the student needs to study more:
At this level, the student is instructed to find his/her weak points at the Easy and Medium levels. The student doesn't have to worry about how he/she did at the Hard level. All he/she needs to find are the areas that he/she missed at the Easy and Medium levels. For this purpose, the student should just ignore the last column in the chart to determine the subjects he/she is weak at. They are:

Numbers Between –1 and 1, Angles, Triangles, Circles, Coordinate Geometry, 3-D Objects, One Variable Equations, Multiple Unknowns, Functions, Tables, Charts, Graphs, Statistics, Probability and Word Questions.

Strong Areas:
At this level, the student needs to identify the areas in which he/she is strong at. These are the subjects in which the student could answer all the Easy and Medium level questions and at least one of the questions at the Hard level. If there is no such subject, then the student should identify the subjects in which he/she answered all the questions correctly at the Easy and Medium levels only. In this example, these are:

Fractions & Ratios, Percentages, Divisibility, Points, Lines, Rectangles, Squares, Equations with Powers, Linear Functions and Basic Counting.

Note that Addition-Subtraction-Multiplication-Division is not listed because there are no questions asked on this subject at the Medium level.

Name: Future Star

Subject Table - Test 2

Date: 7/5/2005

Question Number	Categories - Subjects			Difficulty Level
1.1	Algebra I-One Variable Equations			Easy
1.2	Arithmetic-Basic Arithmetic			Easy
1.3	Geometry-Triangles,	Geometry-Coordinate Geometry		Easy
1.4	Others-Tables, Charts, Graphs			Easy
1.5	Others-Statistics	WQ-Regular		Medium
1.6	Geometry-Squares			Easy
1.7	Arithmetic-Basic Arithmetic			Easy
1.8	Arithmetic-Fractions & Ratios			Easy
1.9	Algebra I-One Variable Equations	Algebra I- Equations with Powers		Medium
1.10	Algebra I-One Variable Equations	WQ-Regular		Medium
1.11	**Geometry-Circles**	**Algebra II-Functions**		**Medium**
1.12	Others-Tables, Charts, Graphs			Medium
1.13	Others-Sequences			Medium
1.14	**Geometry-Points, Lines**	**Geometry- Coordinate Geometry**	WQ-Formulation Only	**Medium**
1.15	Arithmetic-Percentages	Others-Tables, Charts, Graphs		Medium
1.16	**Geometry-3-D Objects**			**Medium**
1.17	**Algebra I-Multiple Unknowns**	**WQ-Formulation Only**		**Medium**
1.18	**Geometry-Coordinate Geometry**	**Algebra II-Functions**		**Hard**
1.19	Algebra I-Expressions with Absolute Value	WQ-Formulation Only		Hard
1.20	**Others-Basic Counting**			**Hard**
2.1	Algebra I-One Variable Equations			Easy
2.2	Geometry-Triangles			Easy
2.3	Arithmetic-Divisibility			Medium
2.4	Geometry-Rectangles	Geometry-Circles		Medium
2.5	**Others-Probability**			**Medium**
2.6	**Others-Tables, Charts, Graphs**	**Others-Probability**		**Medium**
2.7	**Arithmetic-Basic Arithmetic**	**WQ-Formulation Only**		**Hard**
2.8	Geometry-Coordinate Geometry			Medium
2.9	Algebra I-Multiple Unknowns			Medium
2.10	Arithmetic-Percentages	WQ-Regular		Easy
2.11	Geometry-Triangles			Medium
2.12	**Others-Statistics**			**Medium**
2.13	**Algebra II-Functions**			**Medium**
2.14	**Others-Permutations, Combinations**			**Hard**
2.15	**Geometry-Angles, Triangles**			**Medium**
2.16	Others-Defined Operators			Medium
2.17	Geometry-Coordinate Geometry	Algebra II- Linear Functions	Algebra II-Functions	Hard
2.18	**Arithmetic-Basic Arithmetic**	**WQ-Regular**		**Hard**
3.1	Algebra I-Multiple Unknowns			Easy
3.2	Algebra I-Proportionality	WQ-Regular		Easy
3.3	**Algebra I-Multiple Unknowns**			**Easy**
3.4	Geometry-Points, Lines			Easy
3.5	**Others-Tables, Charts, Graphs**	**WQ-Regular**		**Medium**
3.6	**Algebra I-One Variable Equations**	**Others-Statistics**		**Medium**
3.7	Algebra II-Linear Functions			Medium
3.8	Algebra II-Linear Functions			Medium
3.9	**Geometry-Triangles**			**Medium**
3.10	Arithmetic-Percentages	Algebra I-One Variable Equations	WQ-Regular	Medium
3.11	Arithmetic-Fractions & Ratios	Geometry-Squares	Others-Basic Counting	Medium
3.12	**Arithmetic-Numbers Between –1 and 1**	**Algebra I-Multiple Unknowns**		**Medium**
3.13	**Others-Statistics**	**WQ-Regular**		**Medium**
3.14	**Algebra I-Equations with Powers**			**Hard**
3.15	**Algebra II-Functions**			**Hard**
3.16	**Others-Sets**			**Hard**

Skipped: 1.14, 2.7, 2.13, 2.14, 2.15, 3.5, 3.13, 3.14, 3.16

Wrong: 1.11, 1.16, 1.17, 1.18, 1.20, 2.5, 2.6, 2.12, 2.18, 3.3, 3.6, 3.9, 3.12, 3.15

Name: Future Star

Analysis Chart - Test 2

Date: 7/5/2005

Category	Subject	Easy	Medium	Hard
Arithmetic	Basic Arithmetic	1.2, 1.7		**2.7**, ***(2.18)***
	Decimals			
	Fractions & Ratios	1.8	3.11	
	Percentages	2.10	1.15, 3.10	
	Powers			
	Square Root			
	Radicals			
	Negative Numbers			
	Numbers Between –1 and 1		***(3.12)***	
	Divisibility		2.3	
	Even & Odd Numbers			
	Absolute Value			
Geometry	Points, Lines	3.4	1.14	
	Angles		**2.15**	
	Polygons			
	Triangles	1.3, 2.2	2.11, **2.15**, ***(3.9)***	
	Quadrangles			
	Rectangles		2.4	
	Squares	1.6	3.11	
	Circles		***(1.11)***, 2.4	
	Trigonometry			
	Coordinate Geometry	1.3	**1.14**, 2.8	***(1.18)***, 2.17
	Symmetry			
	3-D Objects		***(1.16)***	
Algebra I	One Variable Equations	1.1, 2.1	1.9, 1.10, ***(3.6)***, 3.10	
	Multiple Unknowns	3.1, ***(3.3)***	***(1.17)***, 2.9, ***(3.12)***	
	Equations with Powers		1.9	**3.14**
	Radical Equations			
	Inequalities			
	Expressions with Absolute Value			**1.19**
	Proportionality	3.2		
Algebra II	Functions		***(1.11)***, **2.13**, 1.18	***(1.18)***, 2.17, ***(3.15)***
	Linear Functions		3.7, 3.8	2.17
	Quadratic Functions			
Others	Rounding			
	Tables, Charts, Graphs	1.4	1.12, 1.15, 2.6, **3.5**	
	Sets			**3.16**
	Defined Operators		2.16	
	Logic			
	Statistics		1.5, ***(2.12)***, ***(3.6)***, **3.13**	
	Sequences		1.13	
	Sums			
	Basic Counting		3.11	***(1.20)***
	Permutations, Combinations			**2.14**
	Exclusive Events			
	Independent Events			
	Probability		***(2.5)***, ***(2.6)***	
WQ	Regular	2.10, 3.2	1.5, 1.10, **3.5**, 3.10, **3.13**	***(2.18)***
	Formulation Only		**1.14**, ***(1.17)***	**1.19**, **2.7**
	Describing Figures			

Skipped: 1.14, 2.7, 2.13, 2.14, 2.15, 3.5, 3.13, 3.14, 3.16

Wrong: (1.11), (1.16), (1.17), (1.18), (1.20), (2.5), (2.6), (2.12), (2.18), (3.3), (3.6), (3.9), (3.12), (3.15)

APPENDIX B

MEASURING DISTANCES & ANGLES

If you can’t find the answer in the “proper way” For some geometry questions, you can measure the distances and angles to answer them.

In most cases, it takes longer time to measure than to solve the question by regular methods. In some other cases, especially for the “Hard” geometry questions, measuring may be the quickest way to determine the correct answer.

Measuring is one of the very few methods available to answer the grid-in questions correctly even if you don’t know the proper solution.

The methods described in this appendix takes some time to practice, but it is worth the effort to learn them. They will also give you a different perspective to think about a question.

Measuring Distances

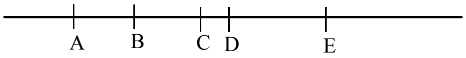
To be able to measure a distance in a figure, two conditions must be satisfied:

1. The figure has to be drawn to scale. If the figure is not drawn to scale, you must redraw it and make it to scale.
2. There has to be a known distance on the figure to use as a reference.

If these conditions are satisfied, you can mark the known distance on your answer sheet or on your second pencil. Then you can compare the known distance with the unknown.

At the end of the test, make sure to erase all the extra marks on your answer sheet.

Example: (Easy)

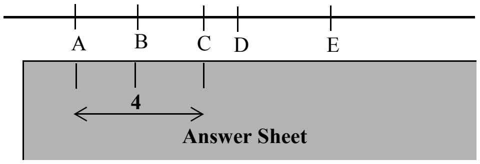


In the above figure, B bisects $\overline{AC}$ and D bisects $\overline{BE}$. AC = 4 and BE = 6. What are CD and AE?

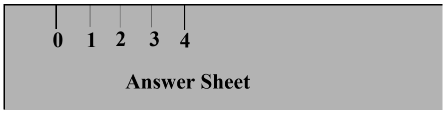
Solution by Measuring CD:

If you don't know how to answer this question, you can measure the distances CD and AE. Here is how:

Step 1:
Align your answer sheet with the line as shown in the below figure and mark the distance AC and the mid-point B.

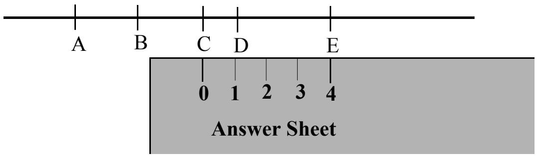


Step 2:
Make a ruler from the two marks by finding the mid-points as shown in the figure. Be as accurate as possible.



Step 3:
Use your hand-made ruler and measure the length of CD as shown in the below figure.

You will see that the points C and D correspond to 0 and 1 on your ruler, respectively. So CD = 1



The Proper Solution:

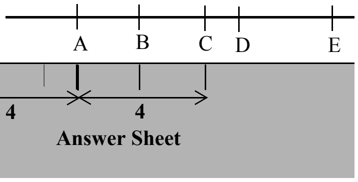
AC = 4 ➔ BC = AC/2 = 2
BE = 6 ➔ BD = BE/2 = 3
CD = BD - BC = 3 - 2 = 1

Solution by Measuring AE:

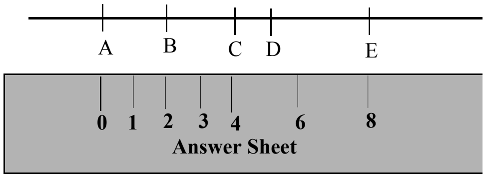
Since AE is longer than your ruler, you can calculate AE by measuring smaller distances and adding them together.

You can also make a longer ruler. Here is how:

Step 1:
Align your hand-made ruler with the line on your question booklet to add another 4 units to your ruler, as shown in the figure, and mark the distance AC and the mid-point B.



Step 2:
Align your ruler on the paper with AE as shown below and measure it. You will see that AE is equal or very close to 8.



The Proper Solution:

Since D bisects $\overline{BE}$, DE = BE/2 = 6/2 = 3
AE = AC + CD + DE = 4 + 1 + 3 = 8

Measuring Angles

To be able to measure an unknown angle, the figure must be drawn to scale and you need to create a known angle and compare the two.

There are two ways you can measure certain angles.

1.Use the Figures in your Question Booklet

Look at any question with a figure drawn to the scale and with a known angle. Note that the figure does not have to belong to the question which you are trying to answer. If you see a known angle that you might be able to use anywhere in the math section you are working on, mark it on your answer sheet.

You can always find special angles, 90^{o}, 60^{o}, 45^{o}, 30^{o} at the beginning of each math section. They are in the special triangles in the Reference Information section.

Don't fold your answer sheet or your test booklet to create angles, distances etc..

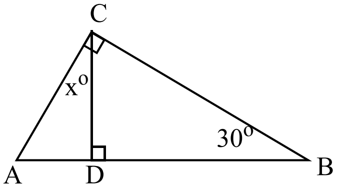
Example: (Medium)

In the figure, what is the value of x?

Solution by Measuring:

Step 1:

Since one of the angles, $\angle B = 30^{o}$, is provided in the figure, copy this angle onto your answer sheet.

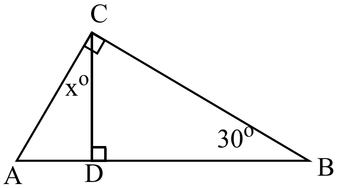


Step 2:

The unknown angle, $\angle ACD$, looks same as the known angle, $30°$. So align the arms of the $30°$ angle that you created on your answer sheet with the arms of $\angle ACD$ and see that it is also $30°$.

The Proper Solution:

Consider the triangles ΔABC and ΔACD. $\angle A$ is common to both triangles and $\angle ADC = \angle ACB = 90°$

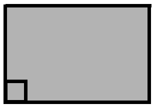


Since the addition of all 3 angles of any triangle is $180°$, their 3rd angles must be the same as well. Therefore, $\angle ACD = x° = \angle ABC = 30°$

2.Use your Answer Sheet to Measure Special Angles

Getting 90^{o}

Each corner of your answer sheet is 90 degrees, as shown in the figure.



Determining a Range for any Angle

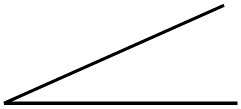
You can use your answer sheet as a ruler and easily determine any angle's minimum and maximum possible values. We will explain how this is done with an example.

To be able to demonstrate the power of the techniques in this section, we purposefully choose a question impossible to answer: an angle, drawn to scale, with no other information provided. You are asked to find the measure of the angle.

A word of advice: The solution seems too long to be useful. However, to use the techniques provided is easier done than said. The ideas behind these techniques will also teach you a lot about special triangles which appear in many SAT math questions.

Example: (Hard)

What is the measure of the angle in the figure?

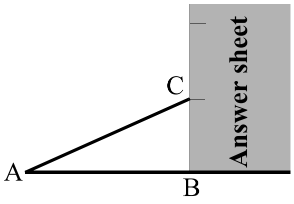


(A) 10^{o}
(B) 15^{o}
(C) 20^{o}
(D) 25^{o}
(E) 30^{o}

Solution:

Compare $\angle A$ with 30^{o} by creating a 30^{o} angle as follows:

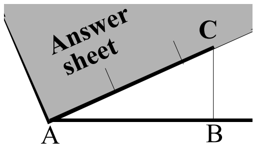
Align the edge of your answer sheet with one arm ($\overline{AB}$)of the angle as shown in the figure and draw a perpendicular line, $\overline{BC}$, from any point you choose on $\overline{AB}$.



In the figure $\overline{BC} \perp \overline{AB}$ and ΔABC is a right triangle.

Mark the distances BC and $2 \times BC$ on your answer sheet.

Align the marked edge of your answer sheet with $\overline{AC}$ as shown in the figure. You can clearly see that AC is longer than $2 \times BC$.



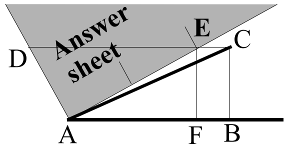
If AC were equal to $2 \times BC$, then ΔABC would be a 30^o- 60^o- 90^o special triangle with $\angle A = 30°$. Since AC is longer than $2 \times BC$, $\angle A < 30°$. You can eliminate case (E) for sure.

If you know the approximate size of a 30^o angle, you would know that $\angle A$ is less, but close to 30^o. Hence you could conclude that the answer must be 25^o, case (D). If you did that, you would have been right.

However, you can go one step further and get a 30^o angle to compare it with $\angle A$.

Getting 30^o:

Draw a line, $\overline{CD}$, passing through point C and parallel to $\overline{AB}$ as shown in the figure.



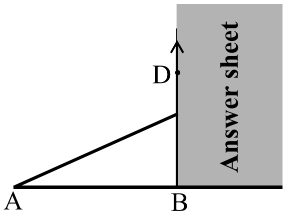
Place the beginning of your self-made ruler on point A. Then rotate your answer sheet so that second mark on your sheet is on $\overline{CD}$. Draw $\overline{EF} \perp \overline{AB}$ In the figure, $\angle EAB = 30°$, because $AE = 2 \times EF$.

At this point you can decide that $\angle A$ is smaller than, but close to 30^o, and choose (D), 25^o, as your answer. It is indeed the correct answer.

On the other hand, you can take one step further and prove that $\angle A$ is larger than 20^o. To achieve this, we will first obtain a 45^o angle and then $45/2 = 22.5^o$ angle to compare with $\angle A$.

Getting 45^o:

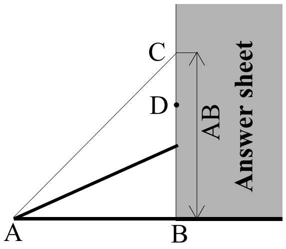
Align the edge of your answer sheet with one arm ($\overline{AB}$) of the angle as shown in the figure and draw a ray ($\overrightarrow{BD}$) perpendicular to $\overline{AB}$.



You can choose any point on $\overline{AB}$ to draw $\overrightarrow{BD}$.

Next, align your answer sheet with $\overline{AB}$ and mark the distance AB on it.

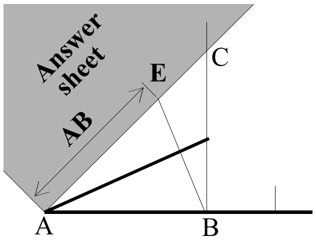
Then, align the edge of your answer sheet with $\overrightarrow{BD}$ one more time and mark the distance AB on $\overrightarrow{BD}$ at point C, as shown in the figure.



In the figure ΔABC is a 45^o- 45^o- 90^o special right triangle with $\angle CAB = 45°$, because $AB = BC$ and $\angle B = 90°$

Getting 22.5^o:

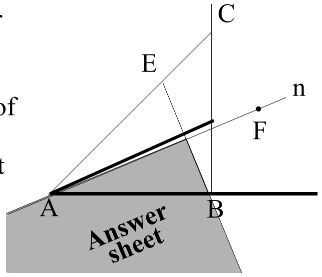
Align your answer sheet with $\overline{AC}$ and mark the distance AB at point E on it, as shown in the figure.



Draw $\overline{BE}$.

In the figure, ΔABE is an isosceles triangle with $AB = AE$.

Finally, align one edge of your answer sheet with $\overline{BE}$ and slide it so that the perpendicular edge of your answer sheet passes through point A, as shown in the figure.



Draw the line n.

$\angle FAB = \frac{\angle BAE}{2} = \frac{45}{2} = 22.5°$, because $\overline{AF}$ is the angle bisector of $\angle BAE$.

It is clear that $\angle A$, both arms are shown by thick lines, is slightly more than 22.5^o.

Therefore, you can eliminate cases (A), (B) and (C). The answer is (D) as we concluded before.

We could choose to compare $\angle A$ with 22.5^o first. This would have been a better choice, since $\angle A$ is closer to 22.5^o than 30^o. It also eliminates three cases, (A), (B) and (C), not just one case, (E).

Below is a list of angles that you can get in addition to 30^o, 45^o and 22.5^o, by using similar techniques.

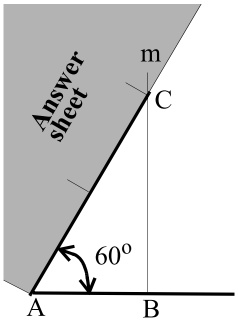
Getting 60^o:

Draw a line and place two points, A and B, on it.

Draw a line, m, passing through point B and perpendicular to $\overline{AB}$ as shown in the figure.

Mark the distances AB and $2 \times AB$ on your answer sheet.

Place the corner of your answer sheet on point A and rotate it until $2 \times AB$ mark is on line m.



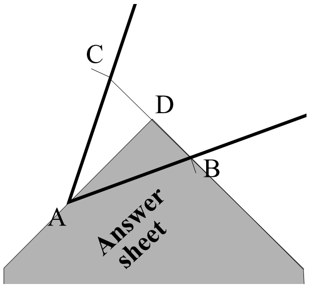
Draw $\overline{AC}$.

$\angle A = 60°$, because $AC = 2 \times AB$.

Getting Half of Any Angle:

You can do this in two different ways. Both are equally simple.

First method:
Starting from the vertex, A, of the angle, mark equal distances, AB and AC, on both arms of the angle, as shown in the figure.



Draw $\overline{BC}$.

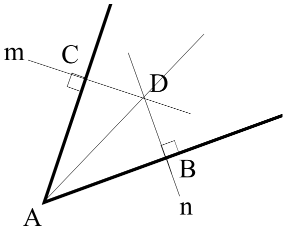
Align one edge of your answer sheet with $\overline{BC}$ and slide it until the other edge passes through point A.

Draw $\overline{AD}$.

$$\angle DAB = \angle CAD = \frac{\angle A}{2}$$

Second method:
Starting from the vertex, A, of the angle, mark equal distances, AB and AC, on both arms of the angle, as shown in the figure.

Draw lines, m and n, passing through points C and B, and perpendicular to each arm of the angle.



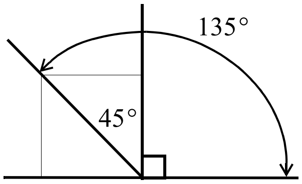
Draw $\overline{AD}$.

$$\angle DAB = \angle CAD = \frac{\angle A}{2}$$

You can get several combinations of the angles provided in this appendix.

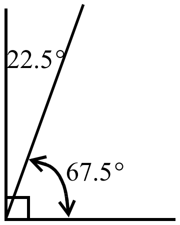
Getting 90 + 45 = 135°

Draw a 90^o angle on your question booklet by using the edge of your answer sheet and add 45 degrees to it as shown in the figure.



Getting $90^o - 22.5^o = 67.5^o$

Draw a 90^o angle on your question booklet by using the edge of your answer sheet and subtract 22.5^o from it as shown in the figure.



Getting 112.5^o, 202.5^o and 157.5^o

Use similar methods to get the angles as follows:

$90^o + 22.5^o = 112.5^o$

$180^o + 22.5^o = 202.5^o$ (note that 180 degrees is a straight line)

$180^o - 22.5^o = 157.5^o$

Getting 6.5^o, 7.5^o, 15^o, 37.5^o, 52.5^o, 75^o, 82.5^o, 105^o, 120^o, 150^o, 165^o, 195^o, 210^o, 240^o

Here is how you can draw these angles as the addition or subtraction of the "known" angles:"

$22.5 - 15 = 6.5^o$

$30 - 22.5 = 7.5^o$

$30/2 = 15^o$

$22.5 + 15 = 37.5^o$ or $60 - 22.5 = 37.5^o$

$30 + 22.5 = 52.5^o$

$90 - 15 = 75^o$ or $60 + 15 = 75^o$

$60 + 22.5 = 82.5^o$

$90 + 15 = 105^o$

$180 - 60 = 120^o$

$180 - 30 = 150^o$

$180 - 15 = 165^o$

$180 + 15 = 195^o$

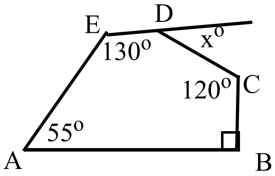
$180 + 30 = 210^o$

$180 + 60 = 240^o$

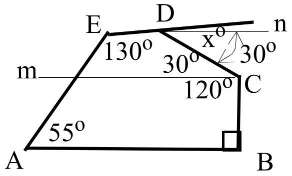
Even if you can not measure the angle in question accurately, you can at least use these methods to eliminate some of the answers.

Examples:

1. (Medium)
In the figure, what is the value of the outer angle, x?



Solution by Measurement:

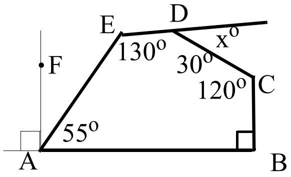


It looks like x is close to 30. You know that $\angle C = 120°$. You can get 30^o by drawing parallel lines, m and n, to $\overline{AB}$ from points C and D, respectively, as shown in the figure.

Now it is clear that x is close to but larger than 30.

The measures of the three known angles in question, $\angle B$, $\angle C$ and $\angle E$, are multiples of 10^o and only one of them, $\angle A$, is 55^o. Because the answer is usually obtained by adding or subtracting the known angles in the question and multiples of 180^o (180^o, 360^o, 540^o etc.), the answer most probably ends with 5. So, you can guess correctly that x = 35.

Another Solution by Measurement:



You could also mark the 35^o angle, $\angle FAE$ in the figure, on your answer sheet and compare it to x^o to see that x^o is 35^o.

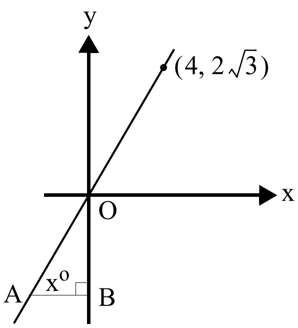
Use any angle available to you to make your measurement/guess.

The Proper Solution:

The inner angles of a 5-sided polygon is $(5 - 2)180 = 540^o$ ➔
$\angle D = 540 - 90 - 120 - 130 - 55 = 145^o$ ➔
$x = 180 - 145 = 35^o$

2. (Medium)
In the figure, what is the value of x?

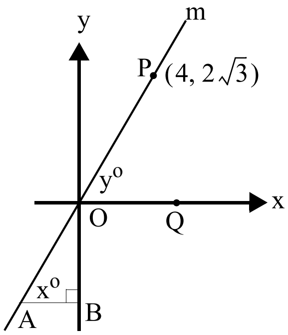
Solution by Measurement:



On your answer sheet, mark the distances AB and $2 \times AB$.

Align your answer sheet with $\overline{AO}$ and compare $2 \times AB$ with AO. You will easily see that $AO = 2 \times AB$. Hence ΔABO is a 30^o - 60^o - 90^o special triangle with $x = 60$.

The Proper Solution:



The slope, s, of line m is $\frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.

On the other hand, $s = \tan(y) = \frac{\sqrt{3}}{2}$ ➔
$y = 60$

In the figure, $\overline{AB}$ is parallel to x-axis. ➔
$x = y = 60$

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